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Primordial Black Holes and Cosmic String Loop

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The elliptic loop equation of cosmic string is studied in the Minkowski space and during the radiation-dominated era in the Robertson-Walker universe. A remarkable fact about this equation is that a loop may never contract to one with a Schwarzschild radius, if it is sufficiently non-circular.

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The cosmic string scenario for the formation of large scale structure has many observable features.[1] We have proved that topologically nontrivial string solutions exist naturally in the flipped $SU(5)$ model from heterotic superstring.[2] Vilenkin has discussed the circular loop equation of cosmic string and then showed a solution in the case of Minkowski space.[3] We have studied analytically the circular loop equation during the radiation-dominated era in Robertson-Walker universe.[4] It is well known that a sufficiently smooth, circular cosmic string loop may collapse to form a black hole.[5] During the evolution of a network of cosmic strings, some cosmic string loops may collapse to form black holes. In this case, the observational restrictions on a population of primordial black holes may be used to restrict such a cosmic string scenario. Observational restrictions are used to restrict f, the fraction of cosmic string loop which collapse to form black holes, and μ, the string mass per unit length. The strongest restriction on the parameters f and μ is due to the energy density in 100 MeV photons radiated by the black holes.[6] The γ-ray background limit requires $f \leq 10^{-17}$ for $G\mu = 10^{-6}$. Here, G is Newton constant. If the rough estimates of the magnitude of $f \sim 10^{-12}$ are reliable,[5,6] then these results would indicate that $G\mu \leq 10^{-11}$ is necessary for compatibility with γ-ray background. In this ease, one could rule out the cosmic string scenario of large scale structure formation, which demands $G\mu \sim 10^{-6}$. We are not confident in these crude estimates of f, however, as the authors have indicated in Refs. 5 and 6. In fact, a loop may never contract to a loop with a Schwarzschild radius, if it is sufficiently non-circular. Clearly, a more detailed investigation is necessary to study elliptic loops.

The purpose of this essay is to study the elliptic loop equation of cosmic string in the Minkowski space and Robertson-Walker universe. If the loops expand with Hubble flow at the time of formation of loops, the circular loops contract always under its own tension to those within its Schwarzschild radius, it will form a black hole. For the elliptic loop, if the major semi-axis is larger than the minor semi-axis by several Schwarzschild radii, it should not collapse to form a black hole.

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The Robertson-Walker metric is written as

$$ds^2 = c^2dt^2 - R^2(t)[d\omega^2 + \sigma_k^2(\omega)(d\theta^2 + \sin^2\theta d\psi^2)] , \tag{1}$$

where $R(t)$ is the scale factor and $\sigma_k(\omega)$ are taken as $\sigma_1 = \sin\omega$, $\sigma_0 = \omega$ and $\sigma_{-1} = \sinh\omega$. Let us assume for simplicity that the string lies in the hypersurface $\theta = \pi/2$, then the space-time coordinates $\xi^i (i = 0 \text{ and } 1)$ of the world-sheet parametrized by $\xi^0 = t$, $\xi^1 = \psi$ become $x = (ct, \omega(t,\psi), \pi/2, \psi)$. In the case of an elliptic loop $\omega^2(t,\psi) = a^2(t)\cos^2\psi + b^2(t)\sin^2\psi$, the action functional of an elliptic string loop is given by

$$S = -\mu \int\int dt d\psi R(t) \left[c^2(\frac{\partial\omega}{\partial\psi})^2 + c^2\sigma_k^2(\omega) - R^2(t)\sigma_k^2(\omega)(\frac{\partial\omega}{\partial t})^2\right]^{\frac{1}{2}} , \tag{2}$$

which leads to the following equations of motion:

$$\begin{aligned}
&(c^2 - R^2\dot{a}^2)(a^2 - b^2) + \sigma_k^2(a)aR^2\left[\ddot{a} - \frac{2R\dot{R}}{c^2}\dot{a}^3 - \delta_k(a)\dot{a}^2 + \frac{3\dot{R}\dot{a}}{R} + \delta_k(a)\frac{c^2}{R^2}\right] = 0 , \\
&(c^2 - R^2\dot{b}^2)(b^2 - a^2) + \sigma_k^2(b)bR^2[\ddot{b} - \frac{2R\dot{R}}{c^2}\dot{b}^3 - \delta_k(b)\dot{b}^2 + \frac{3\dot{R}\dot{b}}{R} + \delta_k(b)\frac{c^2}{R^2}] = 0 ,
\end{aligned} \tag{3}$$

where $\delta_k(\omega) = [d\sigma_k(\omega)/d\omega]/\sigma_k(\omega)$. The physical loop size is

$$R_p(t,\psi) = R(t)\sigma_k(\omega) , \tag{4}$$

and $a_p(t) = R(t)\sigma_k(a) = R(t)\omega(t,0)$, $b_p(t) = R(t)\sigma_k(b) = R(t)\omega(t,\pi/2)$.

For the circular loop, Eqs. (3) are reduced to

$$\ddot{a} - \frac{2R\dot{R}}{c^2}\dot{a}^3 - \delta_k(a)\dot{a}^2 + 3\frac{\dot{R}}{R}\dot{a} + \delta_k(a)\frac{c^2}{R^2} = 0 . \tag{5}$$

We have the solutions of the time evolution of circular loops during the radiation-dominated era in the $k = 0$ Robertson-Walker universe.[4] We assume $\dot{a}(t_0) = 0$, i.e., the loops expand with Hubble flow by neglecting the peculiar velocities of loops at the time of formation of loops. From Eq. (5), we have $\ddot{a} < 0$ so there is a maximum value of function $a(t)$ at $t = t_0$. When $t > t_0$, $a(t)$ will continue to decrease monotonically before $a(t)$ reaches a zero point where string loop collapses to a point. We may prove this argument by contradiction. Suppose $a(t)$ decreased and reached a minimum and re-increased. At its imagined minimum we have $\dot{a} = 0$ and $\ddot{a} > 0$. Looking at Eq. (5) and the condition $\dot{a} = 0$ at minimum point

$$\ddot{a} = -\delta_k(a)\frac{c^2}{R^2} , \tag{6}$$

the left-hand side is positive and the right-hand side is negative in expanding universe. (In the $k = 0$ and -1 cases, $\delta_k(a)$ is always positive. In the $k = +1$ case, $\delta_{+1}(a) > 0$ for $a < \pi/2$; $\delta_{+1}(a) < 0$ for $a > \pi/2$ which corresponds to collapsed universe.) We now have proved a theorem as follows:

Theorem 1 In the general Robertson-Walker universe, if the loops expand with Hubble flow at the time of formation of loops, the cosmic string loops occur always collapsing.

Furthermore, we can also prove following theorem using the solution of Eq. (5).

Theorem 2 In the general Robertson-Walker universe, if the loop expands with Hubble flow at the time of formation of loop, the ratio $c\tau_{\rm p}/R_{\rm p}(t_0)$ tends to $\pi/2$ for the little initial radius limit $[R_{\rm p}(t_0) \ll R(t_0)]$, where $\tau_{\rm p}$ is the lifetime of loop.

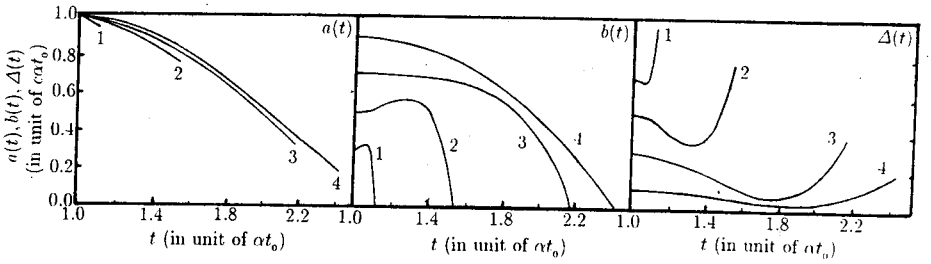

Fig. 1. Evolution of loop size versus the different a_0/b_0 values. Curves 1, 2, 3 and 4 correspond to 10/3, 2, 10/7 and 10/9, respectively.

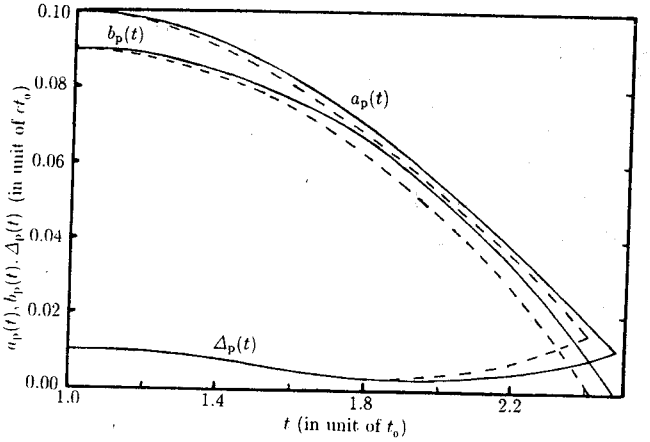

Fig. 2. Evolution of loop size ($a_{\rm p}(0) = 0.1$ and $b_{\rm p}(0) = 0.09$) in the $k = 0$ Robertson-Walker universe. The dashed line is the corresponding curves in the Minkowski space.

For the elliptic loop in the Minkowski space, Eqs. (3) are reduced to

$$a^3\ddot{a} - (2a^2 - b^2)\dot{a}^2 + (2a^2 - b^2)c^2/R^2 = 0 ,$$
$$b^3\ddot{b} - (2b^2 - a^2)\dot{b}^2 + (2b^2 - a^2)c^2/R^2 = 0 . \tag{7}$$

Equations (7) is invariant under the transformation $\tilde{a} = \alpha a$, $\tilde{b} = \alpha b$ and $\tilde{t} = \alpha t$. Therefore, the initial values can be taken as $a(t_0) = 1$, $b(t_0) = b_0$, $0 < b_0 < 1$ and $\dot{a}(t_0) = \dot{b}(t_0) = 0$. Equations (7) can be solved numerically by a Runge-Kutta algorithm. The evolution curves $\Delta(t) \equiv a(t) - b(t)$ are plotted in Fig. 1. The evolution curves end at a point $t = t_{\rm c}$ because the minor axis contracts always to $b(t_{\rm c}) = 0$. It has been seen that there is a minimum $\Delta_{\rm min}(b_0)$ for the function $\Delta(t)$ versus the different $a(t_0)$ and $b(t_0)$. Using numerical results, we have

$$\min[\Delta_{\rm min}(b_0)] = 0.205\Delta(t_0) . \tag{8}$$

Furthermore, we can prove analytically that the function $\Delta(t)$ reached a minimum at the stationary point t_* by using $\dot{a}^2(t_*) = \dot{b}^2(t_*)$ and Eqs. (7). Therefore, we now have shown an argument as follows:

Theorem 3 In the Minkowski space, if the loop is static at the time of formation of loop, and when the major semi-axis is longer than the minor semi-axis by several Schwarzschild radii, it should not collapse to form a black hole.

Next, we shall discuss the $k = 0$ Robertson-Walker universe in the radiation-dominated era, for which the scale factor $R(t)$ is

$$R(t) = R_0(2H_0t)^{1/2} , \tag{9}$$

where $R_0 = R(t_0)$ and H_0 is the Hubble constant.

By using Eqs. (3) and (9), we have

$$\begin{aligned}
&(\frac{c^2}{2H_0R_0^2t} - \dot{a}^2)(a^2 - b^2) + a^2(\frac{c^2}{2H_0R_0^2t} + \frac{3a\dot{a}}{2t} - \frac{2H_0R_0^2a\dot{a}^3}{c^2} - \dot{a}^2 + a\ddot{a}) = 0 , \\
&(\frac{c^2}{2H_0R_0^2t} - \dot{b}^2)(b^2 - a^2) + b^2(\frac{c^2}{2H_0R_0^2t} + \frac{3b\dot{b}}{2t} - \frac{2H_0R_0^2b\dot{b}^3}{c^2} - \dot{b}^2 + b\ddot{b}) = 0 .
\end{aligned} \tag{10}$$

Equations 10 can be solved numerically by a Runge-Kutta algorithm. The evolution curves are plotted in Fig. 2. We find that the revisions of the evolution curves are tiny in comparison with ones in the Minkowski space. For an elliptic and planar cosmic string loop of mass m, the loop equation indicate that such a string will expand and contract under its own tension, with a maximum girth $L_{\max} = m/\mu$. When major semi-axis $a(t)$ contracts to string's Schwarzschild radius $R_s = 2Gm = 2G\mu L_{\max}$, string loop will form a black hole. If the elliptic loop expand with Hubble flow at the time of formation of loop, we have $R_s/a(t_0) \sim G\mu \sim 10^{-6}$. On the other hand, using numerical result, we have $\Delta(t_0)/\Delta_{\min} \approx 5$. Therefore, in the Robertson-Walker universe, if the elliptic loop expand with Hubble flow at the time of formation of loop, and when $\Delta(t_0)$ is larger than a few Schwarzschild radii, the cosmic string loop should not collapse to form a black hole.

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