Perturbative Influence on the Shockwave of Burgers Equation

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From the point of view of evolution equations with soliton solutions, we present a general way for the study of the shockwave of one-dimensional Burgers equation under the action of perturbations. Apart from the damping case which needs a somewhat special treatment, we formulate the effects induced by other general perturbations unifyingly.

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The Burgers equation is one of the simplest nonlinear diffusion equations. Since it was originally proposed by Burgers[1] as a model for irrotational or vorticity-free hydrodynamics, it is found to arise naturally in a number of diverse areas.[2-6] It is known that the Burgers equation has a shockwave solution[5] which is an analog of the soliton. In the present work, we consider the perturbative influence on this shockwave. Now, we start with the perturbed Burgers equation in the form:

$$u_t = 2uu_x + u_{xx} + \varepsilon R[u], \tag{1}$$

subject to the initial condition

$$u(x,0) = -a(1 - \tanh ax). \tag{2}$$

Following the main line in our recent work devoted to the study of soliton perturbation problems,[7-9] we shall linearize Eq. (1) by virtue of the derivative expansion method. At first, we assume an asymptotic expansion in the form:

$$u(x,t) = u^{(0)}(x,t) + \varepsilon u^{(1)}(x,t) + \cdots, \tag{3}$$

where we take

$$u^{(0)}(x,t) = -a[1 - \tanh a(x - \frac{x_0}{\varepsilon} - x_1)]. \tag{4}$$

And then, we should introduce a slow time scale $\tau = \varepsilon t$, since the shockwave parameters a, x_0 and x_1 will depend and only depend on the slow scale τ under the action of perturbations. Transforming into a new reference frame moving with the shockwave, considering the slow scale τ and substituting expansion (3) into Eq. (1), we obtain a system of equations for each order of ε,

$$ax_{0\tau}u_z^{(0)} + 2au^{(0)}u_z^{(0)} + a^2u_{zz}^{(0)} = 0, \tag{5}$$

$$\begin{aligned} &u_t^{(1)} - [a^2u_{zz}^{(1)} + 2au^{(0)}u_z^{(1)} + 2au_z^{(0)}u^{(1)} + ax_{0\tau}u_2^{(1)}] \\ &\quad = R[u^{(0)}] - u_\tau^{(0)} - \frac{1}{a}a_\tau zu_z^{(0)} + ax_{1\tau}u_z^{(0)}, \\ &\qquad \cdots\cdots, \end{aligned} \tag{6}$$

where we have defined $z = a(x - x_0/\varepsilon - x_1)$. Here, it is not difficult to verify that the lowest-order equation (5) will be satisfied only if

$$x_{0\tau} = 2a. \tag{7}$$

Thus, the first-order equation (6) can be rewriten as

$$\begin{aligned} &u_t^{(1)} - a^2[u_{zz}^{(1)} + 2\tanh zu_z^{(1)} + 2\mathrm{sech}^2 zu^{(1)}] \\ &\quad = R[u^{(0)}] + a_\tau(1 - \tanh z - z\mathrm{sech}^2 z) + a^2x_{1\tau}\mathrm{sech}^2 z. \end{aligned} \tag{8}$$

By means of a transformation $u^{(1)} = V\mathrm{sech}z$, we transform Eq. (8) into an equivalent equation:

$$\begin{aligned} V_t - a^2\hat{L}V = &[R + a_\tau(1 - \tanh z)]\mathrm{ch}z \\ &- a_\tau z\mathrm{sech}z + a^2x_{1\tau}\mathrm{sech}z, \end{aligned} \tag{9}$$

where we have defined operator $\hat{L} = \partial^2/\partial z^2 + (2\mathrm{sech}^2 z - 1)$. It appeared in the study of sine-Gordon equation, and its eigenstates consist of a continuous spectrum $\Psi(z,k)$ and a discrete state $\Psi_0(z)$ with zero eigenvalue, analytic expressions of them are[10]

$$\Psi(z,k) = \frac{1}{\sqrt{2\pi(k^2+1)}}(\tanh z - \mathrm{i}k)\mathrm{e}^{\mathrm{i}kz}, \quad \text{for} \quad \lambda = -(k^2+1), \tag{10}$$

$$\Psi_0(z) = \frac{1}{\sqrt{2}}\mathrm{sech}z, \quad \text{for} \quad \lambda = 0. \tag{11}$$

Now, we can solve Eq. (9) by expanding $V(z,t)$ in terms of eigenstates of the operator $\hat{L}$, since they are complete. Firstly, we consider a somewhat special case of damping effect. In this case, $R[u^{(0)}] = -u^{(0)} = a(1 - \tanh z)$, the first term in the right-hand side of Eq. (9) will approach infinite as z, unless we set $a_\tau = -a$, and this condition leads to

$$a = a_0\mathrm{e}^{-\tau}. \tag{12}$$

Subsequently, Eq. (9) reduces to

$$V_t - a^2\hat{L}V = az\mathrm{sech}z + a^2x_{1\tau}\mathrm{sech}z \equiv F(z). \tag{13}$$

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As is mentioned above, the completeness of eigenstates (10) and (11) enable us to expand $V(z,t)$ as

$$V(z,t)=\int_{-\infty}^{+\infty}\overline{V}(k,t)\Psi(z,k)\mathrm{d}k+\overline{V}_0(t)\Psi_0(z). \quad (14)$$

Inserting this expansion into Eq. (13), we obtain

$$\overline{V}(k,t)=[1-\mathrm{e}^{-a^2(k^2+1)t}]\frac{\overline{F}(k)}{a^2(k^2+1)}, \quad (15)$$

$$\overline{V}_0=\overline{F}_0 t, \quad (16)$$

where

$$\overline{F}(k)=\int_{-\infty}^{+\infty}F(z)\Psi^*(z,k)\mathrm{d}z, \quad (17)$$

$$\overline{F}_0=\int_{-\infty}^{+\infty}F(z)\Psi_0(z)\mathrm{d}z. \quad (18)$$

Equation (16) will approach infinite as t, this is the so-called secular term in perturbation theory. In order to eliminate this term, we must impose a secular condition $\overline{F}_0=0$ on the solution, which will result in $x_{1\tau}=0$. Thus, Eq. (17) becomes

$$\overline{F}(k)=\int_{-\infty}^{+\infty}a z\mathrm{sech}z\Psi^*(z,k)\mathrm{d}z. \quad (19)$$

And, we finally obtain the first-order correction as

$$u^{(1)}(z,t)=\Big\{\int_{-\infty}^{+\infty}[1-\mathrm{e}^{-a^2(k^2+1)t}]\cdot\frac{\overline{F}(k)}{a^2(k^2+1)}\Psi(z,k)\mathrm{d}k\Big\}\mathrm{sech}z. \quad (20)$$

Secondly, we turn to some more general perturbations such as u_x^2, u_{xxx}, etc. which guarantee $R\mathrm{ch}z$ to approach 0 as z approaches $\pm\infty$. In these situations, we must set $a_\tau=0$ apparently, and Eq. (9) reduces to

$$V_t-a^2\hat{L}V=R\cosh z+a^2x_{1\tau}\mathrm{sech}z\equiv F'(z). \quad (21)$$

Employing the same procedures carried out above, we derive

$$u^{(1)}(z,t)=\Big\{\int_{-\infty}^{+\infty}[1-\mathrm{e}^{-a^2(k^2+1)t}]\cdot\frac{\overline{F}'(k)}{a^2(k^2+1)}\Psi(z,k)\mathrm{d}k\Big\}\mathrm{sech}z, \quad (22)$$

where

$$\overline{F}'(k)=\int_{-\infty}^{+\infty}R\mathrm{ch}z\Psi^*(z,k)\mathrm{d}z, \quad (23)$$

and the secular condition is

$$\overline{F}_0'=\int_{-\infty}^{+\infty}F'(z)\Psi_0(z)\mathrm{d}z=0, \quad (24)$$

this condition will lead to

$$x_{1\tau}=-\frac{1}{2a^2}\int_{-\infty}^{+\infty}R[u^{(0)}]\mathrm{d}z. \quad (25)$$

Since a is a constant in Eq. (25), the shockwave will just undergo a phase shift proportional directly to slow scale $\tau=\varepsilon t$ and remain its height, width and velocity, whereas the height, width and velocity decrease exponentially on εt under the action of damping effect. Besides, the shockwave modifies its shape by a modulated dispersive wave different from that of solitons of integrable evolution equation.[7]

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