Minimal Braid in Applied Symbolic Dynamics *

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Based on the minimal braid assumption, three-dimensional periodic flows of a dynamical system are reconstructed in the case of unimodal map, and their topological structures are compared with those of the periodic orbits of the Rössler system in phase space through the numerical experiment. The numerical results justify the validity of the minimal braid assumption which provides a suspension from one-dimensional symbolic dynamics in the Poincaré section to the knots of three-dimensional periodic flows.

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The periodic flows of the nonlinear differential equation

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) \tag{1}$$

can form some real knots in the three-dimensional phase space. A method of constructing knots applies directly to the edge shift matrix (i.e. Markov matrix) in symbolic dynamics of maps in the Poincaré section. The advantage of the method is to associate the knot with a finite symbolic sequence which is frequently used in the iterative dynamics of one-dimensional maps.[1] However, there still exists uncertainty in the method because the knot of periodic flow constructed by the suspension of the edge shift matrix is not unique. The reason is that the degree of freedom lost in the Poincaré section cannot be recovered by a simple permutation decided by the edge shift matrix. In mathematics, the edge shift matrix equivalent to a permutation cannot uniquely correspond to a knot. Thus how to recover the real knot of the orbit of periodic flow by a simple edge shift matrix is interesting. This paper attempts to present an assumption of the minimal braid to solve the problem. The assumption is verified by the numerical experiment in the Rössler system. The numerical results in the period-doubling (up to period-32) and knots of some low periodic sequences have proved the effectivity of the minimal braid assumption. Although the minimal braid assumption is rather simple, it provides a starting point to construct the complex knots which we may meet in the differential dynamical systems.

Here we take the Rössler system[2] as an example, which is a widely spread prototype of a continuous dynamical system defined by the following three nonlinear ordinary differential equations (ODEs):

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + (x - c)z. \end{aligned} \tag{2}$$

After transients, the three-dimensional trajectory integrated from Eq. (2) settles down on an attractor provided that the initial condition is included in its basin of attractor. Properties and characteristics of the attractor depend on the values of the control parameters (a, b, c). In view of it exhibiting a unimodal or multimodal map at some parameter values in the Poincaré section, the Rössler system can be an inescapable checkpoint to justify the minimal braid assumption. In the following, we first briefly review the symbolic dynamics of maps in the Poincaré section.

We consider a three-dimensional flow of dynamical system (1), which forms a map $f_{\boldsymbol{\lambda}}$ on the interval $I = [c_0, c_N]$ in the Poincaré section, with $N-1$ parameters $(\lambda_1, \lambda_2, \ldots, \lambda_{N-1}) \equiv \boldsymbol{\lambda} \subset \mathcal{R}^{N-1}$. The $N-1$ critical points and the N subintervals are numbered in their natural order in symbolic space, $I_1 \prec C_1 \prec I_2 \prec C_2 \prec \cdots \prec I_{N-1} \prec C_{N-1} \prec I_N$, where $\prec$ refers to the Metropolis–Stein–Stein (MSS) order.[3] For each point $x \in I$, there must be a unique symbolic address $A(x) \in \Sigma_N$ according to the symbol of the set to which x belongs, $x \in A(x)$.

For an arbitrary admissible $N-1$ multiple superstable kneading sequence $W = C_1 X_1 C_2 X_2 \cdots C_{N-1} X_{N-1} = w_1 w_2 \cdots w_{|W|}$ ($|W|$ denotes the length of sequence W), shifts $\varphi^i(W) = (W)_{i+1} = w_{i+1} w_{i+2} \cdots w_{|W|}$ $(i = 1, 2, \ldots, |W|)$ form $|W|$ vertices. On the set $\mathcal{C}^0 = \{\varphi^i(W)\}$ of vertices, a basic shift matrix ω ($|W| \times |W|$ 0–1 matrix), is constructed by the natural shift order: $\omega((W)_i, (W)_{i+1}) = 1$, which is a simple periodic matrix. The vertex set $\mathcal{C}^0$ can be ordered by a permutation $\pi : \mathcal{Z}_+ \to \mathcal{Z}_+$ which makes

$$\varphi^{\pi(1)}(W) \prec \varphi^{\pi(2)}(W) \prec \cdots \prec \varphi^{\pi(|W|)}(W) \tag{3}$$

hold,[4] where a shift map $\varphi : (\Sigma_N)^{\mathcal{Z}_+} \to (\Sigma_N)^{\mathcal{Z}_+}$ is $\varphi(I(x))_i = (I(x))_{i+1}$, $I(x) = A_0(x)A_1(x)A_2(x)\cdots A_n(x)\cdots$.[5] The permutation π can also be written as a $|W| \times |W|$ 0–1 matrix. We

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note that these ordered vertices form a line graph G, while shift maps from vertex $\varphi^{\pi(i)}(W)$ to vertex $\varphi^{\pi(i)+1}(W)$ make a directed graph. Then the adjacency matrix S_v of the directed graph is constructed such that for each $\pi(i)$, $S_v((W)_{\pi(i)},(W)_{\pi(i)+1}) = 1$ holds, which is another $|W| \times |W|$ 0–1 matrix. These two vertex matrices are similar:[6] $S_v = \pi\omega\pi^{-1}$.

Based on the minimal braid assumption, for an arbitrary periodic flow we can obtain an edge shift matrix S_e from the vertex shift matrix S_v. The edges are order pairs $(\pi_W(i), \pi_W(i+1)) = I_{\pi_W(i)} \equiv I_k$ in the line graph G with the vertices $\{\pi_W(i)\}$. By the homological language,[6,7] the vector space of 0-chains, $\mathcal{C}^0$, is spanned by $\{\varphi^{\pi(i)}(W)\}$, and that of 1-chains, $\mathcal{C}^1$, by the edge set $\{I_k\}$ $(k = \{0, 1, \ldots, |W| - 1\})$. From Ref. [6], the vertex shift matrix S_v induces an isomorphism S_p and leads to an edge shift matrix $S_e = |S_p|$ (i.e. the Stefan matrix) which can be written as $S_e = |\mu S_v \mu^{-1}|$, where μ is a nonsquare matrix and the absolute value of matrix $S = [s_{ij}]$ is denoted by $|S| = [|s_{ij}|]$.

According to the ordering rule, by distributing the sequences in Eq. (3) with natural order $1, 2, 3, \ldots, p-1, p$, and replacing shifted sequence with corresponding natural order, we can obtain the dynamical permutation rule listed in Ref. [4]:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & p-1 & p \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(p-1) & \sigma(p) \end{pmatrix}, \quad (4)$$

where $\sigma(i)$ denotes the natural order of sequence which is obtained by shift operation on initial sequence corresponding to natural order i.

The order permutation of orbital points can be carried out in two successive intersections with the surface of section, namely, they can be regarded as two separate parallel lines on which the transition of intervals (edges) occurs. It is convenient to stipulate two regions to recover the knot of three-dimensional flow. In the inner region between two parallel lines the knot is formed by the crossings of the flow, and in the outer region the suspension of the flow is obtained a simple order. However, in order to form a real knot in three-dimensional space we should present a simple rule of the formation of knot. According to the content of Ref. [1], we may summarize it as a minimal braid assumption with the following two rules:

(a) The weaving rule for the type of negative crossings: according to the order of appearances of the oriented lines inside the inner region an oriented line forms, and the latter one always goes under the former one (ones).

(b) The forbidden rule for the twist of lines: any two lines have not more than half twist (see Fig. 1).

For instance, the dynamical permutation of superstable sequence $RLRRRLRC$ is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 8 & 7 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

According to the minimal braid assumption, the minimal braid (knot) of the sequence $RLRRRLRC$ is shown in Fig. 2.

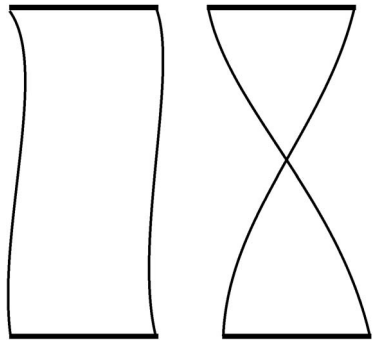


Fig. 1. Admissible cases of two lines.

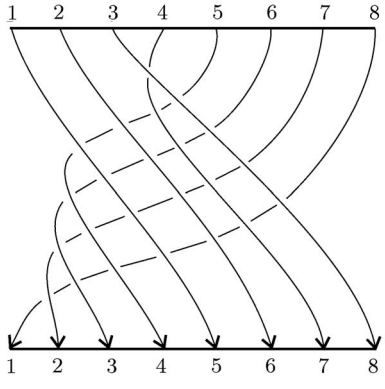


Fig. 2. Minimal braid of sequence $RLRRRLRC$.

The two rules of the minimal braid assumption will obtain uniqueness from the dynamical permutation of interval to the three-dimensional knot. However, the validity of the assumption is doubtful, which was discussed before.[8] For the purpose of verifying the validity of the minimal braid assumption, the method is applied to one-dimensional symbolic dynamics. In the case of the Rössler system, topological structure of some periodic orbits is compared with that of corresponding dynamical knots. The result shows that the minimal braid assumption is successful in a special class of ODE systems.

Given a periodic sequence, we can construct three-dimensional flow based on the minimal braid assumption. Then topological structure of periodic orbit can be described by the number and type of crossings. We can find a standard convention of crossing types in a projective plane in knot theory (see Fig. 3). The period-doubling sequence in unimodal map is a typical example to demonstrate the construction. Thus

we choose $RC, (RC)^{*2}, \ldots, (RC)^{*5}$ to construct three-dimensional flows. For sequences which satisfy admissible condition of kneading sequence, procedure of computing the number of crossings in a dynamical knot obtained by the minimal braid assumption can be programmed in computer code. The result for sequences $RC, (RC)^{*2}, \ldots, (RC)^{*5}$ is 1, 5, 23, 97, 399. Besides the dynamical knots of period-doubling, some low periodic knots which can be found in periodic orbits of the Rössler system are also constructed for comparison. As to types of crossings, the rule (a) implies that the type of all crossings in a dynamical knot is negative (-1).

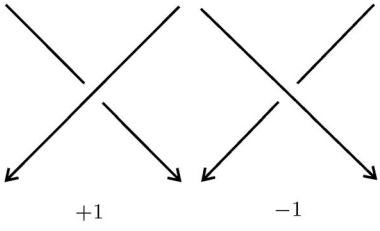


Fig. 3. Positive and negative crossing.

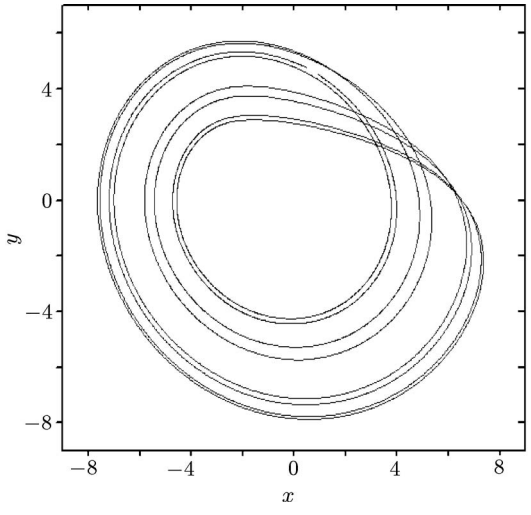


Fig. 4. Period-8 orbit of the Rössler system $(a = 0.2, b = 0.2, c = 4.2)$.

In the Rössler system, for some fixed control parameters (a, b, c), after selecting appropriate initial values of state parameters (x, y, z) and iterating them according to the fourth-order Runge–Kutta method, one can obtain a set of space points whose trajectory is a strange attractor. When the control parameters $a = 0.2$ and $b = 0.2$ are fixed while the parameters c is varied, the Rössler system exhibits a period-doubling scenario. To acquire a periodic orbit of the attractor, we can iterate from a point after transient and stop calculation when iterated point is in its neighbourhood. Then we can obtain points of the periodic orbit whose itinerary is a loop in three-dimensional space. For convenience, we choose xy plane as the projective plane to calculate topological structure of the periodic orbit in the Rössler system. For example, projective figure of period-8 orbit at control parameter values $(a = 0.2, b = 0.2, c = 4.2)$ is shown in Fig. 4.

After periodic orbital points are obtained in the numerical calculations, it is not easy to judge the number of crossings and type of each crossing in three-dimensional space. The difficulties come from the discreteness of orbital points in the numerical calculations. Thus, the crossings of discrete points have some uncertainty, as the number of crossings increases exponentially with increasing the period of symbolic sequence. At this time, large numbers of crossings concentrate in a small region with a high density. In the procedure of calculating the number of crossings, pseudo-crossings and repeated counting of the same crossing must be excluded. We can use orientation of iterated point as orientation of arrows in the convention of crossing and then obtain the type of each crossing through algebraic calculation, which is negative (-1). In the case of high density of orbital points, detailed information of the method to determine the type and number of crossings is described in Ref. [9]. Due to the restriction of the numerical method and complexity of differential system caused by shrink of periodic window and noise degradation for a high period, numerical calculations in period-doubling orbits are up to period-32. Comparison of theoretical calculation and numerical experiment of period-doubling sequences (up to period-32) is listed in Table 1.

Table 1. Comparison of theoretical calculation and numerical experiment of period-doubling sequences for the Rössler system.

Periodic sequence	Number of crossings (theor)	Type of crossings (expt)	Number of crossings (expt)
RC	1	-1	1 ± 0
$(RC)^{*2}$	5	-1	5 ± 0
$(RC)^{*3}$	23	-1	23 ± 0
$(RC)^{*4}$	97	-1	97 ± 0
$(RC)^{*5}$	399	-1	399 ± 1

Table 2. Comparison of theoretical calculation and numerical experiment of Metropolis–Stein–Stein (MSS) sequences (up to period-7) for the Rössler system.

Periodic sequence	Number of crossings (theor)	Type of crossings (expt)	Number of crossings (expt)
RLC	2	-1	2 ± 0
$RLLC$	3	-1	3 ± 0
$RLRRC$	8	-1	8 ± 0
$RLLRC$	6	-1	6 ± 0
$RLLLC$	4	-1	4 ± 0
$RLLLRC$	7	-1	7 ± 0
$RLRRRC$	9	-1	9 ± 0
$RLLLRRC$	10	-1	10 ± 0
$RLLLLRC$	8	-1	8 ± 0

Besides the knots of period-doubling, results of some low periodic knots are also verified in the numerical experiment. Table 2 gives the comparison of results of some MSS sequences up to period-7. The type of knots can be recognized by the Reidemeister

moves. Some of them are as the same as those in Table 9.1 of Ref. [1], in which some are torus knots and some are rather complex.[10] As an important topological invariant, the Conway polynomials can be programmed in computer code.[11] Furthermore, it is interesting to compare the two topological invariants: one is the Milnor–Thurston characteristic polynomials from Ref. [5], the other is the Conway polynomials.

By the comparison, we can see topological characteristic of knots obtained from the minimal braid assumption is the same as those of periodic orbits obtained from iterative operation in three-dimensional space. Therefore, we can reconstruct high-dimensional flow with the minimal braid assumption based on low-dimensional map and explore dynamical properties and characteristics of differential manifolds in high-dimensional space.

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References

[1] Hao B L and Zheng W M 1998 *Applied Symbolic Dynamics and Chaos* (Singapore: World Scientific) ch 9
[2] Rössler O E 1976 *Phys. Lett.* A **57** 397
[3] Metropolis N, Stein M L and Stein P R 1973 *J. Comb. Theory* A **15** 25
[4] Peng S L and Luo L S 1991 *Phys. Lett.* A **153** 345
[5] Milnor J and Thurston W 1988 *Lecture Notes in Mathematics* ed J C Alexander (Berlin: Springer) vol 1342, p 465
[6] Lampreia J P and Sousa Ramos J 1993 *Int. J. Bifurc. Chaos* **3** 1607
[7] Lampreia J P and Sousa Ramos J 1997 *Portugaliae Math.* **54** 1
[8] Ge M L and Zheng W M 2001 A discussion with the report of Peng S L: '*Star Products and Knots*' (New Era Nonlinear Science Conf., Tianjin)
[9] Zhang Y G and Zhang C 2003 *J. Yunnan University* at press (in Chinese)
[10] Zhang Cheng 2003 *M.S. Thesis* (Yunnan University) (in Chinese)
[11] Zhang Chuan 2002 *M.S. Thesis* (Yunnan University) (in Chinese)