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Diagnosing 2D Symmetry Protected Topological States Via Mixed State Anomaly

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Symmetry-protected topological phases are short-range entangled quantum states characterized by anomalous edge behavior, a manifestation of the bulk-boundary correspondence for topological phases. Moreover, the Li-Haldane conjecture posits that the entanglement spectrum exhibits the same anomaly as the physical edge spectrum, thereby serving as an entanglement-based fingerprint for identifying topological phases. In this work, we extend the entanglement-based diagnostic tools by demonstrating that the edge anomaly is manifested not only in the entanglement spectrum but also in the reduced density matrix itself, a phenomenon we refer to as the mixed-state anomaly. For the two-dimensional $\mathbb{Z}_2$ SPT phase, we derive a topological contribution in the disorder parameter as a consequence of this mixed-state anomaly. Furthermore, with additional time-reversal symmetry, the anomalous reduced density matrix induces a spontaneous-symmetry-breaking type long-range order in the presence of the disorder operator.

Introduction

Symmetry and topology are fruitful guiding principles in the discovery and understanding of quantum many-body systems, leading to the construction and classification of symmetry-protected topological (SPT) phases [1−4]. Nontrivial SPT states are short-range entangled states that cannot be transformed into trivial product states using symmetric finite-depth local unitaries, and are characterized by anomalous edge excitations. For instance, in 2D spin systems with $\mathbb{Z}_2$ Ising symmetry, besides the conventional Ising paramagnet, a topological Ising paramagneta exists that hosts symmetry-breaking or gapless edge modes [5−7].

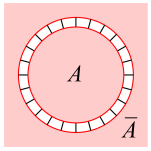
Conventional methods for diagnosing SPT phases include introducing physical edges to investigate anomalous edge states [8−11], or couple the system to gauge fields to study its response [12−16]. These "Hamiltonian-based" approaches, while effective, require specific modifications to the physical systems. "Wavefunction-based" methods—such as strange correlators [17] and wavefunction surgeries [18−22]—provide alternatives but demand knowledge of the full wavefunction.

An alternative approach relies on entanglement-based diagnostics that require only the reduced density matrix (RDM) of the target state on local patches larger than the correlation length. The insight that entanglement encodes topological properties traces back to topological entanglement entropy [23,24], and was advanced by Li and Haldane's work on the entanglement spectrum [25], which mirrors the edge spectrum as a manifestation of the bulk-boundary correspondence [26−28]. However, while valuable for numerical studies, these entanglement-based diagnostic are not directly measurable in experiments, limiting their practical utility.

In this work, we overcome these limitations by demonstrating that the reduced density matrix (RDM) itself, when analyzed within the "entanglement Hilbert space" [29], encodes richer topological information of SPT phases than its derived quantities such as entanglement entropy or entanglement spectrum. While previous studies have shown that the Hall conductance can be extracted from the entanglement Hamiltonian $-\ln\rho$ in chiral topological phases [30,31], we report a novel physical signature for the 2D $\mathbb{Z}_2$ SPT phase: the disorder parameter [32,33] exhibits a universal topological contribution beyond the area law. While disorder parameters have been employed to study various quantum phases—including 1D SPT phases [18], point-group SPT phases [20,34], symmetry-enriched topological orders [35], gapless systems [36−43], and 1D mixed state phases with anomalous symmetries [32,33], our discovery of such universal topological signature for the 2D onsite SPT phases represents a new result in the field. Remarkably, in the presence of additional time-reversal symmetry, this results in spontaneous symmetry-breaking-like long-range order in the presence of the $\mathbb{Z}_2$ disorder operator.

The Setup

Consider a ground state wavefunction $|\psi\rangle$ of a 2D SPT phase protected by a global $\mathbb{Z}_2$ symmetry $G=\{1,g\}$, where g is represented by $U(g)=\bigotimes_j U_j(g)$ with $U_j(g)$ acting on site j. We partition the system into a disk-shaped region A and its complement $\bar{A}$. The RDM for region A is



$$\rho_A=\mathrm{Tr}_{\bar{A}}|\psi\rangle\langle\psi|=\sum_{\alpha=1}^{\chi_A}\exp(-\lambda_\alpha)|\phi_A^\alpha\rangle\langle\phi_A^\alpha|, \quad (1)$$

where the Schmidt decomposition naturally defines a χ_A dimensional entanglement Hilbert space $\mathcal{H}_{\partial A}$, isomorphic to the subspace $\mathrm{span}\{|\phi_A^\alpha\rangle|_{\alpha=1,\ldots,\chi_A}\}$. For the $\mathbb{Z}_2$ SPT, we expect that $\mathcal{H}_{\partial A}$ can be approximated by a tensor-factorized Hilbert space $\mathcal{H}_{\partial A}=\bigotimes_{j=1}^{L_{\partial A}}\mathcal{H}_j$ localized at the 1D entanglement cut ∂A (black bars in the above cartoon), consistent with the entanglement area-law [44]. For a projected entangled pair state (PEPS), for instance, $\mathcal{H}_j$ corresponds to virtual leg j at boundary ∂A [45]. This perspective establishes an isometry the support of the 2D RDM ρ_A and the 1D entanglement Hilbert space $\mathcal{H}_{\partial A}$, allowing us to represent ρ_A as a 1D mixed state $\rho_{\partial A}$ on $\mathcal{H}_{\partial A}$ [27,46] [see Section A of the Supplementary Materials (SM) for details].

A key feature of SPT phases is that the induced symmetry action

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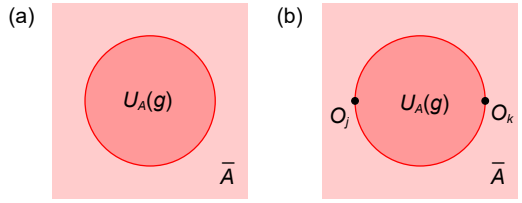


Fig. 1. Diagnostics proposed for detecting 2D $\mathbb{Z}_2$ SPT: (a) disorder parameter $\langle U_A(g)\rangle$; (b) The $\mathbb{Z}_2$ charge correlator at the boundary of the disorder operator, denoted as $\langle O_jO_k\rangle_A^g$.

on $\mathcal{H}_{\partial A}$, $W_{\partial A}(g)$, does not factorize trivially into a product of on-site operators on each $\mathcal{H}_j$. This implies that $\rho_{\partial A}$ is a mixed state with an anomalous global symmetry, a phenomenon we refer to as the mixed state anomaly [47–53]. We now present the central consequences of this anomaly, followed by concrete examples, heuristic arguments from field theory, and a more rigorous analysis using tensor networks.

Main Results: Observables from the Mixed-State Anomaly

Our central results concern two physical observables to probe the $\mathbb{Z}_2$ SPT phase. The first one is the disorder parameter, defined as the expectation value of symmetry operator $U_A(g)=\otimes_{j\in A}U_j(g)$ restricted to a large patch A [depicted in Fig. 1(a)]. This quantity is equivalent to the trace of a symmetry-twisted operator $\rho_A^g\equiv\rho_A U_A(g)$. Our analysis of this quantity for a $\mathbb{Z}_2$ SPT state yields the following universal structure:

$$-\ln\langle U_A(g)\rangle=\alpha L_{\partial A}-\underbrace{\ln D}_{\text{top.}}-\ln\cos(\theta L_{\partial A})+\ldots, \tag{2}$$

where $L_{\partial A}$ is the boundary length of region A, α, and θ are non-universal coefficients, and $D\geqslant 2$ is an integer. This form first appeared in Ref. [33] for the thermal states of SPT edges.

While the leading area-law term in Eq. (2) reflects the symmetry-preserving nature of the SPT state [54], our key prediction is that the universal topological contribution $-\ln D$ with $D\geq 2$ provides a robust diagnostic for the nontrivial $\mathbb{Z}_2$ SPT phase. When expressed in the $\mathcal{H}_{\partial A}$, ρ_A^g becomes its 1D counterpart $\rho_{\partial A}^g:=\rho_{\partial A}W_{\partial A}(g)$, and the integer D originates from a protected degeneracy in the transfer matrix spectrum for $\rho_{\partial A}^g$—a direct manifestation of the the mixed-state anomaly. When additional symmetries present, the degeneracy D can exceed 2. A notable case occurs when time-reversal symmetry is present, as in the Levin-Gu wavefunction discussed below, where we find $D=4$. In such case, the dominant eigenvectors of the symmetry twisted transfer matrix carry distinct $\mathbb{Z}_2$ quantum numbers. This leads to a second observable: a long-range correlation of $\mathbb{Z}_2$-charged operators at the boundary of the disorder operator [depicted in Fig. 1(b)]:

$$\langle O_jO_k\rangle_A^g:=\frac{\langle U_A(g)O_jO_k\rangle_\psi}{\langle U_A(g)\rangle_\psi}=\frac{\mathrm{Tr}(\rho_{\partial A}^g O_jO_k)}{\mathrm{Tr}(\rho_{\partial A}^g)}\sim O(1), \tag{3}$$

where $O_{j/k}$ are *o*dd under the $\mathbb{Z}_2$ symmetry, localized at ∂A but separated by a large distance from each other.

Example (Levin-Gu State)

Now, we demonstrate explicitly Eqs. (2) and (3) on the Levin-Gu (LG) state $|\Psi^{\mathrm{LG}}\rangle$, a fixed point wavefunction of the 2D $\mathbb{Z}_2$ SPT phase [6]. It is defined on a triangular lattice of Ising spins as a signed superposition of all spin configurations:

$$|\Psi^{\mathrm{LG}}\rangle=\frac{1}{2^{N/2}}\sum_{\text{all spin configs.}}(-1)^{N_{\mathrm{dw}}(c)}|c\rangle. \tag{4}$$

where $N_{\mathrm{dw}}(c)$ counts the number of domain walls in spin configuration c. The LG state can be generated by a finite depth unitary circuit, denoted as U^{LG}, acting on a direct product state $|\Omega^+\rangle$:

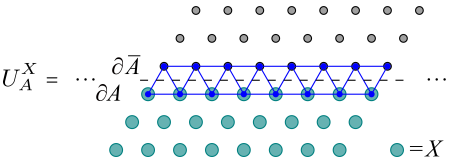
$$|\Psi^{\mathrm{LG}}\rangle=\underbrace{\prod_{f\in\text{faces}}\mathrm{CCZ}_f\prod_{e\in\text{edges}}\mathrm{CZ}_e\prod_{v\in\text{vertices}}Z_v}_{=:U^{\mathrm{LG}}}\underbrace{|+\rangle^{\otimes N}}_{=:|\Omega^+\rangle}. \tag{5}$$

Here, we have assigned a Pauli-Z to each vertex, control-Z to each edge and control-control-Z to each face. The global $\mathbb{Z}_2$ symmetry in question is generated by $U^X:=\prod_{v\in\text{vertices}}X_v$. Then the expectation value of the symmetry restricted on A, i.e. $U_A^X=\prod_{v\in A}X_v$ is given as follows

$$\langle U_A^X\rangle=\langle\Psi^{\mathrm{LG}}|U_A^X|\Psi^{\mathrm{LG}}\rangle=\langle\Omega^+|(U^{\mathrm{LG}})^\dagger U_A^X U^{\mathrm{LG}}|\Omega^+\rangle. \tag{6}$$

Since $U_A^X|\Omega^+\rangle=|\Omega^+\rangle$, the disorder parameter then equals the commutator between U_A^X and U^{LG}, whose non-trivial part is expected to be localized near the boundary.

Indeed, explicit calculation (presented in Section B of the SM) shows that the U_A^X for the lower half plane

$U_A^X=$  , (7)

has the following commutator with U^{LG}:

$$(U_A^X)^\dagger(U^{\mathrm{LG}})^\dagger U_A^X U^{\mathrm{LG}}=\prod_e\mathrm{CZ}_e\prod_v Z_v, \tag{8}$$

where the edges e and vertices v are taken near the boundary, denoted as the blue truss marked in the above lattice. Therefore, equation (6) reduces to the expectation value of a 1D operator. Explicit calculation yields

$$-\ln\langle U_A^X\rangle=\ln 2\cdot L_{\partial A}-\ln(4\cos(\pi L_{\partial A}/3)), \tag{9}$$

with $L_{\partial A}$ the length of the boundary. Compared with Eq. (2), we have $\theta=\pi/3$ and $D=4$. Note that $D>2$ is due to the presence of additional time-reversal symmetry.

As shown in Eq. (3), time-reversal symmetry induces an extra observable

$$\langle Z_jZ_k\rangle_A^g=\frac{1}{3}+\frac{2}{3}\cos\frac{2\pi(j-k)}{3}+\frac{2}{3}\tan\frac{\pi L_{\partial A}}{3}\sin\frac{2\pi(j-k)}{3}, \tag{10}$$

which matches the form in Eq. (3) (See Sections B and C of the SM).

We emphasize that although the fixed-point wavefunction provides a clear illustration, our central results in Eqs. (2) and (3) hold universally throughout the SPT phase. In the following sections, we support this claim with both a heuristic argument based on conformal field theory (CFT) and a more rigorous treatment using tensor network methods, demonstrating the generality of our approach beyond fixed-point models. Additionally, in Section D of the SM, we present explicitly calculations of disorder parameters for wavefunctions deformed from fixed-point, further confirming the generality of our results.

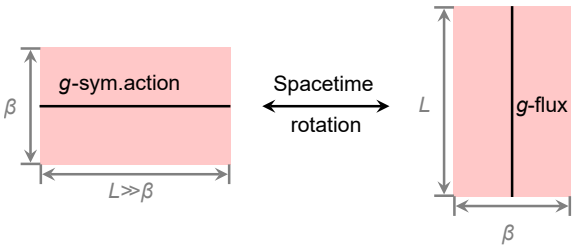
Heuristic CFT Argument

For short-range correlated SPT states, a heuristic but nontrivial assumption is that its RDM can be approximated by a high temperature conformal field theory, $\rho_A\propto\exp(-\beta H_{\mathrm{CFT}})$ [23,25]. More specifically, for the $\mathbb{Z}_2$ SPT in question, the representative entanglement Hamiltonian H_{CFT} has been identified as the free boson CFT [6,7,55]:

$$H_{\mathrm{CFT}}=\frac{1}{8\pi}\int_0^L\mathrm{d}x\left(R^2(\partial_x\phi)^2+\widetilde{R}^2(\partial_x\theta)^2\right). \tag{11}$$

Generalizing Cardy's argument [56], under modular transformation, the high temperature (i.e. $\beta\sim\Delta^{-1}\ll L$, with $L=L_{\partial A}$) partition

function with a symmetry twist is transformed to a low temperature one with a flux line inserted:



On the right hand side, the $\mathbb{Z}_2$-flux effectively generates an anti-periodic boundary condition for the Hamiltonian, denoted as $H_{\mathrm{CFT}}(g)$. For free boson theory in Eq. (11), $H_{\mathrm{CFT}}(g)$ has 4-fold ground state degeneracy [57,58]. So the disorder parameter yields

$$\langle U_A(g)\rangle \sim 4\exp(-LE_0(g))+\cdots, \tag{12}$$

which matches Eq. (2). $D=4$ here is consistent with the fact that equation (11) is time-reversal symmetric.

We remark here that lattice models are generally more complicated than their continuum field theory counterparts—particularly due to the absence of explicit space-time symmetry on the lattice. For instance, as demonstrated in Section E of the SM, UV-complete lattice realizations of the $\mathbb{Z}_2$ SPT edge can produce ground state degeneracies that differ from the field-theoretic predictions, thereby invalidates the heuristic argument. Given these subtleties, we now turn to a more rigorous treatment based on tensor network formalism that is native on lattice.

RDM as a 1D Anomalous Mixed State

As motivated earlier, the RDM ρ_A of a $\mathbb{Z}_2$ SPT state is isomorphic to a 1D mixed state $\rho_{\partial A}$ on the entanglement Hilbert space $\mathcal{H}_{\partial A}$. As detailed in Section A of the SM, the global symmetry $U(g)$ of the pure state induces a weak symmetry $U_A(g)$ on $\rho_{\partial A}$:

$$\rho_{\partial A} = W_{\partial A}(g)\cdot\rho_{\partial A}\cdot W^{\dagger}_{\partial A}(g). \tag{13}$$

Crucially, for the $\mathbb{Z}_2$ SPT phase, $W_{\partial A}(g)$ does not factorize into a product of on-site $\mathbb{Z}_2$ symmetry action, sharing the same anomaly as a physical edge [5,8,59–61]. This anomaly can be systematically captured by embedding the $\mathbb{Z}_2$ action within a larger, on-site $\mathbb{Z}_4$ symmetry in an extended Hilbert space, followed by a projection back to the effective entanglement space.

For illustration, we consider a minimal example that realizes the $\mathbb{Z}_2\to\mathbb{Z}_4$ extension. Let each site of $\mathcal{H}_{\partial A}$ be two qubits labeled as l and r. $\mathbb{Z}_4$ symmetry acts on site j as $W_j(g)=X^l_jX^r_j\exp[\mathrm{i}(\pi/4)(1-Z^l_jZ^r_j)]$. The effective entanglement Hilbert space is defined by the projector $P=\prod_j(1+Z^r_jZ^l_{j+1})/2$, in which $\mathbb{Z}_4$ symmetry becomes $\widetilde{W}(g)=(\prod_j\widetilde{X}_j)\cdot\exp[\mathrm{i}(\pi/4)\sum_j(1-\widetilde{Z}_j\widetilde{Z}_{j+1})]$. Here, $\widetilde{X}_j:=X^r_jX^l_{j+1}$, and $\widetilde{Z}_j:=Z^r_j=Z^l_{j+1}$ serving as the Pauli operators on the effective qubits. This procedure reproduces the anomalous $\mathbb{Z}_2$ symmetry for the edge theory of the Levin-Gu state [5,6]. Physically, the projector P defines a low energy space where the so-called 1D $\mathbb{Z}_4$ intrinsic gapless SPT is realized [62–64].

In more general settings, the local Hilbert space $\mathcal{H}_j$ at each site of the entanglement cut is enlarged. The effective entanglement space is defined by projector $P:=\prod_j(1+\lambda^r_j\lambda^l_{j+1})/2$ for commuting operators $\lambda^{l/r}$ satisfying $(\lambda^{l/r})^2=1$. This then imposes local constraints on $\rho_{\partial A}$:

$$\rho_{\partial A}=\lambda^r_j\lambda^l_{j+1}\cdot\rho_{\partial A}=\rho_{\partial A}\cdot\lambda^r_j\lambda^l_{j+1}\,. \tag{14}$$

Together with onsite weak $\mathbb{Z}_4$ symmetry $W_j(g)$, they satisfy the algebraic relations [60]:

$$\begin{aligned}&[W_j(g)]^2=D_j\equiv\lambda^l_j\cdot\lambda^r_j\,,\\ &W_j(g)\cdot\lambda^{l/r}_j=-\lambda^{l/r}_j\cdot W_j(g)\,.\end{aligned} \tag{15}$$

This algebraic structure is the hallmark of the $\mathbb{Z}_2$ SPT anomaly, which works beyond the fixed-point wavefunction.

This framework also provides generic PEPS construction for the $\mathbb{Z}_2$ SPT phase, offering a bulk interpretation of anyon condensation [60]. The strong $\mathbb{Z}_2$ symmetry (D_j) originates from a virtual-leg symmetry in PEPS tensors, corresponding to an m-anyon loop of the Toric Code topological order [65] (see Section A of the SM). For an open string $M\in\partial A$, the operator $W_M(g):=\prod_{j\in M}W_j(g)$ creates g-flux line terminating at g-defects. Equation (15) implies the fusion rule $[W_M(g)]^2=D_M$, indicating that two g-defects fuse into an m-anyon. When the decomposisition $D_j=\lambda^l_j\lambda^r_j$ is imposed with $\lambda^{l/r}$ carrying g-charge [from the second line of Eq. (15)], equation (14) reveals that the tensor wavefunction is invariant under $\lambda^l_1\cdot D_M\cdot\lambda^r_n$ on virtual legs. This signals a condensation of bound states of m-anyon and g-charge, resulting a $\mathbb{Z}_2$ SPT phase [66,67].

Having established the anomalous symmetry on the entanglement Hilbert space $\mathcal{H}_{\partial A}$, we are ready to study the mixed state $\rho_{\partial A}$ wherein. $\rho_{\partial A}$ hosts the following properties: (1) As $\rho_{\partial A}$ is the RDM of an SPT state, all correlators in $\rho_{\partial A}$ decay exponentially; (2) $\rho_{\partial A}$ possesses a weak $\mathbb{Z}_4$ symmetry generated by $W_{\partial A}(g)$, and a strong $\mathbb{Z}_2$ subgroup symmetry generated by $[W_{\partial A}(g)]^2=D_{\partial A}$; (3) The projection constraint (14) implies a string order parameter,

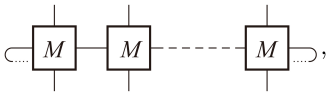
$$\lim_{|l|\to\infty}\mathrm{Tr}\left[\rho_{\partial A}\cdot\left(\lambda^r_1\cdot\left(\prod_{j=2}^{l}D_j\right)\cdot\lambda^l_{l+1}\right)\right]=1\,, \tag{16}$$

identified as a $\mathbb{Z}_2$ strong symmetry string decorated with double-$\mathbb{Z}_4$-charged endpoints. These properties identify $\rho_{\partial A}$ as a realization of the recent proposed 1D $\mathbb{Z}_4$ intrinsic average SPT (iASPT) phase [68]. This realization stands in contrast to previous studies, which suggest that iASPT phases require explicit disorder or decoherence.

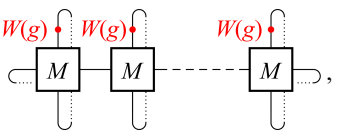
In Section F of the SM, we generalize this correspondence between 2D SPT phases and 1D intrinsic average SPT phases to any finite group using the mathematical framework of crossed module [69].

Observables from Anomalous RDM

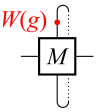
We now proceed to sketch (full details are shown in Section G of the SM) our derivation of the main results Eqs. (2) and (3) based on properties of $\rho_{\partial A}$. We assume $\rho_{\partial A}$ to be translation-invariant and representable as a Matrix Product Density Operator (MPDO):

$\rho_{\partial A}=\mathrm{Tr}\left[M^{L_{\partial A}}\right]=$  , (17)

where M denotes the local tensor of the MPDO. Hence, the disorder parameter reads

$\langle U_A(g)\rangle=\mathrm{Tr}(\underbrace{\rho_{\partial A}W_{\partial A}(g)}_{=:\rho^g_{\partial A}})=$  , (18)

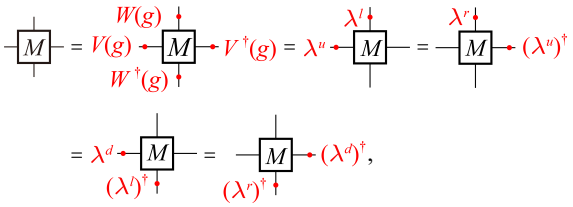
namely, the trace of the symmetry twisted density matrix $\rho^g_{\partial A}$ can be expressed by the symmetry-twisted transfer matrix

$T(g):=$ 

Consequently, $\langle U_A(g)\rangle$ is determined by the dominant eigenvalues of $T(g)$ when $L_{\partial A}\to\infty$.

Next, we show that the spectrum of $T(g)$ exhibits degeneracy due to the anomalous symmetry action, leading to the topological term in the disorder parameter. Symmetry constraints in Eqs. (13) and (14)

induce gauge transformations on the internal lines of M:

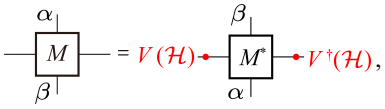


(19)

where algebraic relations between $V(g)$ and $\lambda^{u/d}$ follows Eq. (15):

$$[V(g)]^2 = \lambda^u\lambda^d\,, \quad [\lambda^{u/d}]^2 = 1,$$
$$V(g)\cdot\lambda^{u/d} = -\lambda^{u/d}\cdot V(g)\,. \tag{20}$$

We then impose Hermiticity condition $\rho_{\partial A} = \rho_{\partial A}^\dagger$, which can be viewed as an anti-unitary $\mathbb{Z}_2$ symmetry that interchanges the ket and bra space. Thus, it imposes an extra constraint on M:



(21)

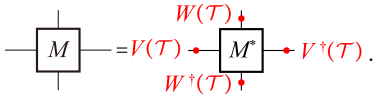
where $V(\mathcal{H})\mathcal{K}$ commutes with $V(g)$, and $V(\mathcal{H})\mathcal{K}\cdot\lambda^u\cdot(V(\mathcal{H})\mathcal{K})^\dagger = \lambda^d$. It is then straightforward to derive that $T(g)$ is invariant under anti-unitary operation $V(\mathcal{H})\mathcal{K}\lambda^u$. As $(V(\mathcal{H})\mathcal{K}\lambda^u)^2\cdot T(g) = \lambda^u\lambda^r\cdot T(g) = -T(g)$, spectrum of $T(g)$ are pairs of Kramers doublets, whose eigenvalues are conjugate to each other.

Denoting the dominant eigenvalues of $T(g)$ as $\exp(-\alpha\pm\mathrm{i}\theta)$, for large $L_{\partial A}$, the disorder parameter is then given by

$$-\ln\langle U_A(g)\rangle = \alpha\cdot L_{\partial A} - \ln 2 - \ln\cos(\theta L_{\partial A}) + \dots\,, \tag{22}$$

which contains the promised topological term beyond the conventional area law.

Next, we show that additional time-reversal symmetry $\mathcal{T}$ could further shift the topological term, as well as give rise to a spontaneous-symmetry-breaking–type long-range order. Time reversal symmetry imposes the following local tensor constraint:



(23)

Here, we consider the case where $W(\mathcal{T})\mathcal{K}$ commutes with $W(g)$, which also implies $V(\mathcal{T})\mathcal{K}$ and $V(g)$ commute.

From Eq. (20), we have $[V(g)]^2\cdot T(g) = -T(g)$. Therefore, for an arbitrary eigenstate of $T(g)$, denoted as $|\psi\rangle$, we have $V(g)|\psi\rangle = \pm\mathrm{i}|\psi\rangle$. As the anti-unitary symmetry $V(\mathcal{T})\mathcal{K}$ commutes with $V(g)$, $V(\mathcal{T})\mathcal{K}|\psi\rangle$ carries opposite $V(g)$ quantum number from $|\psi\rangle$ [70]. Therefore, with additional $\mathcal{T}$ symmetry, the spectrum of $T(g)$ is at least four fold degeneracy in terms of modulus, and can be partitioned into two pairs carrying opposite $V(g)$ quantum number $\pm\mathrm{i}$. Such degeneracy gives $D = 4$ in Eq. (2). These degenerate eigenstates with opposite $V(g)$ quantum number lead to further predictions as Eq. (3): for local operator O carrying g-charge, we have

$$\frac{\langle U_A(g)O_jO_k\rangle}{\langle U_A(g)\rangle} \xrightarrow{|j-k|\to\infty} O(1)\,, \tag{24}$$

where j,k are sites near ∂A [71].

Summary and Discussion

This work reveals a deep connection between topological phases of matter and lower-dimensional mixed-state quantum phases. By extending the bulk-boundary correspondence, we demonstrate that the RDM of a 2D SPT phase can be expressed as a 1D mixed state encoding the characteristic edge anomaly. This framework generalizes beyond the previous Lieb–Schultz–Mattis-type anomalies in mixed states [32,72] to encompass broader 't Hooft anomalies. Focus on the $\mathbb{Z}_2$ SPT phase, we show that the mixed-state anomaly produces measurable signatures including a robust topological term in the disorder parameter in Eq. (2) and long-range order in twisted mixed state in Eq. (3) when additional time-reversal symmetry is present. The underlying RDM formalism is general and provides a foundation for constructing similar diagnostic quantities for other SPT phases, including topological insulators [73], which we will explore in future work.

Looking forward, we plan to extend this framework to topological phases with anyonic excitations. In such systems, one-form anyon symmetries [74,75] manifests as strong symmetries within the 1D mixed state representation of the RDM, and deconfinement of anyons gives a strong-to-weak symmetry breaking [76,77] pattern in this RDM [78]. Moreover, since global symmetries are mapped to weak symmetry in the 1D RDM, nontrivial symmetry-enriched topological (SET) phases will induce an interplay between strong and weak symmetries. This interplay offers potential novel diagnostic signatures for SET orders.

The corresponding between topological phases and lower-dimensional mixed-state phases may also provide insight into fundamental questions in mixed-state physics—such as classification and detection of mixed-state phases [79–86], and phase transitions between mixed states [76,87,88].

Note Added

Upon completing this work, we became aware of a related paper by Sala, Pollmann, Oshikawa, and You [89] that constructs the 2D SPT fixed-point wavefunction using 1D mixed states, and relates bulk strange correlators to twisted Rényi correlators to characterize to the mixed-state topology.

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[70] In contrast, the anti-unitary symmetry $V(\mathcal{H})\mathcal{K}\lambda^u$ anti-commutes with $V(g)$, so $V(\mathcal{H})\mathcal{K}\lambda^u|\psi\rangle$ and $|\psi\rangle$ share the same $V(g)$ quantum number.
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