Internal Subharmonic Resonance in Faraday Experiment *

WANG Xin-Long(王新龙)[1,2], CHEN Yi-Huang(陈毅煌)[2], WEI Rong-Jue(魏荣爵)[1,2]
[1]*State Key Laboratory of Modern Acoustics and Institute of Acoustics, Nanjing University, Nanjing 210093*
[2]*Department of Electronic Science and Engineering, Nanjing University, Nanjing 210093*

(Received 23 December 1999)

The classical Faraday experiment is a tractable problem in nonlinear physics, and its rich spectrum of nonlinear phenomena has made it a prototype in studying nonlinear wave dynamics in general. Here we report the observation of a new type of subharmonic resonances which occurs at some smaller fractional frequencies, such as $\nu/3$, $\nu/4$, and $\nu/6$, where ν is the forcing frequency. As a result of these resonances, the lower-order surface-wave modes whose eigenfrequencies are close to the fractional frequencies participate into the wave motions. Our theory involving quadratically nonlinear mode couplings shows that an already excited mode itself can act as a parametric source on the lower-order primary modes, thus leading to the cascading subharmonic resonances.

PACS: 43.25.Gf, 43.35.Pt, 47.20.Ky, 47.35.+i

Parametric resonance is one of the fundamental phenomena in physics[1-3] and the related fields.[4] A typical example is the famous Faraday resonance in fluid dynamics.[5] When a vessel containing liquid is subject to a simple vibration of frequency ν, a pattern of standing waves is formed on the free surface. What is unusual is that the excited waves oscillate at the subharmonic frequency $\nu_{\rm F} = \nu/2$, i.e., half that of the drive. In the linear framework,[6] the description of a small disturbance on the plane surface of liquid was reduced to a set of Mathieu's equations which imply the parametric instability. However, the parametric amplification soon becomes saturated by the strong nonlinearity of the surface-waves, and hence, the parametrically excited waves are intrinsically nonlinear.[7] The studies in recent decades further showed that the nonlinear standing waves can exhibit a rich variety of nonlinear behaviors,[8] in particular, the formations of the solitary-waves[9-15] and complex patterns.[16-18] Because of these, the classical Faraday experiment has become a feasible and effective way for understanding the nonlinear wave dynamics in general.

Mathieu's equation also implies the resonances of higher-order harmonics at frequencies $3\nu/2$, $5\nu/2$, $\cdots$. These resonances have been found, for example, in a microelectromechanical system,[19] where damping is sufficiently weak and the parameters can be precisely controlled. What motivates us here is the excitation of other subharmonics than the usual $\nu/2$ one. Such a subharmonic bifurcation is generally believed to contribute to the nonlinear complexity of the wave dynamics. It is also supposed to be a possible scenario to chaos or turbulence, other than the well-known ones.

We improve the Faraday experiment by using a rectangular plexiglass container of water. The fluid is confined to the region $0 < x < l$, $0 < y < b$, and $-d < z < 0$ in the static state. For simplicity, in what follows we consider only the purely longitudinal modes $(n, 0)$, with wavenumbers $k_n = n\pi/l$ and eigenfrequencies $\nu_n = \omega_n/2\pi$, where n is the modal index. According to the linear theory,[6] ω_n is related to k_n via the dispersion formula,

$$\omega_n^2 = g k_n \tanh k_n d, \tag{1}$$

where g is the acceleration of gravity. Here we have ignored the surface tension of water. To excite the one-dimensional surface-waves, we use a narrow trough $(l < b)$ and adjust Faraday frequency $\nu_{\rm F}$ to be smaller than the cutoff frequency of the transverse modes.

Let us first see the result for $l \times b \times d = 10 \times 3 \times 2$ cm. We choose the forcing frequency $\nu = 7.4$ Hz. At a sufficient large driving amplitude a, e.g., 0.8 mm, the Faraday resonance results in the excitation of mode $(2, 0)$, whose natural frequency $\nu_2 \approx 3.64$ Hz. However, at a slightly larger a, say, 1.0 mm, we observe the excitation of the $\nu/4$ subharmonic, as is shown in Fig. 1. The disappearance of the $\nu/4$ component at $x = l/2$, as shown in Fig. 1(b), illustrates that the trough center is a wave node of the new component. In fact, it is identified to be the $(1, 0)$ mode whose natural frequency ν_1 is about 2.08 Hz, close to $\nu/4 = 1.85$ Hz . Therefore, the dominant portion of surface displacement $\xi(x, t)$ can be expressed as the superposition of the primary modes, i.e.,

$$\xi \sim a_0 \sum_{n=1}^{N} \phi_n \cos(k_n x)\, {\rm e}^{-{\rm i} J_n \omega t} + {\rm c.c.}, \quad \left(J_n \equiv \frac{n}{2N}\right), \tag{2}$$

where c.c. denotes the complex conjugate of the previous terms, N is the number of participating modes (here $N = 2$), ω the forcing angular frequency $(\omega = 2\pi\nu)$, a_0 a scaling constant factor, and $\phi_{1,2}$ the rescaled complex amplitudes (see below).

*Supported by the National Natural Science Foundation of China under Grant Nos. 19925414, 19774029, and 19834040.
©by the Chinese Physical Society

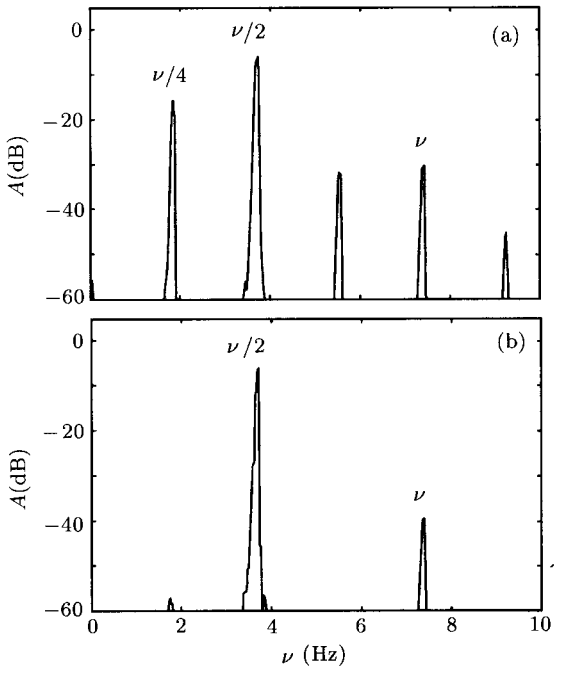

Fig. 1. Power spectra of water surface displacement $\xi(x,t)$ in a trough of $l \times b \times d = 10 \times 3 \times 2$ cm at $\nu = 7.4$ Hz and $a = 1$ mm. The response is first detected by an electrode made up of a pair of conducting needles, and then the signal reflecting the surface elevation is fed to the input of HP-3562A signal analyser. Detecting position: (a) at one end wall: $x = 0$, (b) at the trough center: $x = 5$ cm.

As another example, Fig. 2 shows the spectra of the surface response for $l \times b \times d = 10 \times 2.5 \times 2$ cm and $\nu = 6.8$ Hz. As we can see from the figure, in this case there appears a subharmonic sequence consisting of three fractional frequencies, $\nu/6$, $\nu/3$, and $\nu/2$. The disappearance of both $\nu/6$ and $\nu/3$ spectral lines in Fig. 2(c) shows that the central position ($x = l/2$) is the wave nodes of corresponding wave modes. Actually, we observe the simultaneous excitations of the three modes, (1,0), (2,0) and (3,0). The natural frequencies of these modes, as calculated from the relation (1), are $\nu_1 \approx 1.43$ Hz, $\nu_2 \approx 2.67$ Hz, and $\nu_3 \approx 3.64$ Hz, respectively, close to the fractional frequencies $\nu/6$, $\nu/3$, and $\nu/2$. The resulting wave motion can also be expressed by the relation (2), with $N = 3$. The figure also shows the presence of the $\nu/12$ subharmonic component, indicating the onset of the period-doublings in the (1,0) mode.

From what we have described, we see that the fractional-frequency subharmonic resonances occur to those eigenmodes whose characteristic quantities (ν_n, k_n) satisfy

$$\nu_n : \nu_F \approx k_n : k = 2J_n, \tag{3}$$

where k is the wavenumber of the wave mode oscillating at Faraday frequency ν_F. The requirements are easily realized for those modes $(n,0)$ satisfying the shallow water condition, i.e., $k_n d \ll 1$, since, in the limiting case, the dispersion relation (1) is simplified to $\omega_n \approx \sqrt{gd}k_n$.

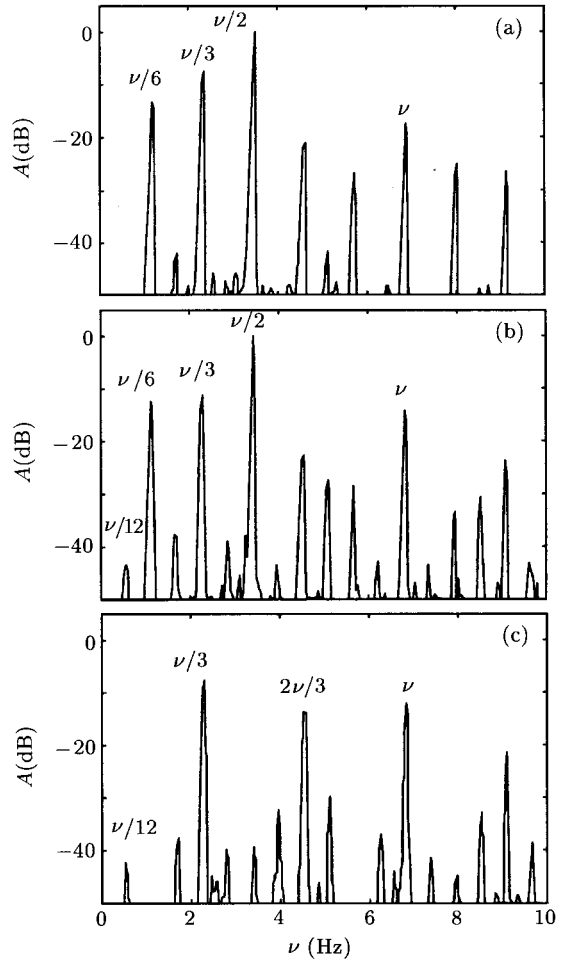

Fig. 2. Power spectra of water surface displacement $\xi(x,t)$ in a trough of $l \times b \times d = 15 \times 2.5 \times 2$ cm at $\nu = 6.8$ Hz and $a = 1$ mm. Probing position: (a) $x = 0$, (b) $x = 5$ cm, (c) $x = 7.5$ cm.

The fractional frequency resonances can be understood as the result of the nonlinear couplings among the primary wave modes. Let us consider the simplest case, i.e., the excitation of double modes ($N = 2$). We assume (a) $k_n d \ll 1$ (shallow water), (b) small amplitude motion $a_0 = O(\varepsilon)$[for convenience, we set $a_0 = (\varepsilon l/\pi) \tanh(\pi d/l)$ in the expression (2)], where ε is a small parameter ($0 < \varepsilon \ll 1$), (c) slowly varying amplitude, i.e., $\mathrm{d}\phi_n(\tau)/\mathrm{d}\tau = O(\phi_n)$, where the slow time $\tau \equiv \varepsilon\omega t$, and (d) (ν_n, k_n) satisfying (3), with $J_n = n/4$. Define the frequency detuning β_n and the dimensionless driving strength γ as

$$\varepsilon\beta_n \equiv \frac{1}{2}\left[1 - \left(\frac{\omega_n}{J_n\omega}\right)^2\right], \quad \varepsilon\gamma \equiv \frac{\omega^2 a}{4g}. \tag{4}$$

The last requirement (d) implies that $\beta_n = O(1)$. We further require that $\gamma = O(1)$. Then, starting from the basic hydrodynamic equations together with the boundary conditions,[6] and applying the multiple-scale

technique to them, we derive a couple of ordinary differential equations for $\phi_1(\tau)$ and $\phi_2(\tau)$:

$$\mathrm{i}(4\phi_1' + \alpha_1\phi_1) + \beta_1\phi_1 + 2\mu\phi_2\phi_1^* = 0, \tag{5a}$$

$$\mathrm{i}(2\phi_2' + \alpha_2\phi_2) + \beta_2\phi_2 + \gamma\phi_2^* + \mu\phi_1^2 = 0, \tag{5b}$$

where the primes on $\phi_{1,2}$ mean the derivatives with respect to τ, the asterisk "$*$" denotes complex conjugate of $\phi_{1,2}$, α_n is the damping coefficient for the n-th mode, and the coupling coefficients $\mu \approx \mathrm{sech}^2(\pi d/l)$. Equation (5b) implies the usual Faraday resonance.[7] As a result, $\phi_1 = 0$ but $\phi_2 \neq 0$ as γ exceeds the threshold $\gamma_2 = \sqrt{\alpha_2^2 + \beta_2^2}$. It immediately follows from Eq. (5a) that a small disturbance on the lower order mode is described by

$$\mathrm{i}(4\delta\phi_1' + \alpha_1\delta\phi_1) + \beta_1\delta\phi_1 + 2\mu\phi_2\delta\phi_1^* = 0. \tag{6}$$

It turns out that the already excited mode, $(2,0)$, acts as an internal parametric source on the lower order $(1,0)$. With the amplification of mode $(2,0)$ at $\gamma > \gamma_2$, mode $(1,0)$ becomes linearly unstable, and is resonantly amplified as $2\mu|\phi_2|$ exceeds the threshold $\gamma_1 = \sqrt{\alpha_1^2 + \beta_1^2}$. The steady-state amplitudes of the double-mode motion can be given explicitly by

$$|\phi_1| = (2\mu)^{-1}\left[\beta_1\beta_2 - \alpha_1\alpha_2 + \sqrt{(\gamma_1\gamma)^2 - (\alpha_1\beta_2 + \alpha_2\beta_1)^2}\right]^{1/2}, \tag{7a}$$

$$|\phi_2| = (2\mu)^{-1}\gamma_1. \tag{7b}$$

The independency of γ from $|\phi_2|$ indicates that the higher-order mode plays the role of an energy router, and the additional energy via the increasing γ is all absorbed by the lower mode. For the parameters used in the first experiment (see Fig. 1), we have $\mu \approx 0.69$, $\varepsilon(\beta_1, \beta_2, \gamma) \approx (-0.135, 0.016, 0.055)$, and the estimated damping coefficients, $\varepsilon\alpha_1 \approx \varepsilon\alpha_2 \approx 0.042$. We thus calculate from Eqs. (7a) and (7b) that $20\lg(|\phi_1/\phi_2|) \approx -9.93\,\mathrm{dB}$, in good agreement with the measured result shown in Fig. (a). A simple stability analysis shows that the double-mode motion can be maintained in a certain range of control strength γ.

Similarly, under the internal pumping of the already excited mode $(3,0)$, the parametric instability can simultaneously occurs to both $(1,0)$ and $(2,0)$ modes, thus forming the triple-mode wave motion. In principle, more subharmonic wave modes can be parametrically excited in a vertically vibrating thin layer of fluid, as long as their (ν_n, k_n) satisfy the resonant conditions (3). We therefore conclude that, because of the quadratic nonlinear couplings, an already excited wave can itself acts as an internal parametric source on the lower-order modes, thus leading to the excitations of the subharmonic sequences of fractional frequencies.

REFERENCES

[1] J. A. Burns, Am. J. Phys. 38 (1970) 920.
[2] L. Ruby, Am. J. Phys. 64 (1996) 39.
[3] P. B. Umbanhowar, F. Melo and H. L. Swinney, Nature, (London) 382 (1996) 793.
[4] D. P. Howson and R. B. Smith, Parametric Amplifiers (McGraw-Hill, New York, 1990) p. 1.
[5] M. Faraday, Philos. Trans. R. Soc. London, 121 (1831) 299.
[6] T. B. Benjamin and F. Ursell, Proc. R. Soc. London, A 225 (1954) 505.
[7] J. W. Miles, J. Fluid Mech. 146 (1984) 285.
[8] S. Cilliberto and J. P. Gollub, Phys. Rev. Lett. 52 (1984)922.
[9] J. R. Yan *et al.*, Phys. Fluids, A5 (1993) 1651.
[10] X. L. Wang and R. J. Wei, Phys. Rev. Lett. 78 (1997) 2744.
[11] WANG Xin-Long, Chin. Phys. Lett. 14 (1997) 109.
[12] WANG Xin-Long and WEI Rong-Jue, Chin. Phys. Lett. 15 (1998) 275.
[13] WANG Xin-Long, Chin. Phys. Lett. 15 (1998) 859.
[14] X. L. Wang, Physica, D 127 (1999) 12.
[15] I. V. Barashenkov and E. V. Zemlyanaya, Phys. Rev. Lett. 83 (1999) 2568.
[16] D. Binks and Van de Water, Phys. Rev. Lett. 78 (1997) 4043.
[17] P. L. Chen and Vinals, Phys. Rev. Lett. 79 (1997) 2670.
[18] J. L. Rogers *et al.*, Phys. Rev. Lett. 84 (2000) 87.
[19] L. Kimberly *et al.*, Nature, (London) 396 (1999) 149.