New Three-Mode Charge-Related Entangled State Representation*

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(Received 21 January 2002)

We construct a new kind of three-mode charge-related entangled state $|q,k;n\rangle_\theta$ in three-mode Fock space, which possesses the completeness relation and orthogonal property. It can describe when photons emitted by two energy levels simultaneously are absorbed by the third energy level. The relation between $|q,k;n\rangle_\theta$ and the tripartite entangled state $|\eta,\sigma\rangle_\theta$ [Chin. Phys. Lett. 18 (2001) 1301] is pointed out.

PACS: 03. 65. Bz, 42. 50. Dv

It is well known that coherent states are an important concept in both quantum mechanics and quantum optics.[1] Bhaumik *et al.*[2] established the so-called charged boson coherent state $|\alpha,q\rangle$ in two-mode Fock space

$$|\alpha,q\rangle = N_q \sum_{n=\max(0,-q)}^{\infty} \frac{\alpha^n}{\sqrt{n!(n+q)!}}|n+q,n\rangle,$$

where N_q is the normalization constant, $|\alpha,q\rangle$ is the common eigenvector of the Abelian "charge" operator $a_1^\dagger a_1 - a_2^\dagger a_2$ and the two-mode annihilation operator $a_1 a_2$, since $[a_1^\dagger a_1 - a_2^\dagger a_2, a_1 a_2] = 0$. Later, we introduced the $SU(3)$ charged hypercharged coherent state.[3] Since $a_1^\dagger a_2^\dagger$ also obeys the commutator $[a_1^\dagger a_1 - a_2^\dagger a_2,\ a_1^\dagger a_2^\dagger] = 0$, we even introduced their common eigenvector, called the "dual pair coherent state."[4] In quantum optics theory, such a state has been called the "pair coherent state" by Agawal.[5] $a_1^\dagger a_1 - a_2^\dagger a_2$ can be viewed as the number of photons emitted from one energy level and then absorbed by another energy level, called the $SU(1,1)$ coherent state by Perelomov.[6] A question thus naturally arises. If photons simultaneously emitted by two energy levels in an optical cavity are absorbed by the third energy level, how can we construct a corresponding eigenstate? Because we now tackle the three-mode case, another two operators which commute with $a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3$ must be found to make up a complete operator set. Here we search for the common eigenvector $|n;q,k\rangle_\theta$ of three compatible operators: $a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3$,

$$\begin{aligned} K &\equiv (a_1^\dagger \cos\theta + a_2^\dagger \sin\theta - a_3)(a_1 \cos\theta + a_2 \sin\theta - a_3^\dagger), \\ R &\equiv (-a_1^\dagger \sin\theta + a_2^\dagger \cos\theta)(-a_1 \sin\theta + a_2 \cos\theta), \end{aligned} \tag{1}$$

as

$$\begin{aligned} \left[a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3, K\right] &= 0, \\ \left[a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3, R\right] &= 0, \\ [K,R] &= 0. \end{aligned} \tag{2}$$

The meaning of $a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3$ is that photons simultaneously emitted by two energy levels are absorbed by a third energy level, and the operator $A \equiv a_1 \cos\theta + a_2 \sin\theta$ describes the situation when the A-mode light falls on a beamsplitter or a polarizer to decompose into the a_1-mode and a_2-mode. $C \equiv -a_1 \sin\theta + a_2 \cos\theta$ represents the decomposition of the photon mode by another kind of beamsplitter (it is noted that there is a $\pi/2$ phase difference between the decompositions of A and C). We want to search for the common eigenvector $|q,k;n\rangle_\theta$ which obeys the eigenfunction equations

$$\left(a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3\right)|q,k;n\rangle_\theta = (q+n)|q,k;n\rangle_\theta, \tag{3}$$

$$R|q,k,n\rangle_\theta = C^\dagger C|q,k;n\rangle_\theta = n|q,k;n\rangle_\theta, \tag{4}$$

$$\begin{aligned} K|q,k,n\rangle_\theta &= (A^\dagger - a_3)(A - a_3^\dagger)|q,k;n\rangle_\theta \\ &= k|q,k;n\rangle_\theta, \quad k \geq 0, \end{aligned} \tag{5}$$

where we have set the eigenvalue $k \geq 0$, since $K = K^\dagger$ and

$$\langle |K| \rangle = |(a_1 \cos\theta + a_2 \sin\theta - a_3^\dagger)|\rangle|^2 \geq 0. \tag{6}$$

We call $|q,k;n\rangle_\theta$ the charge-related state. To search for $|q,k;n\rangle_\theta$ we notice that due to

$$\begin{aligned} &A \equiv a_1 \cos\theta + a_2 \sin\theta,\ B \equiv a_3, \\ &C \equiv -a_1 \sin\theta + a_2 \cos\theta, \end{aligned} \tag{7}$$

we can derive

$$\begin{aligned} a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3 &= A^\dagger A + C^\dagger C - B^\dagger B, \\ K &= (A^\dagger - B)(A - B^\dagger), \end{aligned} \tag{8}$$

and the eigenvector equations (3)–(5) become

$$\begin{aligned} &(A^\dagger A + C^\dagger C - B^\dagger B)|q,k;n\rangle_\theta = (q+n)|q,k;n\rangle_\theta, \\ &C^\dagger C|q,k;n\rangle_\theta = n|q,k;n\rangle_\theta, \\ &K|q,k;n\rangle_\theta = k|q,k;n\rangle_\theta, \quad k \geq 0. \end{aligned} \tag{9}$$

* Supported by the National Natural Science Foundation of China under Grant No10175057.

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Hence we see that the decomposition of the required state $|q,k;n\rangle_\theta$ may have the form

$$|q,k;n\rangle_\theta = |q,k\rangle_{AB}\otimes|n\rangle_C, \tag{10}$$

where the subscript C denotes that $|n\rangle_C$ belongs to the Fock space spanned by

$$|n\rangle_C = \frac{C^{\dagger n}}{\sqrt{n!}}|0\rangle_C. \tag{11}$$

Now we search for $|q,k\rangle_{AB}$ which obeys the eigenvector equations. For this purpose we introduce the following entangled state (such a form has been introduced before for exploring new squeezed states[7] and been applied to quantum teleportation[8])

$$\begin{aligned}|\eta\rangle_{AB} =&\exp\Big\{-\frac{|\eta|^2}{2}+\eta A^\dagger-\eta^* B^\dagger+A^\dagger B^\dagger\Big\}|\tilde{0}\tilde{0}\rangle,\\ \eta =&|\eta|\mathrm{e}^{\mathrm{i}\alpha},\end{aligned} \tag{12}$$

where the state $|\tilde{0}\tilde{0}\rangle$ is annihilated by either $A|\tilde{0}\tilde{0}\rangle=0$ or $B|\tilde{0}\tilde{0}\rangle=0$. Here $\eta=|\eta|\mathrm{e}^{\mathrm{i}\alpha}$ is a complex number, and $|\eta\rangle_{AB}$ obeys

$$\begin{aligned}(A^\dagger-B)|\eta\rangle_{AB}&=\eta^*|\eta\rangle_{AB},\\ (A-B^\dagger)|\eta\rangle_{AB}&=\eta|\eta\rangle_{AB}.\end{aligned} \tag{13}$$

Since

$$\begin{aligned}(A^\dagger A-B^\dagger B)|\eta\rangle_{AB}=&(\eta A^\dagger+\eta^* B^\dagger)|\eta\rangle_{AB}\\ =&-\mathrm{i}\frac{\partial}{\partial\alpha}|\eta\rangle_{AB},\end{aligned} \tag{14}$$

we have

$$\begin{aligned}{}_{AB}\langle\eta|(A^\dagger A-B^\dagger B)|q,k\rangle_{AB}=&q_{AB}\langle\eta|q,k\rangle_{AB}\\ =&\mathrm{i}\frac{\partial}{\partial\alpha}{}_{AB}\langle\eta|q,k\rangle_{AB}.\end{aligned} \tag{15}$$

The solution of Eq. (15) is

$${}_{AB}\langle\eta|q,k\rangle_{AB}=f(|\eta|,q,k)\mathrm{e}^{-\mathrm{i}q\alpha}, \tag{16}$$

where $f(|\eta|,q,k)$ is to be determined. Using the unique condition of the wavefunction

$$\mathrm{e}^{-\mathrm{i}q\alpha}|_{\alpha=0}=\mathrm{e}^{-\mathrm{i}q\alpha}|_{\alpha=2\pi},$$

we know $q=0,\pm1,\pm2\cdots$. Furthermore, from Eqs. (9) and (13) we have

$$\begin{aligned}{}_{AB}\langle\eta|K|q,k\rangle_{AB}=&{}_{AB}\langle\eta|q,k\rangle_{AB}|\eta|^2\\ =&{}_{AB}\langle\eta|q,k\rangle_{AB}k,\end{aligned} \tag{17}$$

or

$${}_{AB}\langle\eta|q,k\rangle_{AB}(|\eta|^2-k)=0. \tag{18}$$

Combining Eq. (16) with (18) we know

$$\begin{aligned}f(|\eta|,q,k)=&\delta(|\eta|^2-k)\\ =&\frac{1}{2\sqrt{k}}[\delta(|\eta|-\sqrt{k})-\delta(|\eta|+\sqrt{k})]\\ =&\frac{1}{2\sqrt{k}}\delta(|\eta|-\sqrt{k}),\end{aligned} \tag{19}$$

where we have used $k>0$ in the last step. In the following we neglect the subscript belonging to $|q,k\rangle_{AB}$ and $|\eta\rangle_{AB}$ for brevity. Using

$$\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta|=1, \tag{20}$$

we have

$$\begin{aligned}\langle q',k'|q,k\rangle=&\int\frac{\mathrm{d}^2\eta}{\pi}\langle q',k'|\eta\rangle\langle\eta|q,k\rangle\\ =&\int\frac{\mathrm{d}^2\eta}{\pi}\frac{1}{4\sqrt{k'k}}\delta(|\eta|-\sqrt{k'})\delta(|\eta|-\sqrt{k})\\ &\cdot\mathrm{e}^{\mathrm{i}(q'-q)\alpha}\\ =&\delta(k-k')\delta_q^{q'}.\end{aligned} \tag{21}$$

Since

$$\begin{aligned}&\int_0^\infty\mathrm{d}k\sum_{q=-\infty}^{\infty}\langle\eta'|q,k\rangle\langle q,k|\eta\rangle\\ =&\int_0^\infty\mathrm{d}k\sum_{q=-\infty}^{\infty}\delta(|\eta|^2-k)\mathrm{e}^{-\mathrm{i}q\alpha'}\delta(|\eta'|^2-k)\mathrm{e}^{\mathrm{i}q\alpha}\\ =&\int_0^\infty\delta(|\eta|^2-k)\delta(|\eta'|^2-k)\mathrm{d}k\sum_{q=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}q(\alpha-\alpha')}\\ =&\delta(|\eta'|^2-|\eta|^2)\sum_{i=-\infty}^{\infty}2\pi\delta(\alpha'-\alpha-2\mathrm{i}\pi)\\ =&\frac{\pi\delta(|\eta'|-|\eta|)\delta(\alpha'-\alpha)}{|\eta|}\\ =&\pi\delta^{(2)}(\eta'-\eta)=\langle\eta'|\eta\rangle,\end{aligned} \tag{22}$$

we have the completeness relation

$$\int_0^\infty\mathrm{d}k\sum_{q=-\infty}^{\infty}|q,k\rangle\langle q,k|=1. \tag{23}$$

We now deduce the explicit form of $|q,k\rangle$, using Eq. (20) we have

$$\begin{aligned}|q,k\rangle=&\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta|q,k\rangle\\ =&\frac{1}{2\sqrt{k}}\int_0^\infty\int_0^{2\pi}\frac{|\eta|\mathrm{d}|\eta|\mathrm{d}\alpha}{\pi}\exp\Big\{-\frac{|\eta|^2}{2}\\ &+\eta A^\dagger-\eta^*B^\dagger+A^\dagger B^\dagger\Big\}|\tilde{0}\tilde{0}\rangle\delta(|\eta|^2-k)\mathrm{e}^{-\mathrm{i}q\alpha}\\ =&\frac{1}{2\pi}\mathrm{e}^{-k/2}\int_0^{2\pi}\mathrm{d}\alpha\mathrm{e}^{-\mathrm{i}q\alpha}\exp\{\sqrt{k}\mathrm{e}^{\mathrm{i}\alpha}A^\dagger\\ &-\sqrt{k}\mathrm{e}^{-\mathrm{i}\alpha}B^\dagger+A^\dagger B^\dagger\}|\tilde{0}\tilde{0}\rangle.\end{aligned} \tag{24}$$

Letting $e^{i\alpha} = z$, and using the contour integration technique, we obtain its explicit form

$$|q,k\rangle = \frac{e^{-k/2}}{2\pi}\oint \frac{dz}{iz} z^{-q}\exp\left\{\sqrt{k}A^\dagger z - \frac{\sqrt{k}B^\dagger}{z} + A^\dagger B^\dagger\right\}|\tilde{0}\tilde{0}\rangle$$
$$= \sum_{s=\max(0,q)}^{\infty} e^{-k/2}\frac{(-1)^{s-q}}{s!(s-q)!}(\sqrt{k}A^\dagger)^s(\sqrt{k}B^\dagger)^{s-q}e^{A^\dagger B^\dagger}|\tilde{0}\tilde{0}\rangle$$
$$= \sum_{s=\max(0,q)}^{\infty} e^{-k/2}k^{-q/2}\frac{(-1)^{s-q}k^s A^{\dagger s}B^{\dagger s-q}}{s!(s-q)!}e^{A^\dagger B^\dagger}|\tilde{0}\tilde{0}\rangle. \quad (25)$$

When working back into the original three-mode Fock space we have

$$|q,k;n\rangle_\theta = |q,k\rangle_{AB}\otimes|n\rangle_C$$
$$= \frac{e^{-k/2}k^{-q/2}}{\sqrt{N_c!}}\sum_{s=\max(0,q)}^{\infty}\{[(-1)^{s-q}k^s$$
$$\cdot(a_1^\dagger\cos\theta + a_2^\dagger\sin\theta)^s(-a_1^\dagger\sin\theta + a_2^\dagger\cos\theta)^n$$
$$\cdot a_3^{\dagger s-q}e^{(a_1^\dagger\cos\theta + a_2^\dagger\sin\theta)a_3^\dagger}|000\rangle][s!(s-q)!]^{-1}\}, \quad (26)$$

and Eqs. (22) and (24) become

$${}_\theta\langle q',k';n'|q,k;n\rangle_\theta = \delta(k-k')\delta_q^{q'}\delta_n^{n'}, \quad (27)$$
$$\int_0^\infty dk\sum_{q=-\infty}^{\infty}\sum_{n=0}^{\infty}|q,k;n\rangle_{\theta\theta}\langle q,k;n| = 1. \quad (28)$$

From the structure of Eq. (26) we see that $|q,k;n\rangle$ is a new entangled state, which can be used as a quantum channel.

We now briefly discuss the relation between $|q,k;n\rangle_\theta$ and the newly constructed three-mode entangled state $|\eta,\sigma\rangle_\theta$ in Ref. [9], where

$$|\eta,\sigma\rangle_\theta = \exp\Big[-\frac{1}{2}(|\eta|^2+|\sigma|^2) + a_3^\dagger(a_1^\dagger-\eta^*)\cos\theta$$
$$+ a_3^\dagger(a_2^\dagger-\sigma^*)\sin\theta + \eta a_1^\dagger + \sigma a_2^\dagger\Big]|000\rangle. \quad (29)$$

η and σ are the two complex numbers, and $|\eta,\sigma\rangle_\theta$ obeys the following three eigenvector equations

$$(a_1 - a_3^\dagger\cos\theta)|\eta,\sigma\rangle_\theta = \eta|\eta,\sigma\rangle_\theta, \quad (30)$$
$$(a_2 - a_3^\dagger\sin\theta)|\eta,\sigma\rangle_\theta = \sigma|\eta,\sigma\rangle_\theta, \quad (31)$$
$$(a_3 - a_1^\dagger\cos\theta - a_2^\dagger\sin\theta)|\eta,\sigma\rangle_\theta$$
$$= -(\eta^*\cos\theta + \sigma^*\sin\theta)|\eta,\sigma\rangle_\theta. \quad (32)$$

Letting $\eta = |\eta|e^{i\alpha}$, $\sigma = |\sigma|e^{i\beta}$, then from Eq. (29) we have

$$\frac{\partial}{\partial\alpha}|\eta,\sigma\rangle_\theta = i|\eta|(e^{-i\alpha}a_3^\dagger\cos\theta + e^{i\alpha}a_1^\dagger)|\eta,\sigma\rangle_\theta$$
$$= i(a_3^\dagger\eta^*\cos\theta + a_1^\dagger\eta)|\eta,\sigma\rangle_\theta, \quad (33)$$
$$\frac{\partial}{\partial\beta}|\eta,\sigma\rangle_\theta = i|\sigma|(e^{-i\beta}a_3^\dagger\sin\theta + e^{i\beta}a_2^\dagger)|\eta,\sigma\rangle_\theta$$
$$= i(a_3^\dagger\sigma^*\sin\theta + a_2^\dagger\sigma)|\eta,\sigma\rangle_\theta. \quad (34)$$

By using Eqs. (30)–(34) we observe

$$(a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3)|\eta,\sigma\rangle_\theta$$
$$= [a_1^\dagger\eta + a_2^\dagger\sigma + a_3^\dagger(\eta^*\cos\theta + \sigma^*\sin\theta)]|\eta,\sigma\rangle_\theta$$
$$= -i\Big(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\Big)|\eta,\sigma\rangle_\theta \quad (35)$$

As a result of Eqs. (35) and (3) we see

$${}_\theta\langle\eta,\sigma|(a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3)|q,k;n\rangle_\theta$$
$$= i\Big(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\Big){}_\theta\langle\eta,\sigma|q,k;n\rangle_\theta$$
$$= (q+n){}_\theta\langle\eta,\sigma|q,k;n\rangle_\theta,$$

which yields

$${}_\theta\langle\eta,\sigma|q,k;n\rangle_\theta = F(|\eta|,|\sigma|,k)\exp[-i(q\alpha+n\beta)], \quad (36)$$

where $F(|\eta|,|\sigma|,k)$ can be determined by using Eq. (5). In fact, from Eqs. (30) and (31) we have

$$(a_1\cos\theta + a_2\sin\theta - a_3^\dagger)|\eta,\sigma\rangle_\theta = (\eta\cos\theta + \sigma\sin\theta)|\eta,\sigma\rangle_\theta. \quad (37)$$

Combining Eqs. (32), (37) and (5) we have

$${}_\theta\langle\eta,\sigma|(a_1^\dagger\cos\theta + a_2^\dagger\sin\theta - a_3)$$
$$\cdot(a_1\cos\theta + a_2\sin\theta - a_3^\dagger)|q,k;n\rangle_\theta$$
$$= |\eta\cos\theta + \sigma\sin\theta|^2\,{}_\theta\langle\eta,\sigma|q,k;n\rangle_\theta$$
$$= k|q,k;n\rangle_\theta. \quad (38)$$

Thus

$$F(|\eta|,|\sigma|,k) = \delta(|\eta\cos\theta + \sigma\sin\theta|^2 - k). \quad (39)$$

Using the completeness relation of $|\eta,\sigma\rangle_\theta$

$$\int\frac{d^2\eta d^2\sigma}{\pi^2}|\eta,\sigma\rangle_{\theta\theta}\langle\eta,\sigma| = 1, \quad (40)$$

and considering Eq. (36) as well as Eq. (39) we have

$$|q,k;n\rangle_\theta = \int\frac{d^2\eta d^2\sigma}{\pi^2}|\eta,\sigma\rangle_{\theta\theta}\langle\eta,\sigma|q,k;n\rangle_\theta$$
$$= \int\frac{d^2\eta d^2\sigma}{\pi^2}|\eta,\sigma\rangle_\theta\delta(|\eta\cos\theta + \sigma\sin\theta|^2 - k)$$
$$\cdot e^{-i(q\alpha+n\beta)}, \quad (41)$$

which is a kind of Radon transform due to the existence of the Dirac δ-function. One can check the validity of Eq. (41) by re-examining Eq. (5). In fact, using Eqs. (1), (32) and (37) we have

$$K|q,k;n\rangle_\theta = \int\frac{d^2\eta d^2\sigma}{\pi^2}|\eta\cos\theta + \sigma\sin\theta|^2|\eta,\sigma\rangle_\theta$$
$$\cdot\delta(|\eta\cos\theta + \sigma\sin\theta|^2 - k)e^{-i(q\alpha+n\beta)}$$
$$= k|q,k;n\rangle_\theta, \quad (42)$$

which recovers Eq. (5). As an application of the new state $|q,k;n\rangle_\theta$ we consider the following special Hamiltonian model

$$\begin{aligned} H_1 =& a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 - [a_3^\dagger(a_1^\dagger\cos\theta + a_2^\dagger\sin\theta) \\ &+ a_3(a_1\cos\theta + a_2\sin\theta)], \end{aligned} \tag{43}$$

where the bilinear interaction indicates that mode $a_3^\dagger$ couples with the coherent production of both mode $a_1^\dagger$ and mode $a_2^\dagger$; such an interaction may exist in some nonlinear medium in the parametric approximation. Using Eq. (1) we can rewrite Eq. (43) as

$$H_1 = K + R - 1. \tag{44}$$

Thus from Eqs. (4) and (5) we know that its energy eigenvector is $|q,k;n\rangle_\theta$ with the spectrum being $k+n-1$. Next, we consider

$$\begin{aligned} H_2 =& \omega(a_1^\dagger a_1 + a_2^\dagger a_2) + \Omega a_3^\dagger a_3 \\ &- \frac{\omega+\Omega}{2}[a_3^\dagger(a_1^\dagger\cos\theta + a_2^\dagger\sin\theta) \\ &+ a_3(a_1\cos\theta + a_2\sin\theta)]. \end{aligned} \tag{45}$$

Using Eq. (1) we can rewrite Eq. (45) as

$$\begin{aligned} H_2 =& \frac{\omega-\Omega}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3) \\ &+ \frac{\omega+\Omega}{2}(K+R-1). \end{aligned} \tag{46}$$

As a result of Eqs. (3)–(5) we see

$$\begin{aligned} & H_2|q,k;n\rangle_\theta \\ =& \left[\frac{\omega-\Omega}{2}(q+n) + \frac{\omega+\Omega}{2}(k+n-1)\right]|q,k;n\rangle_\theta \\ =& \left[\omega n + \frac{\omega-\Omega}{2}q + \frac{\omega+\Omega}{2}(k-1)\right]|q,k;n\rangle_\theta. \end{aligned} \tag{47}$$

From Eqs. (1)–(5) we can establish a new number-difference-phase uncertainty relation with regard to $a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3$ and

$$\mathrm{e}^{\mathrm{i}\phi} \equiv \sqrt{\frac{a_3^\dagger - a_1\cos\theta - a_2\sin\theta}{a_3 - a_1^\dagger\cos\theta - a_2^\dagger\sin\theta}}. \tag{48}$$

The latter is proposed in Ref. [9] as the three-mode generalization of the Noh–Fougères–Mandel operational phase operator.[10] Since

$$\begin{aligned} & [a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3, a_1\cos\theta + a_2\sin\theta - a_3^\dagger] \\ =& -(a_1\cos\theta + a_2\sin\theta - a_3^\dagger), \end{aligned} \tag{49}$$

$$\begin{aligned} & [a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3, a_1^\dagger\cos\theta + a_2^\dagger\sin\theta - a_3] \\ =& a_1^\dagger\cos\theta + a_2^\dagger\sin\theta - a_3, \end{aligned} \tag{50}$$

we obtain

$$\begin{aligned} [a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3, \mathrm{e}^{\mathrm{i}\phi}] &= -\mathrm{e}^{\mathrm{i}\phi}, \\ [a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3, \mathrm{e}^{-\mathrm{i}\phi}] &= \mathrm{e}^{-\mathrm{i}\phi}. \end{aligned} \tag{51}$$

Thus the corresponding number-difference-phase uncertainty relation is

$$\Delta(a_1^\dagger a_1 + a_2^\dagger a_2 - a_3^\dagger a_3)\Delta\cos\phi \geq \frac{1}{2}|\sin\phi|. \tag{52}$$

In summary, we have found a new three-mode orthonormal and complete quantum mechanical representation, which can describe when photons simultaneously emitted by two energy levels are absorbed by another energy level. This state is the characteristic of two discrete variables (q,n) and one continuum variable k. We call it the generalized "number-difference related" or "charge-related" entangled state. We have also pointed out that $|q,k;n\rangle_\theta$ is a special Radon transform of the three-mode entangled state $|\eta,\sigma\rangle_\theta$, which means that they are partners. For recent works on Radon transform of the Wigner operator, we refer to Refs. [11] and [12].

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