Lower Bounds on Negative Energy Densities for the Scalar Field in Flat Spacetime *

SHU Wei-Xing(舒维星)[1], YU Hong-Wei(余洪伟)[2**], REN Zhong-Zhou(任中洲)[1], WU Pu-Xun(吴普训)[2], LI Fei(李飞)[1]

[1]*Department of Physics, Nanjing University, Nanjing 210093*
[2]*Department of Physics, Hunan Normal University, Changsha 410081*

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We obtain a lower bound on the spacetime-weighted average of the energy density for the scalar field in four-dimensional flat spacetime. The bound takes the form of a quantum inequality. The inequality does not rely on the quantum state and its form is only related to the weights, namely the spacetime sampling functions which are assumed to be smooth, positive and compactly supported. It is found that the inequality is just equal to the temporal quantum energy inequality. When the characteristic length of the temporal sampling function tends to zero, the lower bound becomes divergent. This is consistent with the fact that the spatial restriction on negative energy density does not exist in four-dimensional spacetime.

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It is well known in quantum field theory that there exist states in which the local energy density can be negative.[1] Thus the classical energy conditions, such as the weak energy condition, are violated. In practice, the Casimir effect[2,3] and squeezed states of light[4] gave indirect experimental support to the existence of negative energy densities. Theoretically the negative energy is involved in the evaporation of black holes.[5]

However, if the energy density remained negative with arbitrarily large magnitudes for arbitrarily long time, many bizarre phenomena can occur. These include the violation of the second law of thermodynamics[6] or of cosmic censorship,[7,8] "warp drive"[9] and time machines.[10] Hence, great research has been carried out to study the extent to which quantum fields may violate the classical energy conditions and to search for possible restrictions on negative energy densities.[11−28] One approach is to average the classical energy conditions over an observer's geodesics.[11] Another kind of severe restrictions, originated by Ford and Roman,[12] is the quantum energy inequality (QEI) in which the energy density is averaged with a temporal sampling function along the worldline of an observer. For the scalar field, the energy density measured by an inertial observer obeys the QEI

$$\langle\rho\rangle_T = \int_{-\infty}^{\infty} dt \,\langle :T_{00}(t,\boldsymbol{x}): \rangle_\psi f(t) \geq -\frac{c}{t_0^{n+1}}. \qquad (1)$$

Here $\langle :T_{00}(t,\boldsymbol{x}): \rangle_\psi$ is the renormalized energy density in an arbitrary quantum state $|\psi\rangle$, $f(t)$ is a smooth sampling function for which the time integral is unity and the characteristic width is t_0, n denotes the dimension of space, and c is a constant relevant to $f(t)$. The QEI indicates that the more negative the energy density becomes, the shorter it lasts, and has been used to show that the above-mentioned exotic phenomena are impossible in reality.[13,14] Later Flanagan established the optimal QEI which is free of the choice of sampling functions for the massless scalar field in two-dimensional flat spacetimes.[15] Fewster and colleagues developed an elegant method and derived a state-independent QEI in static curved spacetimes of any dimensions.[16,17] In four-dimensional flat spacetimes the QEI is

$$\langle\rho\rangle_T \geq -\frac{1}{16\pi^3}\int_m^{\infty} du |\widehat{f^{1/2}}(u)|^2 u^4 Q_3\Big(\frac{u}{m}\Big), \qquad (2)$$

where

$$Q_3(x) = \Big(1-\frac{1}{x^2}\Big)^{1/2}\Big(1-\frac{1}{2x^2}\Big) - \frac{1}{2x^4}\log[x+(x^2-1)^{1/2}] \qquad (3)$$

is a bounded non-negative function smaller than unity and $\widehat{f}$ denotes the Fourier transform of f. Following similar ways, various authors have established quantum energy inequalities (QEIs) for the Dirac,[18,21] Maxwell,[19] Proca[20] and Rarita–Schwinger[24] fields.

Compared to the QEIs with the energy densities averaged over time, less work has been carried out for spatial and spacetime QEIs involving averaging over space and a spacetime volume respectively.[25] On the issue, Vollick[21] and Fewster and Hollands[26] found

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** To whom correspondence should be addressed. Email: hwyu@hunnu.edu.cn

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spatial QEIs in two dimensions for scalar fields and conformal fields respectively, but Ford *et al.* constructed a class of states to provide a counterexample to the existence of a spatial QEI in a four-dimensional spacetime.[27] In addition, Borde and colleagues established a spacetime QEI which does not rely on the quantum state in the massless scalar field in a two-dimensional spacetime.[28]

In this Letter, we study the energy density of the quantum scalar field in a four-dimensional Minkowski spacetime and show there exists a lower bound on the spacetime-weighted average of the energy density against the spacetime sampling functions which are assumed to be smooth, positive and compactly supported. Our plan is to extend the approach developed by Fewster and colleagues[16−18] and to obtain a quantum inequality bound which is just equal to Eq. (2). We will also give the quantum inequality for the massless field. Note that we work in units where $c = \hbar = 1$ and take the sign of the metric tensor to be $(+---)$.

We first review the basic knowledge of scalar field theory. The Klein–Gordon equation is

$$(g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi = 0. \tag{4}$$

The quantum field operator ϕ can be expanded as

$$\phi(x) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega_{\boldsymbol{k}}}}(a_{\boldsymbol{k}}e^{-ik\cdot x} + a^\dagger_{\boldsymbol{k}}e^{ik\cdot x}), \tag{5}$$

where the annihilation and creation operators $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$ obey the canonical commutation relations

$$[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}] = [a^\dagger_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}] = 0 \quad [a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}] = (2\pi)^3\delta(\boldsymbol{k} - \boldsymbol{k}'), \tag{6}$$

and $\omega_{\boldsymbol{k}} = \sqrt{\|\boldsymbol{k}\|^2 + m^2}$. The classical energy density for the massive scalar field is

$$T_{00} = \frac{1}{2}\Big[(\partial_0\phi)^2 + \sum_{i=1}^{3}(\partial_i\phi)^2 + m^2\phi^2\Big]. \tag{7}$$

Hence the renormalized expectation value of the energy density in an arbitrary quantum state at a spacetime point $(t, \boldsymbol{x})$ is

$$\begin{aligned}\langle :T_{00}(t,\boldsymbol{x}): \rangle = \int \frac{d^3\boldsymbol{k}d^3\boldsymbol{k}'}{(2\pi)^6}\Big\{ & \frac{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'} + \boldsymbol{k}\cdot\boldsymbol{k}'}{4\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}} \\ & \cdot [\langle a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}'}\rangle e^{i(k-k')\cdot x} - \langle a_{\boldsymbol{k}}a_{\boldsymbol{k}'}\rangle e^{-i(k+k')\cdot x} \\ & + \langle a^\dagger_{\boldsymbol{k}'}a_{\boldsymbol{k}}\rangle e^{-i(k-k')\cdot x} - \langle a^\dagger_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}'}\rangle e^{i(k+k')\cdot x}] \\ & + \frac{m^2}{4\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}[\langle a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}'}\rangle e^{i(k-k')\cdot x} \\ & + \langle a_{\boldsymbol{k}}a_{\boldsymbol{k}'}\rangle e^{-i(k+k')\cdot x} + \langle a^\dagger_{\boldsymbol{k}'}a_{\boldsymbol{k}}\rangle e^{-i(k-k')\cdot x} \\ & + \langle a^\dagger_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}'}\rangle e^{i(k+k')\cdot x}]\Big\}. \end{aligned} \tag{8}$$

Now we extend Fewster's method and prove an inequality which is critical to obtain the lower bound on the spacetime-weighted average of the energy density. We first define the function g as

$$g(A) = \frac{\widehat{f^{1/2}}(A)}{\sqrt{2\pi}}. \tag{9}$$

Here $\widehat{f}$ is the Fourier transform of f, i.e.

$$\widehat{f}(A) = \int_{-\infty}^{\infty} dx\, f(x)e^{-iAx}, \tag{10}$$

and g obeys

$$g \star g = \widehat{f}, \tag{11}$$

where the convolution $\star$ is defined by

$$(h_1 \star h_2)(A) = \int_{-\infty}^{\infty} dA' h_1(A - A')h_2(A'). \tag{12}$$

Following the method of deriving QEI[16−18] we define two families of operators on the Fock space of the Minkowski vacuum by

$$\begin{aligned}\mathcal{O}^{\pm}_{\mu} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}[& \overline{g_0(\mu_0 - \omega_{\boldsymbol{k}})}g_1(\mu_1 - k_1) \\ & \cdot g_2(\mu_2 - k_2)g_3(\mu_3 - k_3)p_{\boldsymbol{k}}a_{\boldsymbol{k}} \\ & \pm \overline{g_0(\mu_0 + \omega_{\boldsymbol{k}})}g_1(\mu_1 + k_1)g_2(\mu_2 + k_2) \\ & \cdot g_3(\mu_3 + k_3)\overline{p}_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}], \end{aligned} \tag{13}$$

where $p_{\boldsymbol{k}}$ is a real valued function about $\boldsymbol{k}$ on $\boldsymbol{R}^3$, growing no faster than polynomially. For convenience we use the following notation:

$$\begin{aligned} & g^{\pm}_0 = g_0(\mu_0 \pm \omega_{\boldsymbol{k}}), \quad g^{\pm}_i = g_i(\mu_i \pm k_i), \\ & g'^{\pm}_0 = g(\mu_0 \pm \omega_{\boldsymbol{k}'}), g'^{\pm}_i = g_i(\mu_i \pm k'_i), \\ & a = a_{\boldsymbol{k}},\ a' = a_{\boldsymbol{k}'},\ a^\dagger = a^\dagger_{\boldsymbol{k}}, a'^\dagger = a^\dagger_{\boldsymbol{k}'}, \\ & p = p_{\boldsymbol{k}},\ p' = p_{\boldsymbol{k}'},\ (i = 1, 2, 3). \end{aligned} \tag{14}$$

Considering the integral symmetry about $\boldsymbol{k}$ and $\boldsymbol{k}'$ and using Eq. (6), we derive

$$\begin{aligned} & \int_{-\infty}^{\infty} d\mu_0 \int_{-\infty}^{\infty} d\mu_1 \int_{-\infty}^{\infty} d\mu_2 \int_{-\infty}^{\infty} d\mu_3 \mathcal{O}^{\pm\dagger}_{\mu}\mathcal{O}^{\pm}_{\mu} \\ & = S^{\pm} + \int_{-\infty}^{\infty} d\mu_0 \int_{-\infty}^{\infty} d\mu_1 \int_{-\infty}^{\infty} d\mu_2 \int_{-\infty}^{\infty} d\mu_3 \\ & \quad \cdot \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}|g^+_0|^2|g^+_1|^2g^+_2|^2|g^+_3|^2|p|^2, \end{aligned} \tag{15}$$

where

$$\begin{aligned} S^{\pm} = \frac{1}{2}\int \frac{d^3\boldsymbol{k}\, d^3\boldsymbol{k}'}{(2\pi)^{2n}}[& F(\boldsymbol{k},\boldsymbol{k}')\overline{p}p'a^\dagger a' \pm G(\boldsymbol{k},\boldsymbol{k}')pp'aa' \\ & + \overline{F(\boldsymbol{k},\boldsymbol{k}')}\overline{pp'}a'^\dagger a \pm \overline{G(\boldsymbol{k},\boldsymbol{k}')}\overline{pp'}a^\dagger a'^\dagger]. \end{aligned} \tag{16}$$

Here the functions F and G are defined by

$$F(\boldsymbol{k},\boldsymbol{k}') = \int_{-\infty}^{\infty} d\mu_0 \int_{-\infty}^{\infty} d\mu_1 \int_{-\infty}^{\infty} d\mu_2 \int_{-\infty}^{\infty} d\mu_3$$

$$[g_0^-\overline{g_1^-g_2^-g_3^-g_0^-}g_1'^-g_2'^-g_3'^- + \overline{g_0^+}g_1^+g_2^+g_3^+g_0'^+\overline{g_1'^+g_2'^+g_3'^+}] \quad (17)$$

and

$$G(\boldsymbol{k},\boldsymbol{k}') = \int_{-\infty}^{\infty}d\mu_0\int_{-\infty}^{\infty}d\mu_1\int_{-\infty}^{\infty}d\mu_2\int_{-\infty}^{\infty}d\mu_3 \cdot [g_0^+\overline{g_1^+g_2^+g_3^+g_0'^-}g_1'^-g_2'^-g_3'^- + \overline{g_0^-}g_1^-g_2^-g_3^-g_0'^+\overline{g_1'^+g_2'^+g_3'^+}]. \quad (18)$$

Using Eq. (11) and $g(u)=\overline{g(-u)}$, we can simplify the above two equations as

$$F(\boldsymbol{k},\boldsymbol{k}') = 2\widehat{f}_0(\omega_{\boldsymbol{k}'}-\omega_{\boldsymbol{k}})\widehat{f}_1(k_1-k_1') \cdot \widehat{f}_2(k_2-k_2')\widehat{f}_3(k_3-k_3') \quad (19)$$

and

$$G(\boldsymbol{k},\boldsymbol{k}') = 2\widehat{f}_0(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})\widehat{f}_1(-k_1-k_1') \cdot \widehat{f}_2(-k_2-k_2')\widehat{f}_3(-k_3-k_3'). \quad (20)$$

Now we calculate

$$\begin{aligned}[\mathcal{O}_\mu^{\pm\dagger},\mathcal{O}_\mu^{\pm}] =& -\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}|g_0(\mu_0-\omega_{\boldsymbol{k}})|^2|g_1(\mu_1-k_1)|^2 \cdot g_2(\mu_2-k_2)|^2|g_3(\mu_3-k_3)|^2|p|^2 \\ &+\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}|g_0(\mu_0+\omega_{\boldsymbol{k}})|^2|g_1(\mu_1+k_1)|^2 \cdot g_2(\mu_2+k_2)|^2|g_3(\mu_3+k_3)|^2|p|^2 \\ \equiv& -H_{\bar\mu_0\bar\mu_1\bar\mu_2\bar\mu_3}+H_{\mu_0\mu_1\mu_2\mu_3}. \end{aligned} \quad (21)$$

And we find that

$$\begin{aligned}S^{\pm} =& \int_0^{\infty}d\mu_0\int_{-\infty}^{\infty}d\mu_1\int_{-\infty}^{\infty}d\mu_2 \cdot\int_{-\infty}^{\infty}d\mu_3(\mathcal{O}_\mu^{\pm\dagger}\mathcal{O}_\mu^{\pm}-H_{\mu_0\mu_1\mu_2\mu_3}) \\ &+\int_{-\infty}^{0}d\mu_0\int_{-\infty}^{\infty}d\mu_1\int_{-\infty}^{\infty}d\mu_2\int_{-\infty}^{\infty}d\mu_3 \cdot(-H_{\bar\mu_0\bar\mu_1\bar\mu_2\bar\mu_3}+\mathcal{O}_\mu^{\pm}\mathcal{O}_\mu^{\pm\dagger}). \end{aligned} \quad (22)$$

Then, for all well manifested states the expectation value of $S^{\pm}$ satisfies

$$\langle S^{\pm}\rangle \geq -\int_0^{\infty}d\mu_0\int_{-\infty}^{\infty}d\mu_1\int_{-\infty}^{\infty}d\mu_2 \cdot\int_{-\infty}^{\infty}d\mu_3(H_{\mu_0\mu_1\mu_2\mu_3}+H_{\mu_0\bar\mu_1\bar\mu_2\bar\mu_3}). \quad (23)$$

In calculating the right-hand side of Eq. (23) we find that the integration of the first term is equal to that of the second. Therefore, we obtain

$$\begin{aligned}\langle S^{\pm}\rangle \geq& -2\int_0^{\infty}d\mu_0\int_{-\infty}^{\infty}d\mu_1\int_{-\infty}^{\infty}d\mu_2\int_{-\infty}^{\infty}d\mu_3 \\ &\cdot\int\frac{d^n\boldsymbol{k}}{(2\pi)^n}|g_0(\mu_0+\omega_{\boldsymbol{k}})|^2|g_1(\mu_1+k_1)|^2 \\ &\cdot|g_2(\mu_2+k_2)|^2|g_3(\mu_3+k_3)|^2|p|^2. \end{aligned} \quad (24)$$

Now we define the spacetime-weighted average of the energy density as

$$\langle\rho\rangle_{ST} = \int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\int_{-\infty}^{\infty}dx_3 \cdot\langle:T_{00}:\rangle f_0(t)f_1(x_1)f_2(x_2)f_3(x_3). \quad (25)$$

Here $f_0,\ldots,f_3$ are assumed to be smooth, non-negative and compactly supported spacetime sampling functions whose integrals about the corresponding coordinates are all unity. Inserting Eq. (8) into the above equation and using Eq. (19) and Eq. (20), we obtain

$$\begin{aligned}\langle\rho\rangle_{ST} =& \frac{1}{4}\int\frac{d^3\boldsymbol{k}d^3\boldsymbol{k}'}{(2\pi)^6}\Big\{\frac{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}+\boldsymbol{k}\cdot\boldsymbol{k}'}{2\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}} \cdot[F(\boldsymbol{k},\boldsymbol{k}')\langle a^\dagger a'\rangle - G(\boldsymbol{k},\boldsymbol{k}')\langle aa'\rangle \\ &+\overline{F(\boldsymbol{k},\boldsymbol{k}')}\langle a'^\dagger a\rangle - \overline{G(\boldsymbol{k},\boldsymbol{k}')}\langle a^\dagger a'^\dagger\rangle] \\ &+\frac{m^2}{2\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}[F(\boldsymbol{k},\boldsymbol{k}')\langle a^\dagger a'\rangle + G(\boldsymbol{k},\boldsymbol{k}')\langle aa'\rangle \\ &+\overline{F(\boldsymbol{k},\boldsymbol{k}')}\langle a'^\dagger a\rangle + \overline{G(\boldsymbol{k},\boldsymbol{k}')}\langle a^\dagger a'^\dagger\rangle]\Big\} \end{aligned} \quad (26)$$

Applying Eq. (24) for the following cases:

$$p_{\boldsymbol{k}}=\frac{\omega_{\boldsymbol{k}}}{\sqrt{2\omega_{\boldsymbol{k}}}},\quad p_{\boldsymbol{k}}=\frac{k_i}{\sqrt{2\omega_{\boldsymbol{k}}}},\quad p_{\boldsymbol{k}}=\frac{m}{\sqrt{2\omega_{\boldsymbol{k}}}}, \quad (27)$$

we obtain the inequality

$$\begin{aligned}\langle\rho\rangle_{ST} \geq& -\int_0^{\infty}d\mu_0\int_{-\infty}^{\infty}d\mu_1\int_{-\infty}^{\infty}d\mu_2\int_{-\infty}^{\infty}d\mu_3 \cdot\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}|g_0(\mu_0+\omega_{\boldsymbol{k}})|^2|g_1(\mu_1+k_1)|^2 \\ &\cdot|g_2(\mu_2+k_2)|^2|g_3(\mu_3+k_3)|^2\omega_{\boldsymbol{k}}. \end{aligned} \quad (28)$$

The right-hand side is a multi-integral. By the way of transformation of integral variables, we can calculate the parts about μ_i and k_i for $i=1,2,3$, respectively. In the mean time, considering $\int_0^{\infty}du|g(u)|^2=1/2$, we obtain

$$\langle\rho\rangle_{ST} \geq -\frac{1}{2\pi^2}\int_0^{\infty}d\mu_0\int_m^{\infty}d\omega \cdot|g_0(\mu_0+\omega)|^2\sqrt{\omega^2-m^2}\omega^2. \quad (29)$$

Changing the integral variables once again, we can simplify further the above inequality. Let $u_0=\mu_0+\omega$, $v=\omega$, then we have

$$\langle\rho\rangle_{ST} \geq -\frac{1}{4\pi^3}\int_m^{\infty}du_0|\widehat{f_0^{1/2}}(u_0)|^2 \cdot\int_m^{u_0}dv\sqrt{v^2-m^2}v^2. \quad (30)$$

We can rewrite Eq. (30) as

$$\langle\rho\rangle_{ST} \geq -\frac{1}{16\pi^3}\int_m^\infty du_0 |\widehat{f_0^{1/2}}(u_0)|^2 u_0^4 Q_3\Big(\frac{u_0}{m}\Big), \quad (31)$$

where

$$Q_3(x) = \Big(1-\frac{1}{x^2}\Big)^{1/2}\Big(1-\frac{1}{2x^2}\Big) - \frac{1}{2x^4}\log[x+(x^2-1)^{1/2}] \quad (32)$$

is a monotonic increasing function for $x \geq 1$ with the maximum 1 as $x \to \infty$. This lower bound is obviously finite since $|\widehat{f}^{1/2}(u_0)|^2$ decays faster than any polynomial in u_0 as $u_0 \to \infty$. Comparing Eqs. (31) and (2), we can see that the quantum inequality is just equal to the temporal QEI.

In the massless case, $Q_3(u_0/m) = 1$ and the lower bound is

$$\langle\rho\rangle_{ST} \geq -\frac{1}{16\pi^3}\int_0^\infty du_0 u_0^4 |\widehat{f_0^{1/2}}(u_0)|^2, \quad (33)$$

which is clearly weaker than the bound Eq. (31) in the massive case. If the Lorentzian function

$$f_0(t) = \frac{t_0}{\pi(t^2+t_0^2)} \quad (34)$$

with the characteristic width t_0 is used to sample the energy density the inequality is

$$\langle\rho\rangle_{ST} \geq -\frac{27}{2048\pi^2 t_0^4}. \quad (35)$$

When $t_0 \to 0$, the inequality is divergent. This fact is consistent with the conclusion that the spatial constraint on negative energy density does not exist in four dimensions.[27]

In summary, we have extended the method in deriving temporal QEI[16–18] and obtained a lower bound on the spacetime-weighted average of the energy density for the scalar field in a four-dimensional Minkowski spacetime. The quantum inequality bound does not depend on the quantum state and is only related to the spacetime sampling functions which are smooth, positive and compactly supported.

From the quantum inequality we come to the following conclusions. Firstly, the quantum inequality is equal to the temporal QEI Eq. (2). This fact indicates that the method we used here yields constraints on the spacetime average of negative energy density no tighter than the temporal QEI. Secondly, the quantum inequality can be written as $\langle\rho\rangle_{ST} \geq -c/t_0^4$ like the QEI and becomes divergent if the characteristic length of time takes the limit $t_0 \to 0$. This result is compatible with the fact that the spatial restriction on negative energy does not exist in a four-dimensional spacetime.[27] Lastly, we find that the inequality for massive scalar fields is stronger than that for the massless one. This can be explained as it needs to conquer the positive energy of the mass for the averaged energy density to become negative for the massive field.

Considering that the characteristic widths of spatial sampling functions do not emerge apparently in the quantum inequality, we still need to make more efforts to search for the optimal spacetime quantum energy inequality.

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