Even and Odd Q-coherent State Representations of the Q-deformed Heisenberg-Weyl Algebra

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We construct explicitly even and odd q-coherent states. These q-coherent states are introduced in terms of the q-functions defined in the paper. It is shown that the even and odd q-coherent states form a kind of representations of the q-deformed Heisenberg-Weyl algebra which is realized in the form of matrix q-differential operators in the even and odd q-coherent state space. We also analyse some different between the even and odd q-CSs and the usual even and odd CSs.

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It is well known that the q-deformed Heisenberg-Weyl algebra (q-HWA) is one of the most fundamental objects in quantum groups.[1-3] The number representation and the Glauber-typed q-coherent state representations of the q-HWA have been well studied by many authors,[3-5] and applied widely to physics and mathematical physics. Here we will present a new type of representations of the q-HWA, called even and odd q-coherent state (q-CS) representations. The q-HWA is generated by three elements a, a^+ and N satisfying the following relations:

$$aa^+ - qa^+a = q^{-N} , \tag{1}$$

$$[N,\ a^+] = a^+, \quad [N,\ a] = -a , \tag{2}$$

where q is a complex number. The operators a, a^+ and N act in a Hilbert space with the basis $|n\rangle$ ($n = 0, 1, 2, \cdots$), such that

$$a|0\rangle = 0 , \quad |n\rangle = \frac{(a^+)^n}{\sqrt{[n]!}}|0\rangle , \tag{3}$$

where the q-factorial $[n]! = [n][n-1]\cdots[1]$, with the q-number

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} . \tag{4}$$

Their actions on the basis vectors are given by

$$a^+|n\rangle = \sqrt{[n+1]}\,|n+1\rangle , \quad a|n\rangle = \sqrt{[n]}|n-1\rangle . \tag{5}$$

The resolution of unity in the Hilbert space is written as

$$I = \sum_{n=0}^{\infty} |n\rangle\langle n| . \tag{6}$$

We now define an even and an odd q-coherent state,[6] respectively,

$$|z\rangle_q^{\mathrm{e}} = N_q^{\mathrm{e}}(z)\cosh_q(za^+)|0\rangle \ , \tag{7a}$$

$$|z\rangle_q^{\mathrm{o}} = N_q^{\mathrm{o}}(z)\sinh_q(za^+)|0\rangle \ , \tag{7b}$$

where $N_q^{\mathrm{e}}(z)$ and $N_q^{\mathrm{o}}(z)$ are normalization constants to be determined, and we have introduced two q-functions:

$$\cosh_q x = \tfrac{1}{2}\left(e_q^x + e_q^{-x}\right) = \sum_{n=0}^{\infty} \frac{x^{2n}}{[2n]!} \ , \tag{8a}$$

$$\sinh_q x = \tfrac{1}{2}\left(e_q^x - e_q^{-x}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{[2n+1]!} \tag{8b}$$

with

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \ .$$

Substituting Eq. (8) into Eq. (7) and making use of Eq. (5), we have

$$|z\rangle_q^{\mathrm{e}} = N_q^{\mathrm{e}}(z) \sum_{n=0}^{\infty} \frac{z^{2n}}{\sqrt{[2n]!}} \ , \tag{9a}$$

$$|z\rangle_q^{\mathrm{o}} = N_q^{\mathrm{o}}(z) \sum_{n=0}^{\infty} \frac{z^{2n+1}}{\sqrt{[2n+1]!}} \ . \tag{9b}$$

We require that the even and odd q-CSs are normalized in the following form:

$${}_q^{\mathrm{e}}\langle z|z\rangle_q^{\mathrm{e}} = 1 \ , \quad {}_q^{\mathrm{o}}\langle z|z\rangle_q^{\mathrm{o}} = 1 \ . \tag{10}$$

Then, the normalization constants are given by

$$N_q^{\mathrm{e}}(z) = \cosh_q^{-\frac{1}{2}}(z\bar{z}) \ , \quad N_q^{\mathrm{o}}(z) = \sinh_q^{-\frac{1}{2}}(z\bar{z}) \ . \tag{11}$$

From Eq. (9) it follows that

$${}_q^{\mathrm{e}}\langle z'|z\rangle_q^{\mathrm{e}} = N_q^{\mathrm{e}}(z')N_q^{\mathrm{e}}(z)\cosh_q(z\bar{z}') \ , \tag{12a}$$

$${}_q^{\mathrm{o}}\langle z'|z\rangle_q^{\mathrm{o}} = N_q^{\mathrm{o}}(z')N_q^{\mathrm{o}}(z)\sinh_q(z\bar{z}') \ , \tag{12b}$$

$${}_q^{\mathrm{e}}\langle z'|z\rangle_q^{\mathrm{o}} = 0 \ . \tag{13}$$

This means that the even (odd) q-CSs are nonorthogonal; however, the even q-CSs and the odd q-CSs are orthogonal to each other.

As is well known, the core of CSs is their completeness relation. In present case, it can be shown that the even (odd) q-CSs do not form a completeness set. However, the even q-CSs together with the odd q-CSs constitute a complete Hilbert space. And their completeness relation takes the form:

$$\frac{[2]}{2\pi}\int d_q^2 z\, e_q^{-z\bar{z}} \left\{\cosh_q(z\bar{z})|z\rangle_q^{\mathrm{e}}\,{}_q^{\mathrm{e}}\langle z| + \sinh_q(z\bar{z})|z\rangle_q^{\mathrm{o}}\,{}_q^{\mathrm{o}}\langle z|\right\} = I \ , \tag{14}$$

where $d_q^2 z = r d_q r d\theta$ with $z = re^{i\theta}$, so the integral over the variable r is a q-integration, while the integral over $d\theta$ is a normal integration.

Proof. Making use of Eqs. (9) and (11), one can write the left-hand side as follows:

$$\begin{aligned}
&\frac{[2]}{2\pi}\int d_q^2 z\, e_q^{-z\bar{z}} \sum_{n,m}\left\{\frac{z^{2n}\bar{z}^{2m}}{\sqrt{[2n]![2m]!}}|2n\rangle\langle 2m| + \frac{z^{2n+1}\bar{z}^{2m+1}}{\sqrt{[2n+1]![2m+1]!}}|2n+1\rangle\langle 2m+1|\right\} \\
&= \frac{[2]}{2\pi}\int d_q^2 z\, e_q^{-z\bar{z}} \sum_{n,m}\frac{z^n\bar{z}^m}{\sqrt{[n]![m]!}}|n\rangle\langle m| \\
&= \frac{[2]}{2\pi}\int_0^{\zeta} r d_q r \int_0^{2\pi} d\theta\, e_q^{-r^2} \sum_{n,m}\frac{r^{n+m}e^{i(n-m)\theta}}{\sqrt{[n]![m]!}}|n\rangle\langle m| \qquad (15) \\
&= \sum_{n=0}^{\infty}\frac{[2]}{[n]!}|n\rangle\langle n|\int_0^{\zeta} r^{2n+1}e_q^{-r^2}\, d_q r \\
&= I\ .
\end{aligned}$$

Here we have used the q-Euler's formula for $\Gamma(x)$ function,[4]

$$\int_0^{\zeta} x^n e_q^{-x}\, d_q x = [n]!\ , \qquad (16)$$

where ζ is the largest zero of the q-exponential function e_q^{-x}.

Making use of Eqs. (5) and (9), one can check easily that $|z\rangle_q^{\mathrm{e}}$ and $|z\rangle_q^{\mathrm{o}}$ are two eigenstates of the operator a^2, i.e.

$$a^2|z\rangle_q^{\mathrm{e}} = z^2|z\rangle_q^{\mathrm{e}}\ ,\quad a^2|z\rangle_q^{\mathrm{o}} = z^2|z\rangle_q^{\mathrm{o}}\ . \qquad (17)$$

From Eqs. (5) and (9), it is also straightforward to get

$$a|z\rangle_q^{\mathrm{e}} = z\tanh_q^{\frac{1}{2}}(z\bar{z})|z\rangle_q^{\mathrm{o}}\ ,\quad a|z\rangle_q^{\mathrm{o}} = z\coth_q^{\frac{1}{2}}(z\bar{z})|z\rangle_q^{\mathrm{e}}\ , \qquad (18)$$

where

$$\tanh_q x = \frac{e_q^x - e_q^{-x}}{e_q^x + e_q^{-x}}\ ,\quad \coth_q x = \frac{e_q^x + e_q^{-x}}{e_q^x - e_q^{-x}}\ . \qquad (19)$$

So that the even q-CSs and the odd q-CSs can be transformed each other by the action of the operator a. As a result of Eqs. (14) and (18), the combination of the even and odd q-CSs, denoted by $\left\{|z\rangle_q^{\mathrm{e}},\ |z\rangle_q^{\mathrm{o}}\right\}$, gives rise to a representation of the q-HWA.

We now consider a realization of the q-HWA. For simplicity, we consider the actions of the elements of the q-HWA on the unnormalized even and odd q-CS space $\left\{\|z\rangle_q^{\mathrm{e}},\ \|z\rangle_q^{\mathrm{o}}\right\}$, defined by

$$\|z\rangle_q^{\mathrm{e}} = \sum_{n=0}^{\infty}\frac{z^{2n}}{\sqrt{[2n]!}}|2n\rangle\ ,\quad \|z\rangle_q^{\mathrm{o}} = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{\sqrt{[2n+1]!}}|2n+1\rangle\ . \qquad (20)$$

If we express $\|z\rangle_q^e$ and $\|z\rangle_q^o$ as a column vector $\begin{pmatrix} \|z\rangle_q^e \\ \|z\rangle_q^o \end{pmatrix}$, the actions of the operators a and a^+ on this column vector may be written as

$$a\begin{pmatrix} \|z\rangle_q^e \\ \|z\rangle_q^o \end{pmatrix} = \rho(a)\begin{pmatrix} \|z\rangle_q^e \\ \|z\rangle_q^o \end{pmatrix}, \quad a^+\begin{pmatrix} \|z\rangle_q^e \\ \|z\rangle_q^o \end{pmatrix} = \rho(a^+)\begin{pmatrix} \|z\rangle_q^e \\ \|z\rangle_q^o \end{pmatrix}, \tag{21}$$

where

$$\rho(a) = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}, \quad \rho(a^+) = \begin{pmatrix} 0 & \frac{d}{d_q z} \\ \frac{d}{d_q z} & 0 \end{pmatrix}, \tag{22}$$

here $\frac{d}{d_q z}$ is a q-derivative operator.

It can be checked that these matrix operators $\rho(a)$ and $\rho(a^+)$ together with the number operator N satisfy the commutation relations of the q-HWA when they act on the even and odd q-coherent state space. Therefore, they gives rise to a realization of the q-HWA. The above discussions show that the even and odd q-CSs together build a new representation of the q-HWA, and the q-HWA can be realized in the even and odd q-CS space in the matrix form.

The even and odd q-CSs are different from the usual even and odd CSs in the following aspects. Firstly, the former is a representation of the q-HWA while the latter is a representation of the HWA. Secondly, the deformation parameter q in the even and odd q-CSs may have important applications in quantum optics.[6–8] In fact, in our further work we will show that the deformation parameter q may be relevant to the degree of the squeezing and antibunching of the light field, and there exist significant differences on aspects of optical statistics properties of the even and odd q-CSs for all q values. Finally, the even and odd q-CSs become the conventional even and odd CSs in the limit $q \to 1$.

It is obvious that the even and odd q-CSs are nontrivally different from the Glauber-typed q-CS.[3] And it can be shown that the Glauber-typed q-CSs can be expanded nonlinearly in terms of the even and odd q-CSs.

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