Concurrence and Wigner–Yanase Skew Information *

TANG Li(唐莉)[1,2,3**], CHEN Ze-Qian(陈泽乾)[4***]

[1] *State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071*

[2] *Center for Cold Atom Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071*

[3] *Graduate School of Chinese Academy of Sciences, Beijing 100049*

[4] *State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics and Laboratory of Mathematical Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071*

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A measure of entanglement on n qubits is defined in terms of Wigner–Yanase skew information. It is shown that the measure coincides essentially with the concurrence on two qubits. This uncovers the information-theoretic meaning of the concurrence of entangled states.

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As is well known, entangled states have become a key concept in quantum mechanics nowadays. On the other hand, from a practical point of view, entangled states have found numerous applications in quantum information.[1] Consequently, quantification of entanglement is necessary to understand and to develop the quantum information theory. There have been established various entanglement measures, including the entanglement of formation,[2] the relative entropy of entanglement,[3] the concurrence,[4] the logarithm of the negativity,[5] Schmidt measure,[6], and so on.[7] No equally versatile and computationally manageable entanglement measure for mixed state is available so far, although those measures have been proposed. However, there is an elegant formula for two qubits in terms of the concurrence,[4] which has received much attention recently.[8]

In this Letter, we present a measure of quantum entanglement based on Wigner–Yanase skew information, which is shown to coincide essentially with the concurrence on two qubits. In the study of measurement theory,[9] Wigner and Yanase[10] introduced the following quantity from an information-theoretic point of view

$$I(\rho, A) = -\frac{1}{2}\mathrm{tr}[\rho^{1/2}, A]^2, \qquad (1)$$

where ρ is a state and A an observable of a quantum system. This quantity $I(\rho, A)$ is called the Wigner–Yanase skew information. It is the amount of information on the values of observables not commuting with (being skew to) A. They proved that this quantity satisfies the requirements that an expression for the information content should satisfy, among which two basic ones are described as follows.

(a) When two different ensembles are united, the skew information decreases. Phrased mathematically, $I(\rho, A)$ is a convex function in ρ—if ρ_1 and ρ_2 are both density operators, then

$$I(\alpha\rho_1 + \beta\rho_2, A) \leq \alpha I(\rho_1, A) + \beta I(\rho_2, A), \qquad (2)$$

where $\alpha + \beta = 1$ for $\alpha, \beta \geq 0$.

(b) The skew information content of the union of two independent systems is the sum of the skew information content of the individuals; namely, let ρ_1 and ρ_2 be two states of the first and second systems, respectively, and A_1 and A_2 the corresponding local observables; then

$$I(\rho_1 \otimes \rho_2, A_1 \otimes 1 + 1 \otimes A_2) = I(\rho_1, A_1) + I(\rho_2, A_2). \qquad (3)$$

The equality (3) can be extended to several independent systems.

There are some earlier works concerning the Wigner–Yanase skew information. These constitute the celebrated Wigner–Araki–Yanase theorem, which puts a limitation on the measurement of observables in the presence of a conserved quantity,[11] and was further investigated by many authors.[12] Recently, Luo and Zhang[13] revealed that the Wigner–Yanase skew information can be regarded as the quantum analog of the Fisher information in the theory of statistical estimation and can be used to establish some results concerning the characterizations of uncertainty relations and measures of entangled states. Moreover, a Bell-type inequality of Wigner–Yanase skew information was proposed in Ref. [14], which gave a hierarchic classification of all states of multipartite quantum systems from separable to fully entangled states. Here we present a measure of entanglement in terms of Wigner–Yanase skew information.

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** Email: tangli@wipm.ac.cn

*** Email: zqchen@wipm.ac.cn

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First of all, note that the Wigner–Yanase skew information can be rewritten as

$$I(\rho, A) = \mathrm{tr}\rho A^2 - \mathrm{tr}\rho^{1/2} A \rho^{1/2} A. \tag{4}$$

In particular, if $\rho = |\psi\rangle\langle\psi|$ is a pure state, then

$$I(|\psi\rangle, A) = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2. \tag{5}$$

From Eqs. (2) and (5) one can conclude

$$I(\rho, A) \leq 1 \tag{6}$$

for each spin observable A ($A^2 = 1$) in all states ρ.

Now, consider a system of n spin-$\frac{1}{2}$ particles and assume that a local spin observable is measured on each particle, and hence there are the corresponding n local spin observables $A_1, \cdots, A_n$. Since separable states are written as a convex sum over the states $\rho_1 \otimes \cdots \otimes \rho_n$, and by Eqs. (3) and (6), one has

$$I(\rho_1 \otimes \cdots \otimes \rho_n, A_1 + \cdots + A_n) = \sum_{j=1}^{n} I(\rho_j, A_j) \leq n,$$

it is concluded from Eq. (2) that

$$I(\rho, A_1 + \cdots + A_n) \leq n \tag{7}$$

for all separable states ρ of the n particles. (We write $A_1, \cdots$, as shorthand for $A_1 \otimes 1 \otimes \cdots \otimes 1$.) The inequality (7) is a Bell-type inequality in terms of Wigner–Yanase skew information, which can be violated by entangled states. Roughly speaking, the more entangled the larger the violation value of Eq. (7), which means that it is easier to measure an observable not commuting with local spin observables on entangled states, thus corresponding to larger skew information. This directly uncovers the information-theoretic meaning of entangled states.[14]

Now, we turn to the problem of how to quantify entanglement by using Wigner–Yanase skew information. We Define

$$I(\rho) = \sup I(\rho, A_1 + \cdots + A_n), \tag{8}$$

where the supremum is taken over all the local spin observables $A_1, \cdots, A_n$. According to the physical meaning of Wigner–Yanase skew information, the quantity $I(\rho)$ is the greatest amount of information on the values of observables not commuting with (being skew to) local spin observables we can obtain on the state ρ. Hence, $I(\rho)$ can be regarded as the nonlocal skew information of ρ. As shown in Ref. [14], the larger the nonlocal skew information, the more entangled or the less separable is the state. This means that we can obtain more information of some nonlocal observables on entangled states than those on separable ones. In this sense, we may regard entangled states as an information-theoretic concept. In particular, by using the nonlocal skew information of states of multipartite systems we obtain an entanglement measure as follows.

The Wigner–Yanase measure of entanglement is now defined as

$$E_{\mathrm{WY}}(|\psi\rangle\langle\psi|) = \max\{I(|\psi\rangle\langle\psi|) - n, 0\} \tag{9}$$

for pure states of n spin-$\frac{1}{2}$ particles. Once E_{WY} is defined for pure states, one can extend the definition to the full state space in a natural way. This is carried out by using a convex roof construction:[2] For a state ρ of n particles, let

$$E_{\mathrm{WY}}(\rho) = \min \sum_i \lambda_i E_{\mathrm{WY}}(|\psi_i\rangle\langle\psi_i|), \tag{10}$$

where the minimum is taken over all the possible convex combinations in the form $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ in terms of pure states $|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, \ldots$, with $0 \leq \lambda_i \leq 1$ for all i and $\sum_i \lambda_i = 1$.

For each separable pure state $|\psi_1\rangle \cdots |\psi_n\rangle$, it is easy to check that $I(|\psi_1\rangle \cdots |\psi_n\rangle) = n$. This concludes that $E_{\mathrm{WY}} \geq 0$, and $E_{\mathrm{WY}}(\rho) = 0$ if ρ is a separable state. On the other hand, as shown in Ref. [14] that $I(\rho) \leq n^2$ and $I(\rho) = n^2$ if and only if $\rho = |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$, one has $E_{\mathrm{WY}} \leq n(n-1)$, and $E_{\mathrm{WY}}(\rho) = n(n-1)$ if and only if $\rho = |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$. Thus, the Wigner–Yanase measure quantifies the maximally entangled states.

Now, we compute the Wigner–Yanase measure of two qubits. For a pure state $|\psi\rangle$ of two qubits we can write by the Schmidt decomposition theorem

$$|\psi\rangle = p|\phi_1\rangle|\chi_1\rangle + q|\phi_2\rangle|\chi_2\rangle, \tag{11}$$

where p and q are two nonnegative numbers satisfying $p^2 + q^2 = 1$ and $\{|\phi_1\rangle, |\phi_2\rangle\}$ and $\{|\chi_1\rangle, |\chi_2\rangle\}$ are orthonormal bases in the two-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of the two particles, respectively. Now, choose a representation for the first particle such that $|\phi_1\rangle = |0\rangle_1, |\phi_2\rangle = |1\rangle_1$, in the z_1 direction, and similarly, $|\chi_1\rangle = |0\rangle_2, |\chi_2\rangle = |1\rangle_2$, in the z_2 direction. Then, a simple computation yields that

$$I(|\psi\rangle, \sigma_x^1 + \sigma_x^2) = 2 + 4pq. \tag{12}$$

On the other hand, for two generic local spin observables $A_1 = \boldsymbol{a} \cdot \sigma_1$ and $A_2 = \boldsymbol{b} \cdot \sigma_2$ with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ and $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$, we have

$$I(|\psi\rangle, A_1 + A_2) = 2 + 4pq(a_1 b_1 - a_2 b_2) + 2a_3 b_3 - (p^2 - q^2)^2 (a_3 + b_3)^2 \leq 2 + 4pq.$$

Therefore, $I(|\psi\rangle) = 2 + 4pq$. Consequently,

$$E_{\mathrm{WY}}(|\psi\rangle\langle\psi|) = 4pq = 2C(|\psi\rangle), \tag{13}$$

thus, for a generic state ρ of two qubits

$$E_{\mathrm{WY}}(\rho) = 2C(\rho), \tag{14}$$

where $C(\rho)$ is the concurrence of entangled states.[4] This concludes that the concurrence coincides with the Wigner–Yanase measure on two qubits and, the Wigner–Yanase measure extends the concurrence from two qubits to n qubits. Since the Wigner–Yanase measure is defined from an information-theoretic point

of view, we reveal the information-theoretic meaning of the concurrence.

We now turn to our measure to see how good it is to quantify the entanglement of the generalized GHZ states of n qubits ($n \geq 3$) $|\psi\rangle = \alpha|0^n\rangle + \beta|1^n\rangle$, where $\alpha, \beta > 0$, and $\alpha^2 + \beta^2 = 1$. Write $A_j = a_{j1}\sigma_x^j + a_{j2}\sigma_y^j + a_{j3}\sigma_z^j$, where $\boldsymbol{a}_j = (a_{j1}, a_{j2}, a_{j3}) \in \mathbb{R}^3$ are all unit vectors. Note that

$$I(|\psi\rangle, A_1 + \cdots + A_n) = n + 2 \sum_{1 \leq i < j \leq n} a_{i3}a_{j3} - (\alpha^2 - \beta^2)^2 \Big(\sum_{i=1}^{n} a_{i3} \Big)^2.$$

Since $(n-1)\Big(\sum_{i=1}^{n} a_{i3}\Big)^2 \geq 2\sum_{1 \leq i < j \leq n} a_{i3}a_{j3}$, one has

$$\begin{aligned} &I(|\psi\rangle, A_1 + \cdots + A_n) \leq n \\ &\quad + \Big[1 - \frac{n}{n-1}(\alpha^2 - \beta^2)^2\Big] n(n-1), \\ &I(|\psi\rangle, \sigma_z^1 + \cdots + \sigma_z^n) = n \\ &\quad + \Big[1 - \frac{n}{n-1}(\alpha^2 - \beta^2)^2\Big] n(n-1); \\ &I(|\psi\rangle) = n + \Big[1 - \frac{n}{n-1}(\alpha^2 - \beta^2)^2\Big] n(n-1), \\ &E_{\mathrm{WY}}(|\psi\rangle) = \max\Big\{\Big[1 - \frac{n}{n-1}(\alpha^2 - \beta^2)^2\Big] n(n-1), 0\Big\}. \end{aligned} \tag{15}$$

It is concluded that whenever

$$\alpha, \beta > \sqrt{\frac{1}{2}\Big(1 - \sqrt{\frac{n-1}{n}}\Big)}, \tag{16}$$

our measure can quantify the entanglement of the generalized GHZ states.

In recent years, various versions of concurrence for multipartite systems have been demonstrated (e.g., see Refs. [15–18] and references herein). So far, however no complete construction of concurrence has been presented for three or more qubits systems, there are only partial results. Our measure of quantum entanglement based on Wigner–Yanase skew information, which is shown to coincide essentially with the concurrence on two qubits, can also be regarded as an extension of the concurrence from two qubits to n qubits, which reveals the information-theoretic meaning of the concurrence. Indeed, according to the physical meaning of the Wigner–Yanase skew information, the Wigner–Yanase measure $E_{\mathrm{WY}}(\rho)$ is essentially the greatest amount of information on the values of observables not commuting with (being skew to) local spin observables we can obtain on the state ρ. $E_{\mathrm{WY}}(\rho)$ is the nonlocal skew information of entangled states, as we can obtain more information of some nonlocal observables on entangled states than on separable ones.

Nowadays, it has been recognized that most physical processes in nature can be formulated in terms of processing of information, and information may be central to understanding quantum theory.[19] The notion of the Wigner–Yanase skew information quantifies the amount of information on the values of observables being skew to other ones,[11,12] and has been used to establish some results concerning the characterizations of uncertainty relations and measures of entangled states.[13] Our results furthermore shed considerable light on relationships between quantum entanglement and information theory in terms of the Wigner–Yanase measure of entanglement. We expect that, similarly to the concurrence,[4] the Wigner–Yanase measure and its various ramifications will play an important role in quantum information.[1]

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