Proper Accelerations of Time-Like Curves near a Null Geodesic *

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It is well known that when given a null geodesic $\gamma_0(\lambda)$ with a point r in (p,q) conjugate to p along $\gamma_0(\lambda)$, there will be a variation of $\gamma_0(\lambda)$ which can give a time-like curve from p to q. Here we prove that the time-like curves coming from the above-mentioned variation (with the second derivative $\beta_2 \neq 0$) have a proper acceleration $A = \sqrt{A^a A_a}$ which approaches infinity as the time-like curve approaches the null geodesic. Because the curve obtained from variation of the null geodesic must be everywhere time-like, we also discuss the constraint of the 'acceleration' B_0^a of the variation vector field on the null geodesic $\gamma_0(\lambda)$. The acceleration B_0^a of the variation vector field Z^a on the null geodesic $\gamma_0(\lambda)$ cannot be zero.

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It is well-known that an observer in "hyperbolic" motion in Minkowski space–time has a constant proper acceleration (the magnitude of the 4-acceleration). Then the proper acceleration of the observer approaches infinity as the observer approaches the null geodesic.[1] This is a well-known fact first written by Rindler in Ref. [1]. The main purpose of the previous paper[2] is to generalize this result to curved space–times, that is, given a future-complete null geodesic $\gamma_0(\lambda)$, up to the first derivative of the variation, there will be a variation of $\gamma_0(\lambda)$ which will give a time-like curve with its acceleration infinity.

The present paper continues the main point of Ref. [2], and primarily concerns the proper acceleration of time-like curve coming from the variation of null geodesic with two end points fixed on the null geodesic, and gives the conclusion that the proper acceleration of this type of time-like curve does approaches infinity as the time-like curve approaches the null geodesic. Because the two end points fixed on the null geodesic, the existence of the time-like curves from variation of γ_0 is in question. There are two theorems that concern the existence of the time-like curves:

Theorem 1. Let $\gamma_0(\lambda)$ be a smooth causal curve and let $p,\ q \in \gamma_0(\lambda)$. Then there does not exist a smooth one-parameter family of causal curves $\sigma(u,\lambda)$ connecting p and q with $\sigma(0,\lambda) = \gamma_0(\lambda)$ and $\gamma_u(\lambda)$ time-like for all $u > 0$ (i.e., $\gamma_0(\lambda)$ cannot be smoothly deformed to a time-like curve) if and only if $\gamma_0(\lambda)$ is null geodesic with no point conjugate to p along $\gamma_0(\lambda)$ between p and q.[3,4]

Theorem 2. If there is a point r in (p,q) conjugate to p along $\gamma_0(\lambda)$, then there will be a variation of $\gamma_0(\lambda)$ which will give a time-like curve from p to q.[3,4]

We therefore suppose that $\gamma_0(\lambda)$ is a null geodesic with a point r in (p,q) conjugate to p along $\gamma_0(\lambda)$ and the existence of the time-like curves connecting p, and q obtained from variation is ensured. Precisely, we have the following definition of the variation of $\gamma_0(\lambda)$:

Let (M, g_{ab}) be a four-dimensional curved space–time and γ_0: $(0,\lambda_q) \to M$ be a null geodesic. We denote the null geodesic by $\gamma_0(\lambda)$ with λ its affine parameter, and suppose $p,\ q \in \gamma_0(\lambda)$. We define a variation of γ_0 to be a C^1 map[3] σ: $(-\varepsilon,\varepsilon)\times[0,\lambda_q] \to M$ such that: (1) $\sigma(0,\lambda) = \gamma_0(\lambda)$. (2) $\sigma(u,0) = p$, $\sigma(u,\lambda_q) = q$. (3) There is a subdivision $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \lambda_q$ of $[0,\lambda_q]$ such that σ is C^3 on each $(-\varepsilon,\varepsilon)\times[\lambda_i,\lambda_{i+1}]$. (4) For each constant $u \in (-\varepsilon,\varepsilon)$ and $u \neq 0$, $\sigma(u,\lambda)$ is a time-like curve and is represented by $\gamma_u(\lambda)$. (5) We require the second derivative of the variation (with respect to u) is not zero (we will explain it later).

Denoted the tangent vector to the curve $\gamma_u(\lambda)$ by $(\frac{\partial}{\partial\lambda})^a_u \equiv v^a_u$, then $(\frac{\partial}{\partial\lambda})^a_0 \equiv v^a_0$ satisfies the null geodesic equation

$$v^b_0 \nabla_b v^a_0 = 0. \tag{1}$$

Let $(\frac{\partial}{\partial u})^a \equiv Z^a(u,\lambda)$ be the tangent vector to the curve $\sigma(u,\lambda)$ with λ = constant, $Z^a(\lambda)$ on $\gamma_0(\lambda)$ by $Z^a(0,\lambda)$, then the Lie derivative of $Z^a(u,\lambda)$ with respect to v^a_u vanishes,[3] i.e.,

$$v^b_u \nabla_b Z^a(u,\lambda) = Z^b(u,\lambda)\nabla_b v^a_u, \tag{2}$$

Denoted $g_{ab}v^a_u v^b_u$ by $-\alpha^2_u$, that is,

$$-\alpha^2_u = g_{ab}v^a_u v^b_u, \tag{3}$$

and decompose $-\alpha^2_u$ into Taylor series

$$-\alpha^2_u = g_{ab}v^a_u v^b_u = -\alpha^2_0 + \beta_1 u + \beta_2 u^2 + 0(u^3), \tag{4}$$

where

$$\alpha^2_0 = g_{ab}v^a_0 v^b_0 = 0. \tag{5}$$

In order to obtain γ_u to be time-like, it is easy to see $\beta_1 = \frac{\partial(-\alpha^2)}{\partial u}|_{u=0} \leq 0$ when $u > 0$; or $\beta_1 \geq 0$ when

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$u<0$. It can be proved $\beta_1=0$ (detailed in Ref. [3]). The following is the reason: with Eq. (2)

$$\begin{aligned}\frac{\partial(-\alpha_u^2)}{\partial u}&=Z^c(u,\lambda)\nabla_c[g_{ab}v_u^a v_u^b]\\&=2g_{ab}v_u^a(Z^c(u,\lambda)\nabla_c v_u^b=2g_{ab}v_u^a v_u^c\nabla_c Z^b(u,\lambda)\\&=2v_u^c\nabla_c[g_{ab}v_u^a Z^b(u,\lambda)]-2g_{ab}Z^b(u,\lambda)v_u^c\nabla_c v_u^a\\&=2\frac{\partial}{\partial\lambda}[g_{ab}v_u^a Z^b(u,\lambda)]-2g_{ab}Z^b(u,\lambda)v_u^c\nabla_c v_u^a,\end{aligned}\tag{6}$$

therefore, with Eq. (1), one obtains

$$\beta_1=\frac{\partial(-\alpha_u^2)}{\partial u}|_{u=0}=2\frac{\partial}{\partial\lambda}[g_{ab}v_0^a Z^b]=2\frac{\mathrm{d}h}{\mathrm{d}\lambda},\tag{7}$$

where $h(\lambda)=g_{ab}v_0^a Z^b$ is continuous at $(0,\lambda_q)$ due to the continuity of Z^a. From the property of the variation map $\sigma(u,\lambda)$, that is, $\sigma(u,0)=p$, $\sigma(u,\lambda_q)=q$, we obtain

$$Z^a(0)=Z^a(\lambda_q)=0.\tag{8}$$

This in turn induces $h(0)=h(\lambda_q)=0$, but $h(0)=h(\lambda_q)=0$ is impossible if $\beta_1\le 0$ when $u>0$; or $\beta_1\ge 0$ when $u<0$. Thus, β_1 must be zero to make γ_u time-like.

Here $\beta_1=0$ induces $h=g_{ab}v_0^a Z^b=0$, that is, the variation vector Z^a is orthogonal to the tangent vector v_0^a of the null geodesic γ_0.

Therefore, one obtains

$$-\alpha_u^2=g_{ab}v_u^a v_u^b=\beta_2 u^2+0(u^3),\tag{9}$$

$\beta_2=\frac{1}{2}\frac{\partial^2(-\alpha_u^2)}{\partial u^2}|_{u=0}$ follows from Eq. (6) (detailed in Ref. [3]),

$$\begin{aligned}\frac{1}{2}\frac{\partial^2(-\alpha_u^2)}{\partial u^2}&=\frac{\partial^2}{\partial\lambda\partial u}[g_{ab}v_u^a Z^b(u,\lambda)]\\&\quad-Z^d(u,\lambda)\nabla_d[g_{ab}Z^b(u,\lambda)v_u^c\nabla_c v_u^a]\\&=\frac{\partial^2}{\partial\lambda\partial u}[g_{ab}v_u^a Z^b(u,\lambda)]\\&\quad-[g_{ab}(v_u^c\nabla_c v_u^a)Z^d(u,\lambda)\nabla_d Z(u,\lambda)^b]\\&\quad-[Z^a(u,\lambda)Z^d(u,\lambda)\nabla_d(v_u^c\nabla_c v_{ua})]\end{aligned}\tag{10}$$

the term $Z^d(u,\lambda)\nabla_d(v_u^c\nabla_c v_{ua})$ in the third part of above equation is simplified:

$$\begin{aligned}&Z^d(u,\lambda)\nabla_d(v_u^c\nabla_c v_{ua})\\&=(Z^d(u,\lambda)\nabla_d v_u^c)\nabla_c v_{ua}+v_u^c Z^d(u,\lambda)\nabla_d\nabla_c v_{ua}\\&=(Z^d(u,\lambda)\nabla_d v_u^c)\nabla_c v_{ua}+v_u^c Z^d(u,\lambda)\nabla_c\nabla_d v_{ua}\\&\quad+R_{dcae}v_u^c v_u^e Z^d(u,\lambda)\\&=(Z^d(u,\lambda)\nabla_d v_u^c)\nabla_c v_{ua}+v_u^c\nabla_c[Z^d(u,\lambda)\nabla_d v_{ua}]\\&\quad-(v_u^c\nabla_c Z^d(u,\lambda))\nabla_d v_{ua}+R_{dcae}v_u^c v_u^e Z^d(u,\lambda)\\&=v_u^d\nabla_d(v_u^c\nabla_c Z_a(u,\lambda))+R_{dcae}v_u^c v_u^e Z^d(u,\lambda),\end{aligned}$$

that is

$$\begin{aligned}Z^d(u,\lambda)\nabla_d\tilde{A}_{ua}&=Z^d(u,\lambda)\nabla_d(v_u^c\nabla_c v_{ua})\\&=v_u^d\nabla_d(v_u^c\nabla_c Z_a(u,\lambda))\\&\quad+R_{dcae}v_u^c v_u^e Z^d(u,\lambda),\end{aligned}\tag{11}$$

where the relation $\nabla_c\nabla_d v_{ua}-\nabla_d\nabla_c v_{ua}=-R_{dcae}v_u^e$ in the second step and Eq. (2) in the fourth step have been used, and $\tilde{A}_{ua}$ is defined by $\tilde{A}_{ua}=v_u^c\nabla_c v_{ua}$ and different from the proper acceleration of the time-like curve $\gamma_u(\lambda)$ (see the following for the definition of the proper acceleration of $\gamma_u(\lambda)$). Therefore, one obtains

$$\begin{aligned}\frac{1}{2}\frac{\partial^2(-\alpha_u^2)}{\partial u^2}&=\frac{\partial^2}{\partial\lambda\partial u}[g_{ab}v_u^a Z^b(u,\lambda)]\\&\quad-[g_{ab}(v_u^c\nabla_c v_u^a)Z^d(u,\lambda)\nabla_d Z^b(u,\lambda)]\\&\quad-[Z^a(u,\lambda)v_u^d\nabla_d(v_u^c\nabla_c Z_a)(u,\lambda)\\&\quad+R_{dcae}v_u^c v_u^e Z^a(u,\lambda)Z^d(u,\lambda)],\end{aligned}\tag{12}$$

therefore,

$$\begin{aligned}\beta_2=\Big[\frac{1}{2}\frac{\partial^2(-\alpha_u^2)}{\partial u^2}\Big]_{u=0}&=\Big[\frac{\partial^2}{\partial\lambda\partial u}(g_{ab}v_u^a Z^b(u,\lambda))\Big]_{u=0}\\&\quad-[Z^a v_0^d\nabla_d(v_0^c\nabla_c Z_a)+R_{dcae}v_0^c v_0^e Z^d Z^a],\end{aligned}\tag{13}$$

where Eq. (1) has been used.

The parameter λ of the time-like curve $\gamma_u(\lambda)$, in general, is not the proper time of the curve. The proper time τ satisfies

$$g_{ab}\Big(\frac{\partial}{\partial\tau}\Big)_u^a\Big(\frac{\partial}{\partial\tau}\Big)_u^b=-1,\tag{14}$$

then

$$\Big(\frac{\partial}{\partial\tau}\Big)^a=\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\Big)\Big(\frac{\partial}{\partial\lambda}\Big)_u^a=\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\Big)v_u^a.\tag{15}$$

With Eq. (3), one has

$$\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\Big)^2=\frac{1}{\alpha_u^2}.\tag{16}$$

The 4-acceleration of the time-like curve γ_u which is defined as

$$\begin{aligned}A^a&=\Big(\frac{\partial}{\partial\tau}\Big)^b\nabla_b\Big(\frac{\partial}{\partial\tau}\Big)^a=\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}v_u^b\nabla_b\Big(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}v_u^a\Big)\\&=\Big(\frac{1}{\alpha_u^2}\Big)\tilde{A}_u^a-\frac{1}{2\alpha_u^4}v_u^a v_u^b\nabla_b\alpha_u^2,\end{aligned}\tag{17}$$

or, one writes the above equation as

$$\tilde{A}_u^a=\alpha_u^2 A^a+\frac{1}{2\alpha_u^2}v_u^a v_u^b\nabla_b\alpha_u^2.\tag{18}$$

We select the pseudo-orthogonal basis E_1^a, E_2^a, E_3^a, E_4^a that are parallel transported along the null geodesic $\gamma_0(\lambda)$ with

$$E_4^a=\Big(\frac{\partial}{\partial\lambda}\Big)_0^a=v_0^a,\ g_{ab}E_i^a E_i^b=1,\ i=1,2;$$

$$g_{ab}E_i^aE_i^b=0,\ i=3,4;$$
$$g_{ab}E_i^aE_j^b=0,\ i=1,2,j=3,4$$
$$g_{ab}E_3^aE_4^b=-1,\ g_{ab}E_1^aE_2^b=0,$$

with Z^a being orthogonal to v_0^a, then it is easy to see

$$Z^a=Z^1E_1^a+Z^2E_2^a+Z^4E_4^a. \tag{19}$$

for constant λ, the above basis is constant vectors, they can be equally parallel transported along the curve $\sigma(u,\lambda)$ with $\lambda=$ constant, which are denoted by $E_i^a(u,\lambda)$, $i=1,2,3,4$ and at $u=0$, $E_i^a(0,\lambda)=E_i^a$. One can rewrite the vector $\widetilde{A}_u^a$ as $\widetilde{A}_u^a=\sum_{i=1}^4\widetilde{A}_u^iE_i^a(u,\lambda)$, and $[(\frac{\partial}{\partial u})^d\nabla_d\tilde{A}_u^a]_{u=0}=\sum_{i=1}^4[E_i^a(u,\lambda)(\frac{\mathrm{d}\tilde{A}_u^i}{\mathrm{d}u})]_{u=0}=\sum_{i=1}^4E_i^a(\frac{\mathrm{d}\tilde{A}_u^i}{\mathrm{d}u})_{u=0}$, with Eq. (11), and taking the limit $u=0$, then

$$\Big[\Big(\frac{\partial}{\partial u}\Big)^d\nabla_d\tilde{A}_u^a\Big]_{u=0}=v_0^d\nabla_d(v_0^c\nabla_cZ^a)+g^{ab}R_{dcbe}v_0^cv_0^eZ^d. \tag{20}$$

We denote the term $v_0^d\nabla_d(v_0^c\nabla_cZ^a)+g_{ab}R_{dcbe}v_0^cv_0^eZ^d$ by C^a, which is $C^a=\sum_{i=1}^4C^iE_i^a$. Using Eq. (19), and the antisymmetry property of the Riemann tensor R_{abcd}, that is, the second term on the left-hand side of Eq. (20) is contracted to zero by the vector v_{0a}, then Eq. (20) implies $(\frac{\mathrm{d}\tilde{A}_u^3}{\mathrm{d}u})_{u=0}=C^3=0$, that is,

$$\Big[\Big(\frac{\partial}{\partial u}\Big)^d\nabla_d\tilde{A}_u^a\Big]_{u=0}=\Big(\frac{\mathrm{d}\tilde{A}_u^1}{\mathrm{d}u}\Big)_{u=0}E_1^a+\Big(\frac{\mathrm{d}\tilde{A}_u^2}{\mathrm{d}u}\Big)_{u=0}E_2^a+\Big(\frac{\mathrm{d}\tilde{A}_u^4}{\mathrm{d}u}\Big)_{u=0}E_4^a \tag{21}$$
$$=C^1E_1^a+C^2E_2^a+C^4E_4^a. \tag{22}$$

The detailed form of $(\frac{\mathrm{d}\tilde{A}_u^i}{\mathrm{d}u})_{u=0}=C^i$, $i=1,2,4$ is determined by Eq. (20). Firstly, one assumes $C^1C^1+C^2C^2>0$ (it is later shown the relation can hold generally), which means

$$\tilde{A}_u^1=uC^1+0(u^2),\ \tilde{A}_u^2=uC^2+0(u^2),$$
$$\tilde{A}_u^4=uC^4+0(u^2) \tag{23}$$
$$g_{ab}\tilde{A}_u^a\tilde{A}_u^b=u^2(C^1C^1+C^2C^2)+0(u^3),$$
$$C^1C^1+C^2C^2>0. \tag{24}$$

Now, by Eqs. (18) and (9), we can calculate

$$g_{ab}\tilde{A}_u^a\tilde{A}_u^b=\alpha_u^4A^aA_a-\frac{1}{4}\alpha_u^2\Big[\frac{1}{\alpha_u^2}\frac{\partial\alpha_u^2}{\partial\lambda}\Big]^2$$
$$=\beta_2^2A^aA_au^4+\frac{1}{4\beta_2}\Big[\frac{\mathrm{d}\beta_2}{\mathrm{d}\lambda}\Big]^2u^2+0(u^3). \tag{25}$$

From Eq. (9), one obtains $\beta_2<0$. Unless $A^2=A^aA_a$ approaches infinity, Eqs. (24) and (25) are not compatible. Therefore, one obtains that the magnitude of the proper acceleration of the time-like curve is infinity. Secondly, one can prove that $C^1C^1+C^2C^2>0$ is retained on some ranges of $[0,\lambda]$. If the assumption is incorrect, then the only selection is $C^1C^1+C^2C^2=0$ on the entire interval $[0,\lambda]$, which means $C^1=C^2=0$ in the range $[0,\lambda]$. With Eqs. (13) and (19), it is easily to obtain

$$\beta_2-\Big[\frac{\partial^2}{\partial\lambda\partial u}(g_{ab}v_u^aZ^b(u,\lambda))\Big]_{u=0}$$
$$=-[Z^av_0^d\nabla_d(v_0^c\nabla_cZ_a)+R_{dcae}v_0^cv_0^eZ^dZ^a]=Z_aC^a$$
$$=Z^1C^1+Z^2C^2=0, \tag{26}$$

which is equivalent to

$$Z^av_0^d\nabla_d(v_0^c\nabla_cZ_a)=-R_{dcae}v_0^cv_0^eZ^dZ^a. \tag{27}$$

By Eq. (27), one concludes that $Z^av_0^d\nabla_d(v_0^c\nabla_cZ_a)$ is continuous on the entire interval $[0,\lambda]$, which induces that Z^a is C^2 vector field on the entire interval $[0,\lambda]$, and the vector field v_u^a is C^1 by Eq. (2). Using Eq. (26), we have

$$\int_0^{\lambda_q}\beta_2\mathrm{d}\lambda=\int_0^{\lambda_q}d\lambda\Big[\frac{\partial^2}{\partial\lambda\partial u}(g_{ab}v_u^aZ^b(u,\lambda))\Big]_{u=0}$$
$$=\Big[\Big[\frac{\partial}{\partial u}(g_{ab}v_u^aZ^b(u,\lambda))\Big]_{u=0}\Big]_0^{\lambda_q}$$
$$=[(v_0^c\nabla_cZ_a(0,\lambda))Z^a(0,\lambda)$$
$$+v_{0a}Z^c(0,\lambda)\nabla_cZ^b(0,\lambda)]_0^{\lambda_q}=0, \tag{28}$$

where one uses the fact that Z^a vanishes at both end points p and q.

The equation contradicts the fact that $\gamma_u(\lambda)$ is time-like curve and $\beta_2<0$ on $[0,\lambda]$, $C^1C^1+C^2C^2=0$ is therefore impossible on the entire interval $[0,\lambda]$, that is, $C^1C^1+C^2C^2>0$ is at least retained on some ranges of $[0,\lambda]$.

Therefore, the proper acceleration of the time-like curve from p to q produced from the variation of the γ_0 approaches infinity on some ranges of $[0,\lambda]$ as $u\to0$.

Because the curve obtained from variation of the null geodesic must be everywhere time-like, we now discuss the constraint of the 'acceleration' B_0^a of the variation vector field on the null geodesic $\gamma_0(\lambda)$.

The variation map $\sigma(u,\lambda)$ is determined by the variation vector field Z^a and its acceleration vector field $B_0^a=[Z^c(u,\lambda)\nabla_cZ^a(u,\lambda)]_{u=0}$. To make $\gamma_u(\lambda)$ time-like everywhere, the acceleration vector field B_0^a of the variation vector field Z^a must not be zero. The proof is in the following.

We suppose that B_0^a is zero, and we now simplifies the term $[\frac{\partial^2}{\partial\lambda\partial u}(g_{ab}v_u^aZ^b(u,\lambda))]_{u=0}$,

$$\Big[\frac{\partial^2}{\partial\lambda\partial u}(g_{ab}v_u^aZ^b(u,\lambda))\Big]_{u=0}$$
$$=\frac{\partial}{\partial\lambda}[Z^c(u,\lambda)\nabla_c(g_{ab}v_u^aZ^b(u,\lambda))]_{u=0}$$
$$=\frac{\partial}{\partial\lambda}[g_{ab}Z^b(u,\lambda)Z^c(u,\lambda)\nabla_cv_u^a]_{u=0}$$

$$
\begin{aligned}
&+\frac{\partial}{\partial\lambda}[g_{ab}v_u^a Z^c(u,\lambda)\nabla_c Z^b(u,\lambda)]_{u=0}\\
&=v_0^d\nabla_d[Z^a(u,\lambda)v_u^c\nabla_c Z_a(u,\lambda)]_{u=0}+\frac{\partial}{\partial\lambda}[g_{ab}v_0^a B_0^b]\\
&=v_0^d\nabla_d Z^a(0,\lambda)(v_0^c\nabla_c Z_a(0,\lambda))+Z^a v_0^d\nabla_d(v_0^c\nabla_c Z_a),
\end{aligned}
\tag{29}
$$

from Eq. (13), one obtains

$$
\begin{aligned}
\beta_2=&[v_0^d\nabla_d Z^a(0,\lambda)][v_0^c\nabla_c Z_a(0,\lambda)]\\
&-R_{dcae}v_0^c v_0^e Z^d Z^a.
\end{aligned}
\tag{30}
$$

With the pseudo-orthogonal basis E_1^a, E_2^a, E_3^a, E_4^a, the above equation becomes

$$
\beta_2=\sum_{i=1}^{2}\left(\frac{\mathrm{d}Z^i}{\mathrm{d}\lambda}\right)^2-R_{dcae}v_0^c v_0^e Z^d Z^a, \tag{31}
$$

meanwhile, at point p, $\lambda=0$, one has $Z^a=0$, therefore, from Eq. (31), $\beta_2<0$ cannot be met at point p, furthermore, the variation map $\sigma(u,\lambda)$ is a piecewise C^3 on each $(-\varepsilon,\varepsilon)\times[\lambda_i,\lambda_{i+1}]$, that is, it is C^3 on $[\lambda_1=0,\lambda_2]$ and the variation vector Z^a is C^2 on $[\lambda_1=0,\lambda_2]$, there must therefore exist $\lambda_o\neq 0$ and $\lambda_o<\lambda_2$ such that in the segment (ε,λ_o) $\beta_2>0$ everywhere, which means $\gamma_u(\lambda)$ is space-like in the segment (ε,λ_o). Similarly, at point q, $\lambda=\lambda_q$, one has $Z^a=0$, therefore, $\beta_2>0$ at point q and there must also be some quantity $\lambda_\rho\neq\lambda_q$ such that in the segment $(\lambda_\rho,\lambda_q]$ $\beta_2>0$ everywhere. Thus, with the second derivative of $g_{ab}v_u^a v_u^b$ by u, there must be some segments (for example (ε,λ_o), $(\lambda_\rho,\lambda_q]$, with $\lambda_o\neq 0$ and $\lambda_\rho\neq\lambda_q$) such that the curve $\gamma_u(\lambda)$ is space-like. Therefore, to make $\gamma_u(\lambda)$ time-like everywhere, the acceleration vector field B_0^a of the variation vector field Z^a must not be zero.

In conclusion, When a null geodesic γ_0 connecting p, q has a point $r\in(p,q)$ conjugate to p, then up to the second derivative, the proper acceleration of the time-like curve from p to q produced from the variation of the γ_0 approaches infinity on some range $[0,\lambda]$ as $u\to 0$. Meanwhile, the acceleration vector field B_0^a of the variation vector field Z^a must not be zero.

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