Chaos Synchronization in Complex Networks *

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We employ several simple numerical experiments to show that the statements that short mean path length or low heterogeneity enhances synchronizability in complex networks could be misleading. Furthermore, we point out that the impacts of the structural parameters on the synchronizability are complicated even if we could change one structural parameter independently.

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During the past years, the pioneering works[1,2] have inspired great interests in the research related to complex networks. It has been pointed out that the topological properties of the complex networks have significant impacts on dynamical processes taking place on the networks. So far, though most of the works has been focused on the topological and geometrical properties of complex networks, the interests on dynamics defined on complex networks have grown rapidly. For example, the epidemic spreading,[3] prisoner's dilemma game,[4] and equilibrium statistical physics models such as the Ising model[5] have been investigated.

Chaos synchronization in regular networks is now a well-known Phenomenon.[6−8] However how structural properties of complex networks impact on chaos synchronization is still a mysterious problem. Barahona and Pecora pointed out that the networks with small world property produce synchronizability more efficiently than standard regular and pure random networks.[9] The general argument underlying this confidence is that the communication is more efficient because of the smaller mean path length in small world network. In a recent paper of Nishikawa *et al.*,[10] the efficiency of smaller mean path length on chaos synchronization is suppressed by heterogeneity of networks. In their work, heterogeneity plays a dominant role in determining the synchronizability of networks. However in discussing chaos synchronization in complex networks, it has to be noted that there are many quantities to characterize the structural properties of networks (we term them as structural parameters). For example, the number of nodes (N), the mean connectivity ($\overline{k}$), the mean path length (D), the clustering coefficient (C) and the heterogeneity ($k_{\max}/k_{\min}$), to name but a few. The statements on the role of either mean path length or heterogeneity in Refs. [9,10] are made with changes of other structure parameters. Therefore, both the views are not convincing. In this Letter, we try to reexamine the roles of heterogeneity and mean path length on synchronizability by some simple numerical experiments. We find that the situations are more complicated even if we succeed in separating one structural parameter from others.

Consider N identical dynamical systems with symmetrical coupling. The equations of motion are

$$\dot{u}_i = f(u_i) + \sigma \sum_{j=1}^{N} L_{ij} H(u_j), \quad j = 1, 2, \cdots, N, \quad (1)$$

where $u_j \in R^n$. The function $f(u)$ is nonlinear and capable of exhibiting chaotic behaviour, $H(u)$ is the output function, and σ is the overall strength of coupling. L is the Laplacian matrix representing the connection topology of the network, defined as $L_{ij} = 1$ if nodes i and j are connected, $L_{ii} = -k_i$ if node i is connected to k_i other nodes, and $L_{ij} = 0$ otherwise. In a symmetrically coupled network, there are two types of isolated dynamics.[11] In one type, synchronicity may be lost by increasing σ; while in the other type, increasing σ only enhances the synchronicity. Here we consider the first type of systems. Based on generic theories,[11,12] the synchronization of coupled oscillators can be analysed through the eigenvalue spectrum of the Laplacian matrix L. Denote $\lambda_1 = 0 \geq \lambda_2 \geq \cdots \geq \lambda_N$ as the eigenvalues of L. λ_N/λ_2 quantifies the synchronizability of the underlying network. The larger the λ_N/λ_2 value, the greater the difficulty to synchronize the network.[9]

The first numerical experiment starts with a globally connected lattice with N nodes and $N-1$ edges per node. Then we randomly cut the edges with probability p provided that the cutting operation does not lead network to disconnected parts. The mean connectivity of the network is $\overline{k} = (N-1)(1-p)$. We let $N = 100$ and vary the mean connectivity $\overline{k}$ in the range of [4, 60]. For each $\overline{k}$, we run tens of realizations. The structural parameters C, $k_{\max}/k_{\min}$, and D, and

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the eigenvalue spectrum for each realization are calculated. The relation between $k_{\max}/k_{\min}$ and D, as shown in Fig. 1(a), indicates that both the parameters change simultaneously and no simple dependence between them can be found.

Synchronizability λ_N/λ_2 as functions of $k_{\max}/k_{\min}$, C, and D are shown in Figs. 1(b)–1(d), respectively. Figures 1(b)–1(d) show the overall tendencies that strong heterogeneity and large mean length make the network more synchronizable while the clustering coefficient behaves in another way. No average over the realizations is made due to the fact that the synchronization transition is a dynamical bifurcation and is not a stochastic problem.

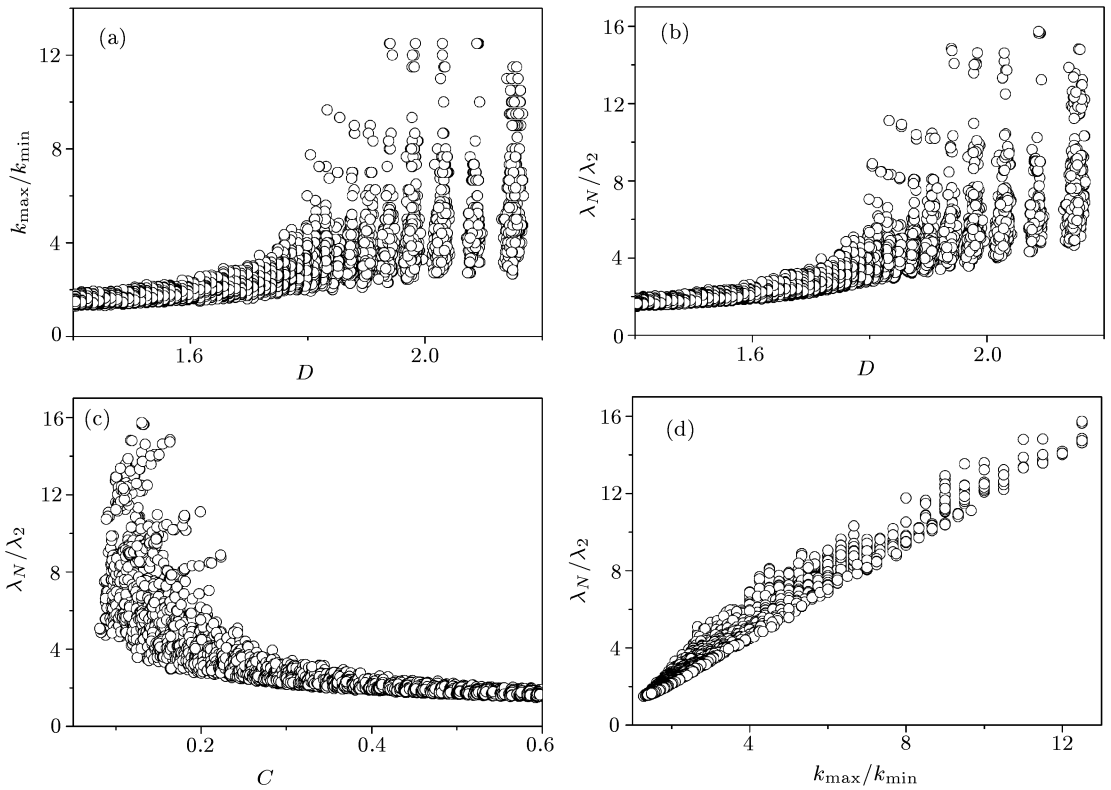

Fig. 1. (a) The relation between the mean path length and the heterogeneity of the network. The relations of the synchronizability with (b) the mean path length, (c) the cluster coefficient, and (d) the heterogeneity of the network, respectively.

The second numerical experiment is made on a simpler case where the network consists of only four nodes and three edges. There are only two possible configurations and we plot them in Fig. 2. These two configurations have the same mean connectivity and cluster coefficient. The heterogeneity for (a) (or (b)) is 3 (or 2) and the mean path length for (a) (or (b)) is 1.5 (or 5/3). The eigenvalue spectrum is $(0, -1, -1, -3)$ for (a) and $(0, -0.5858, -2, -3.4142)$ for (b), that is, the synchronizability is 3 for Fig. 2(a) while 5.83 for Fig. 2(b). In contrary to the statements in Ref. [10], the toy experiment shows that the synchronization becomes more possible for high heterogeneity at the expense of the change of the mean path length.

The synchronizability as a function of structural parameters could be still complicated even if we succeed in separating single structural parameter from others. We take the third numerical experiment. Consider a regular chain with $k = 2$ except for the boundary nodes, e.g., the first and last nodes, we add only one edge connected to the first node to the network. Let $N = 20$, there are 19 possible configurations with the same mean connectivity. If the free end of the new edge is away from the first node with a distance of 3 or more, the clustering coefficients are the same for remaining configurations either. To have the same $k_{\max}/k_{\min}$, the connection of the new edge to the Nth

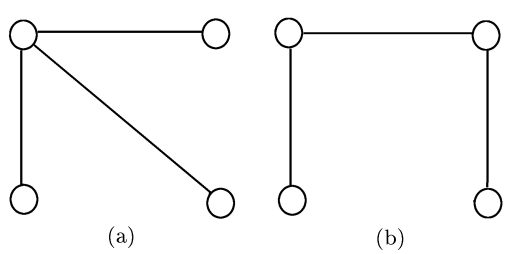

Fig. 2. The two configurations of the network with four nodes and three edges.

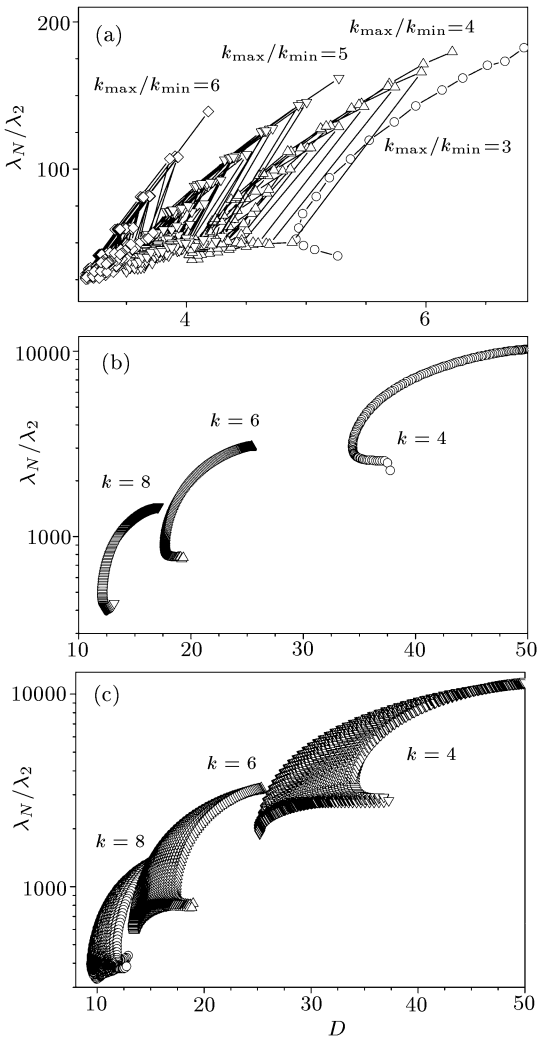

Fig. 3. The synchronizability as functions of the mean path length of the networks are plotted. (a) The influences of the number of the new edges, $N = 20$. (b)–(c) The influences of the connectivity with one new or two edges, $N = 150$.

node is excluded. Therefore, 16 configurations remain where only one structural parameter D remains different. The synchronizability as a function of D is shown in Fig. 3(a). The multi-valued property of the function in Fig. 3(a) is caused by D. Since D is a statistical quantity averaged over the network, it is possible that different configurations of the network have the same D. For example in this case, the mean path length will have nearly the same value if the free end of the new edge is connected to the 14th or 19th node while the synchronizability for them is quite different. Such a discussion is applicable to other results in Fig. 3. Then we add two, three, and four edges to the regular chain, respectively. All the new edges connect to the first node. Cautions are taken to keep other structural parameters unchanged except for D. The results are also presented in Fig. 3(a). With the addition of more new edges, the mean path length decreases and synchronizability increases overall. However the non-monotonic property of the relation between the synchronizability and the mean path length is unchanged. For further investigation, we let $N = 150$ and consider the effects for the different connectivity. We show the results for only one new edge introduced to the chain in Fig. 3(b) while two new edges added in Fig. 3(c). From Figs. 3(b) and 3(c), we know that large connectivity tends to shorten the mean path length and to enhance the synchronizability of the network. However such tendencies are not monotonic either.

In summary, we have employed several simple numerical experiments to show that the statements that short mean path length or low heterogeneity enhances synchronizability in complex networks could be misleading. Furthermore, we point out that the impacts of the structural parameters on the synchronizability are complicated even if we could change one structural parameter independently.

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