Spurious Strange Correlators in Symmetry-Protected Topological Phases

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Strange correlator is a powerful tool widely used in detecting symmetry-protected topological (SPT) phases. However, the result of strange correlator crucially relies on the adoption of the reference state. In this work, we report that an ill-chosen reference state can induce spurious long-range strange correlators in trivial SPT phases, leading to false positives in SPT diagnosis. Focusing on one-dimensional gapped bosonic/spin systems described by matrix product states, we trace the origin of these spurious signals in trivial SPT phases to the magnitude degeneracy of the transfer matrix. We systematically classify three distinct mechanisms responsible for such degeneracy, each substantiated by concrete examples: (1) the presence of high-dimensional irreducible representations (abbreviated as **irrep**) in the eigenspace corresponding to the entanglement spectrum (entanglement space); (2) a phase mismatch in symmetry representations between the target and reference states; and (3) long-range order arising from symmetry breaking. Our findings clarify the importance of the choice of proper reference states, providing a guideline to avoid pitfalls and correctly identify SPT order using strange correlators.

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1. Introduction. Symmetry-protected topological (SPT) phases in strongly interacting systems are short-range entangled (SRE) states that cannot be adiabatically connected to a trivial product state without breaking a specific protecting symmetry.[1] The Affleck–Kennedy–Lieb–Tasaki (AKLT) model serves as a paradigmatic example for understanding SPT phases.[2] Originally proposed as a solvable model for the spin-1 Haldane chain,[3] its ground state can be precisely represented by a matrix product state (MPS)[4–7] constructed from a valence-bond-solid picture.[2,8–11] This framework has been instrumental in elucidating hallmark features of one-dimensional (1D) SPT phases such as string order,[12–15] symmetry-protected edge modes,[2,7,16] hidden symmetries,[15,17–20] and characteristic entanglement spectra.[5,21]

A key challenge in the field is to develop mathematical and physical tools that efficiently distinguish trivial from nontrivial SPT phases, thereby enabling a systematic classification and characterization of SPTs. A non-exhaustive set of representative approaches along this line includes those based on group cohomology,[22] cobordism groups,[23,24] nonlinear sigma models,[25,26] topological field theories,[27–31] conformal field theories,[32–34] the decoration picture,[35] projective or parton constructions,[36–41] topological response and gauged theories,[42–48] and braiding statistics approaches.[49–52] Other frameworks have explored structural features such as low-dimensional edge excitations,[53] interacting systems,[54] and symmetry enrichment.[55] Among these, the **strange correlator**, originally proposed by You *et al.*,[56] has seen rapid development and widespread application.[56–66] Recent advances have extended this tool beyond standard SPTs to diagnose subsystem SPTs,[67], and even the higher-order topological phases generated by cellular automata using multi-point strange correlators.[66]

Defined as a correlation function between a target state $|\Phi\rangle$ and a chosen trivial reference state $|\Omega\rangle$, the strange correlator is given by

$$C(i,j) := \langle O_i^a O_j^b \rangle_S := \frac{\langle \Omega | O_i^a O_j^b | \Phi \rangle}{\langle \Omega | \Phi \rangle}, \tag{1}$$

where O_i^a and O_j^b are local operators. The subscript S emphasizes the distinction from conventional correlators where the bra and ket states are identical. The standard criterion posits that if $|\Phi\rangle$ belongs to a nontrivial SPT phase protected by symmetry group G (i.e., possessing stable edge modes[1]), the strange correlator should exhibit long-range behavior (power-law decay or saturation to a constant). Conversely, if the target state is a trivial SPT phase, the strange correlator is expected to decay exponentially or vanish.

While the strange correlator has achieved significant success, the rigorous criteria for constructing a valid setup—specifically the selection of the trivial reference state $|\Omega\rangle$—have not been systematically addressed. This ambiguity is critical because the conventional converse logic does not hold universally. In this work, we report the existence of **spurious strange correlators**: long-range

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correlations that emerge even when the target state is topologically trivial. A simple yet illustrating counterexample is the case of spin-1/2 trivial product states. Let $|0\rangle$, $|1\rangle$ be the computational basis of a single spin-1/2 Hilbert space and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, then two product states

$$\frac{\langle 0\dots 0|X_iX_j|+\dots+\rangle}{\langle 0\dots 0|+\dots+\rangle} = 1 \tag{2}$$

explicitly exhibit a long-range strange correlator, where $X_i := \sigma_i^x$ is the Pauli-X operator acting on i-th spin. Here, the ratio $\langle\Omega|X_iX_j|\Phi\rangle/\langle\Omega|\Phi\rangle$ equals unity regardless of distance,* exhibiting long-range behavior despite the absence of any SPT order.

This suggests that without a definitive selection rule for reference states, strange correlators may yield false positives. To the best of our knowledge, the mechanisms underlying such spurious behaviors and the guidelines for avoiding them have not been systematically explored. In this manuscript, we address this issue by providing a comprehensive analysis of **strange correlators in 1D gapped bosonic/spin systems**. By developing an MPS framework, we prove theorems establishing that the long-range behavior of the strange correlator can be attributed to the **magnitude degeneracy** of the largest eigenvalue of the transfer matrix (see Sec. 2). Using this framework, we demonstrate that while nontrivial SPT phases necessarily exhibit this degeneracy due to Schur's lemma, they are not the sole source. We systematically classify three distinct mechanisms responsible for spurious long-range correlations, each substantiated by concrete lattice models (see Sec. 3): (1) trivial phases involving high-dimensional irreducible representations, exemplified by the spin-2 AKLT model (first Example and Mechanism therein); (2) a phase mismatch between the symmetry representations of the reference and target states (second Example and Mechanism); and (3) long-range order arising from symmetry-breaking phases (third Example and Mechanism).

Our work provides a rigorous caveat for the application of strange correlators and offers analytic tools to unambiguously diagnose SPT phases by correctly selecting reference states. To preclude spurious signals from Mechanism 2 and Mechanism 3, the trivial reference state $|\Omega\rangle$ should be carefully chosen: it must (i) transform under the same 1D symmetry representation as the target state (preventing phase mismatch), and (ii) preserve the full global symmetry (distinguishing SPT order from symmetry-breaking long-range order). The spurious signals from Mechanism 1 are more intricate, and a more detailed operational guideline is presented in Remark 1 thereafter.

We focus our discussion on 1D systems with global symmetries; extensions to more exotic subsystem symmetries[66] are beyond the scope of this work.

2. Origin of Long-Range Strange Correlators. In this section, we aim to establish the analytical nature of the long-range strange correlator. We first write all correlators in the form of MPSs, then prove that the long-range behavior of the connected strange correlator can be attributed to what we name as magnitude degeneracy (explicit definition will be discussed below).

We introduce the **connected strange correlator**

$$\langle O_i^a O_j^b\rangle_{S,\text{connected}} := \langle O_i^a O_j^b\rangle_S - \langle O_i^a\rangle_S\langle O_j^b\rangle_S \tag{3}$$

to reflect the long-range behavior of the strange correlator, where $O_S := \langle\Omega|O|\Phi\rangle/\langle\Omega|\Phi\rangle$ is named strange expectation value (SEV). There are two reasons to do so, described below.

First, the existence of a long-range connected strange correlator guarantees the existence of operators yielding a long-range ordinary strange correlator [see Supplementary Materials (SM) Sec. IV C for proof for Theorem 1], while the reverse is not true [equation (2) is a typical example]. Formally, we prove that

Theorem 1. Given an MPS target state $|\Phi\rangle$ and a reference state $|\Omega\rangle$, if there exist local operators O^a and O^b such that the connected strange correlator exhibits long-range behavior in the thermodynamic limit, i.e.,

$$\lim_{|i-j|\to\infty}\langle O_i^a O_j^b\rangle_{S,\text{connected}} \neq 0, \tag{4}$$

then there exist (in fact, almost all) operators $\tilde{O}^a$ and $\tilde{O}^b$ such that the ordinary strange correlator also exhibits long-range behavior:

$$\lim_{|i-j|\to\infty}\langle \tilde{O}_i^a \tilde{O}_j^b\rangle_S \neq 0. \tag{5}$$

Second, it is the connected part of strange correlators that relates to the contribution from the nontrivial SPT states, as shown in SM Sec. I. In fact, in many cases discussed in the literature (e.g. AKLT models where $O_i^aO_j^b = S_i^+S_j^-$ and $|\Omega\rangle = |0\dots 0\rangle$, 1D cluster states with $O_i^aO_j^b = \sigma_i^z\sigma_j^z$, etc.), these two correlators share the same value since $\langle O^a\rangle_S = \langle O^b\rangle_S = 0$ in these cases.

Now we introduce the MPS formalism to simplify the discussion. This formalism provides a concise representation of 1D gapped states with arbitrary spins, and later we will see that the long-range behavior of the strange correlators is reflected in the transfer matrix in MPS formalism.

Suppose the translational-invariant target state defined on L spin-S systems is represented by MPS,

$$|\Phi\rangle = \sum_{i_1,\dots,i_L}\text{Tr}\,[M^{i_1}\dots M^{i_L}]|i_1,\dots,i_L\rangle, \tag{6}$$

where each $i_n = -S, -S+1,\dots,S-1,S$ reflects the physical dimension of the spin at each site. Here we have assumed the translational invariance and the periodic boundary condition.

The calculation of the strange correlator in MPS form reduces to calculating the trace of certain matrices,

$$\langle O_i^a O_{i+R+1}^b\rangle_S = \frac{\text{Tr}\,[M_a M_\omega^R M_b M_\omega^{L-R-2}]}{\text{Tr}\,[M_\omega^L]}. \tag{7}$$

*Strictly speaking, both the numerator and denominator vanish in the thermodynamic limit As clarified in Ref. [56], calculations are based on finite system sizes before taking the thermodynamic limit of the ratio, which remains finite.

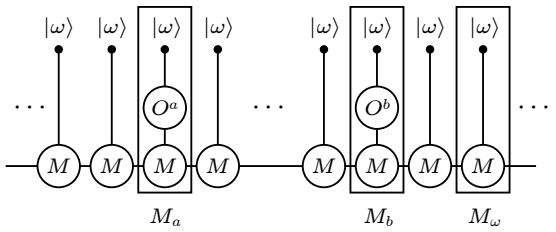

Fig. 1. The formulation of $\langle O^a O^b\rangle_S$ articulated using an MPS representation. The tensor denoted as M, depicted with three interconnected lines, corresponds to the M tensor within the MPS frameworkand is characterized by two horizontal virtual legs and a single vertical physical leg. The black dots are indicative of direct product states, each defined by an individual physical index. The interconnecting lines signify the process of index contraction. The definitions of M_a, M_b, and M_ω are each enclosed within rectangular boundaries.

where L is the length of the system and R is the distance between two operators. We denote the reference state as $|\Omega\rangle = |\omega\rangle \otimes ... \otimes |\omega\rangle$, and define $M_a := \sum_\sigma \langle\omega|O^a|\sigma\rangle M^\sigma$, $M_b := \sum_\sigma \langle\omega|O^b|\sigma\rangle M^\sigma$, and $M_\omega := \sum_\sigma \langle\omega|\sigma\rangle M^\sigma$. We refer to the matrices M_a and M_b as the **operator matrices**, and M_ω as the **transfer matrix**. These matrices are pictorially shown in Fig. 1. Similarly, we can calculate the SEV:

$$\langle O_i^a\rangle_S = \frac{\mathrm{Tr}\left[M_a M_\omega^{L-1}\right]}{\mathrm{Tr}\left[M_\omega^L\right]}. \tag{8}$$

In the MPS form developed above, we can easily observe that the long-range connected strange correlators originate from the magnitude degeneracy of the largest-magnitude eigenvalue of the transfer matrix. Here, we define:

Definition 1. (Magnitude-degeneracy) An eigenvalue λ_0 of a matrix M is said to be D-fold magnitude degenerate iff there exist $\lambda_1, \lambda_2, \ldots, \lambda_{D-1}$ such that they share the same magnitude, i.e., $|\lambda_0| = |\lambda_1| = |\lambda_2| = \ldots |\lambda|_{D-1}$.

Formally, we prove that:

Theorem 2. (Origin of long-range strange correlator). Suppose the eigenvalue of M_ω with the largest magnitude is D-fold magnitude degenerate, i.e., we assume that $|\lambda_0| = |\lambda_1| = \ldots = |\lambda_{D-1}| > |\lambda_D| > \ldots$.

If $D = 1$, then $\forall O^a, O^b$, the connected strange correlators decay exponentially to zero in the long-distance limit,

$$\langle O^a O^b\rangle_{S,\text{connected}} := \langle O^a O^b\rangle_S - \langle O^a\rangle_S \langle O^b\rangle_S = 0. \tag{9}$$

If $D > 1$, then there exist some operator O^a, O^b and an infinite sequence $\{R_i\} := R_1, R_2, \ldots$ where $j > i \iff R_j > R_i > 0$ and $\lim_{i\to\infty} R_i = \infty$, such that

$$\lim_{k\to\infty} \langle O_i^a O_{i+R_k+1}^b\rangle_{S,\text{connected}} \neq 0. \tag{10}$$

We see in Theorem 2 that, similar to the behavior of normal correlation functions in the MPS formalism, the behavior of strange correlators also depends significantly on the eigenvalue with the largest magnitude. The complete proof of the theorem is presented in SM Sec. IV B.

Finally, one may wonder whether it is hard to numerically search for O^a and O^b such that strange correlators exhibit long-range behavior when the largest-magnitude eigenvalue is magnitude degenerate, as described in Theorem 2. We claim that almost every local operator satisfies the requirement above. Formally, we prove that

Theorem 3. If $D > 1$, we define the direct sum of spaces of operator matrices M_a, M_b as

$$\mathcal{M} := \{(M_a, M_b) \,|\, M_a := \sum_\sigma \langle\omega|O^a|\sigma\rangle M^\sigma, \quad M_b := \sum_\sigma \langle\omega|O^b|\sigma\rangle M^\sigma\},$$

then $\langle O_i^a O_j^b\rangle_{S,\text{connected}}$ does not converge to zero (i.e., exhibits long-range behavior) almost everywhere in $\mathcal{M}$, where the exceptions have zero Lebesgue measure in $\mathcal{M}$.

Roughly speaking, this is proved by the fact that the measure of the zero set of a non-identically-zero analytic function is zero (Lemma 2 in the SM), the details of which are presented in Sec. IV C of the SM.

For an SPT state, the virtual bond in the MPS representation carries projective representation (abbreviated as rep) of the symmetry group G.[1] Namely, suppose the symmetry acts uniformly on the system as $u(g) \otimes u(g) \ldots$ on each site, where $u(g)$ is a unitary representation of $g \in G$ in the local Hilbert space, then

$$\sum_j [u(g)]_{ij} M^j = \alpha(g) V^{-1}(g) M^i V(g), \tag{11}$$

where $V(g)$ is a projective rep of G characterized by the invariant $\beta(G)$ being the factor system, and $\alpha(g)$ is a 1D linear rep of G. If the reference state $|\Omega\rangle = |\omega\rangle \otimes \ldots \otimes |\omega\rangle$ is a symmetric trivial state with $|\omega\rangle = \sum_i \omega_i |i\rangle$, $u(g)|\omega\rangle = \sum_{i,j} u_{ij}\omega_j |i\rangle = \mathrm{e}^{\mathrm{i}\theta(g)}|\omega\rangle$, then the transfer matrix M_ω vary as

$$V(g) M_\omega V^{-1}(g) = A(g) M_\omega, \tag{12}$$

with $A(g) = \alpha(g)\mathrm{e}^{-\mathrm{i}\theta(g)}$, a 1D linear rep of G. For finite groups (also for some compact Lie groups), there exist a finite integer N such that $[A(g)]^N = 1$ for all $g \in G$, then one has $V(g)[M_\omega]^N V^{-1}(g) = [M_\omega]^N$.

It can be shown that when the projective rep $V(g)$ belongs to a nontrivial class, the eigenvalues of M_ω are magnitude degenerate (Sec. II of the SM), giving rise to nonzero strange correlator.

3. Spurious Strange Correlators: Examples and Mechanisms. However, the magnitude degeneracy of eigenvalues can also appear in a trivial SPT phase, and so can the long-range strange correlator. This mechanism shares the same mathematical origin (Schur's lemma) as nontrivial SPT phases (Sec. II of the SM), but arises in a topologically trivial phase.

Example 1. ($SO(3)$ $D = 3$ irrep) Consider the ground state of spin-2 AKLT model. This state has $SO(3)$ symmetry and belongs to the trivial SPT phase. The MPS

form of spin-2 AKLT ground state is (see Sec. III of the SM for MPS form of AKLT states)

$$M_{-2}=\begin{pmatrix}0&0&\sqrt{3/5}\\0&0&0\\0&0&0\end{pmatrix},$$
$$M_{-1}=\begin{pmatrix}0&-\sqrt{3/10}&0\\0&0&\sqrt{3/10}\\0&0&0\end{pmatrix},$$
$$M_{0}=\begin{pmatrix}\sqrt{1/10}&0&0\\0&-\sqrt{2/5}&0\\0&0&\sqrt{1/10}\end{pmatrix},$$
$$M_{1}=\begin{pmatrix}0&0&0\\\sqrt{3/10}&0&0\\0&-\sqrt{3/10}&0\end{pmatrix},$$
$$M_{2}=\begin{pmatrix}0&0&0\\0&0&0\\\sqrt{3/5}&0&0\end{pmatrix}. \tag{13}$$

Under $SO(3)$, the matrix M_i transforms as Eq. (11), where $V(g)$ is a $D=3$ linear irrep of $SO(3)$ and $\alpha(g)=1$.

Since the system possesses $SO(3)$ invariance, we require the reference state to also exhibit $SO(3)$ invariance. However, the Hilbert space of a single spin cannot support a spin singlet. Yet, the direct-sum decomposition of the tensor product of two spins contains a spin-singlet component. Therefore, we consider two adjacent spins together, yielding the following $SO(3)$-invariant state: $|\omega\rangle=\frac{1}{\sqrt{5}}(|2,-2\rangle-|1,-1\rangle+|0,0\rangle-|-1,1\rangle+|-2,2\rangle)$, so that $A(g)=\alpha(g)\mathrm{e}^{-\mathrm{i}\theta(g)}=1$ in this case. Note that we have merged two sites into a new site to obtain an $SO(3)$-invariant reference state.

Here both the target state and the reference state are $SO(3)$-symmetric. Explicit calculation of the transfer matrix yields $M_\omega=\frac{1}{\sqrt{5}}\mathbb{I}_{3\times3}$, explicitly showing a 3-fold magnitude degeneracy. Here we choose $O^a=S^+\otimes I$ and $O^b=I\otimes S^-$, where I is the identity; then $\langle O^a\rangle_S=\langle O^b\rangle_S=0$ and

$$\langle O^aO^b\rangle_{S,\text{connected}}=\langle O^aO^b\rangle_S=-3 \tag{14}$$

explicitly showing spurious strange correlator.

Mechanism 1. ($D>1$ irrep) If the MPS represents a trivial SPT phase with $\beta=1$, then in this case the dimensionality of the irreducible linear reps of G can be either 1 or greater than 1. All irreps of Abelian groups are 1D, whereas non-Abelian groups can host higher-dimensional irreps. In this case, high-order magnitude degeneracy is not guaranteed in general. However, if the eigenspace corresponding to the maximal eigenvalue of $(M_\omega)^N$ happens to carry a high-dimensional irrep, then magnitude degeneracy may still be realized.

It is practical to clarify when we can avoid spurious signals from Mechanism 1 and how to numerically choose reference states. We provide a concise procedure below.

Remark 1. (Operational Guideline for Removing Spurious Signals from Mechanism 1) We propose an algebraic protocol for numerical implementation for choosing a reference state that avoids Mechanism 1. Given a target MPS in trivial phase with a virtual bond carrying high-dimensional irreps, one can first decompose its virtual space $\mathbb{V}$ (the representation space of $V(g)$) under the symmetry group G into irreducible sectors: $\mathbb{V}\cong\mathbb{V}_{\text{triv}}\oplus\mathbb{V}_{\text{high}}$, where $\mathbb{V}_{\text{triv}}$ denotes the 1D trivial sector (signal) and $\mathbb{V}_{\text{high}}$ collects all high-dimensional representations (noise). The validity of a reference state $|\Omega\rangle$ relies on the spectral gap of the transfer matrix: $|\lambda_{\text{triv}}|>|\lambda_{\text{high}}|$. An optimal reference state vector $\boldsymbol{\omega}$ can be constructed deterministically by maximizing the signal-to-noise ratio. Define the signal vector $\boldsymbol{t}=\mathrm{Tr}_{\mathbb{V}_{\text{triv}}}[M]$ and the noise covariance matrix $\boldsymbol{K}=\mathrm{Tr}_{\mathbb{V}_{\text{high}}}[M\otimes M^\dagger]$, acting on the physical space, where $\mathrm{Tr}_{\mathbb{V}_{\text{triv/high}}}[M]\equiv\mathrm{Tr}[P_{\text{triv/high}}M]$ with $P_{\text{triv/high}}$ being the projector onto the corresponding space. The optimal reference is given by the generalized inverse solution $\boldsymbol{\omega}_{\text{opt}}\propto\boldsymbol{K}^+\boldsymbol{t}$,[68] where $\boldsymbol{K}^+$ is the Moore–Penrose inverse of $\boldsymbol{K}$. If the single-site $\boldsymbol{\omega}_{\text{opt}}$ is insufficient to ensure $|\lambda_{\text{triv}}|>|\lambda_{\text{high}}|$, one can perform coarse-graining (blocking two or more sites) to enlarge the physical Hilbert space.

We emphasize that the success of this algorithm relies on the existence of at least one 1D irrep in the entanglement space. In the case of nontrivial SPT phases, where the spectrum of transfer matrix M_ω is enforced to be magnitude degenerate due to projective representations, such a unique scalar channel is fundamentally absent.

Now we show another mechanism that also leads to magnitude degeneracy. We provide a clean example below, devoid of any higher-dimensional irreps of $V(g)$, thus Mechanism 1 is not active. First we present following example:

Example 2. ($\mathbb{Z}_2$ phase mismatch) Consider an MPS with physical and virtual spins are both 1/2, leading to a bond dimension $D=2$. Now we suppose the MPS has a global $\mathbb{Z}_2$ spin-flip symmetry (uniform action of Pauli-X operator). This choice excludes the effect of the previously discussed higher-dimensional irreps of $V(g)$, since $H^2(\mathbb{Z}_2,U(1))=1$ and $\mathbb{Z}_2$ is Abelian. This means that $\mathbb{Z}_2$ symmetry cannot support a nontrivial factor system, and cannot admit higher-dimensional irrep due to its Abelian nature.

This symmetry constraint can be explicitly expressed as $XM_0=\alpha(g)ZM_0Z$ and $XM_1=\alpha(g)ZM_1Z$, where X and Z are Pauli-X and Pauli-Z operators, and $\alpha(g)=\pm1$, with g denoting the spin-flip action.

The general form of MPS satisfying such condition can be written as

$$M_0=\begin{pmatrix}u&v\\w&x\end{pmatrix},\quad M_1=\alpha(g)\begin{pmatrix}u&-v\\-w&x\end{pmatrix}, \tag{15}$$

where $u,v,w,x\in\mathbb{C}$.

Such a generally written MPS is always injective except at several fine-tuned points; therefore, we can always find a frustration-free parent Hamiltonian whose unique gapped ground state is the chosen MPS.[69] We choose the reference state as $|\Omega\rangle=\bigotimes_i|\omega\rangle$, where $|\omega\rangle=\omega_0|0\rangle+\omega_1|1\rangle$.

Define $\omega_\pm^*=\omega_0^*\pm\alpha(g)\omega_1^*$ with $*$ symbol denoting the

complex conjugate, then the transfer matrix is

$$M_\omega = \begin{pmatrix} u\omega_+^* & v\omega_-^* \\ w\omega_-^* & x\omega_+^* \end{pmatrix}. \tag{16}$$

Here, $\lambda_\pm = \frac{1}{2}(u+x)\omega_+^* \pm \frac{1}{2}[(u-x)\omega_+^{*2} + 4vw\omega_-^{*2}]^{1/2}$ are the eigenvalues of the tranfer matrix. It is obvious that if we choose $\omega_+^* = 0$, which means $\omega_0 = -\alpha(g)\omega_1$ and $X|\omega\rangle = -\alpha(g)|\omega\rangle$, then it follows that the eigenvalues $\lambda_\pm = \pm\omega_-^*\sqrt{vw}$, explicitly demonstrating the magnitude degeneracy regardless of the choice of u, v, w, and x (assuming $vw \neq 0$), thus causing a long-range connected strange correlator.

If we pick $O_i^a = Z_i$, $O_j^a = Z_j$, $\alpha(g) = 1$, and $|\omega\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)$, then in the thermodynamic limit (with system length L taken to be even to keep the correlator well-defined), the strange correlators reads

$$\langle Z_i Z_{i+R}\rangle_S = \langle Z_i Z_{i+R}\rangle_{S,\text{connected}} = \begin{cases} \frac{u^2+x^2}{2vw}, & R \text{ is odd}, \\ \frac{ux}{vw}, & R \text{ is even}, \end{cases} \tag{17}$$

following the definition of Eqs. (7)–(9).

For randomly selected u, v, w, x, such correlator is almost always nonzero.

The example above can be summarized as follows: the target state and the reference state share the same symmetry, but yield different phases under the symmetry action. We also present an example with parity symmetry.

Mechanism 2. (Phase Mismatch) Suppose the MPS target state has an onsite symmetry group G.

Supposing that the product rep $V(g) \otimes [V^{-1}(g)]^{\mathrm{T}}$ contains the 1D rep $A(g) = \alpha(g)\mathrm{e}^{-\mathrm{i}\theta(g)}$, and the reference state $|\Omega\rangle$ satisfies $g|\omega\rangle = \mathrm{e}^{\mathrm{i}\theta(g)}|\omega\rangle$, or equivalently $\sum_j u(g)_{ij}\omega_j = \mathrm{e}^{\mathrm{i}\theta(g)}R(g)^{-1}\omega_i R(g) = \mathrm{e}^{\mathrm{i}\theta(g)}\omega_i$ with $R(g)$ a 1D representation of G, then one has Eq. (12), namely $M_\omega V^{-1}(g) = A(g)V^{-1}(g)M_\omega$. If $A(g) \neq 1$ for some $g \in G$, then the transfer matrix can exhibit magnitude degeneracy even without the higher-dimensional irrep in $V(g)$. For any eigenvector $M_\omega|\lambda\rangle = \lambda|\lambda\rangle$, $V(g)^{-1}|\lambda\rangle$ is another eigenvector with the eigenvalue $\lambda A(g)$, having the same magnitude: $M_\omega V(g)^{-1}|\lambda\rangle = \lambda A(g)V(g)^{-1}|\lambda\rangle$.

This argument can also be applied to the case of parity symmetry, where the parity transformation is $P = w\otimes w\otimes\ldots\otimes w P_1$, with w being the on-site unitary transformation satisfying $w^2 = 1$ and P_1 being the site-exchange operator. See Example 4 in SM Sec. V. There are also cases where different mechanisms can act simultaneously. Example 5 in SM Sec. V shows the case where Mechanism 1 and Mechanism 2 act simultaneously.

Two mechanisms above have assumed that the MPS target state is the unique symmetric ground state of the system Hamiltonian, i.e., no symmetry is broken. Now we examine the case where the symmetry is broken.

Example 3. (GHZ state) As a concrete example, we consider the 1D Ising model with nearest neighbor ZZ coupling. We choose $|\Phi\rangle = \frac{1}{\sqrt{2}}(|0^{\otimes N}\rangle + |1^{\otimes N}\rangle)$ and $|\Omega\rangle = |+^{\otimes N}\rangle$, and $O_i^a O_j^b = Z_i Z_j$. Now we check explicitly that

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{18}$$

thus $M_\omega = \mathrm{diag}(2^{-1/2}, 2^{-1/2})$, exhibiting magnitude degeneracy.

It follows that

$$\langle Z_i Z_j\rangle_S = \frac{\langle +^{\otimes N}|Z_i Z_j|\Phi\rangle}{\langle +^{\otimes N}|\Phi\rangle} = \frac{2^{-N/2} + 2^{-N/2}}{2^{-N/2} + 2^{-N/2}} = 1 \tag{19}$$

and

$$\langle Z_i\rangle_S = \frac{\langle +^{\otimes N}|Z_i|\Phi\rangle}{\langle +^{\otimes N}|\Phi\rangle} = \frac{2^{-N/2} - 2^{-N/2}}{2^{-N/2} + 2^{-N/2}} = 0, \tag{20}$$

exhibit the long-range strange correlator

$$\langle Z_i Z_j\rangle_{S,\text{connected}} = \langle Z_i Z_j\rangle_S = 1. \tag{21}$$

While a GHZ state is not classified as a nontrivial SPT state, it is indeed highly entangled. We see explicitly here that strange correlators also give a long-range response to such entanglement structure.

Mechanism 3. (Symmetry breaking) In previous sections, we have assumed that the MPS target state is the unique symmetric ground state of the system. In fact, in the symmetry-breaking case, there are also mechanisms that lead to magnitude degeneracy of the transfer matrix and thus to long-range strange correlators.

The origin of the long-range strange correlators can be explained as follows. If the system is in a symmetry-breaking phase, the unique symmetric ground state of the system—some linear combinations of ground state basis (e.g., the GHZ state $|0\ldots0\rangle + |1\ldots1\rangle$ is a symmetric ground state of 1D Ising model with $H = \sum_i Z_i Z_{i+1}$, which preserves the global spin-flip symmetry)—will have a block-diagonal MPS form:[1] $M_i = \mathrm{diag}(M_i^{(0)}, M_i^{(1)}, \ldots)$, with the number of blocks greater than 1 and the double tensor $\mathbb{E}^{(k)} = \sum_i M_i^{(k)} \otimes M_i^{(k)*}$ for each block sharing the same largest eigenvalue. We claim that one can always choose a symmetric reference state such that the transfer matrix exhibits magnitude degeneracy, where $M_\omega = M_i\omega_i^* = \mathrm{diag}(M_i^{(0)}\omega_i^*, M_i^{(1)}\omega_i^*, \ldots)$, as proved in SM Sec. IV D. In Example 3, the reference state is chosen as $|+\ldots+\rangle$, being symmetric under the spin-flip symmetry (i.e., global Pauli-X action).

In this scenario, the magnitude degeneracy comes from different blocksof the MPS tensor, rather than from high-dimensional irreps within a single block. The mechanisms discussed above can exist simultaneously, and only by careful selection of the reference state can one use strange correlators to correctly detect nontrivial SPT phases.

4. Discussion and Outlook. The strange correlator is a widely used probe for SPT order, but its outcome depends crucially on the choice of reference state. Within the MPS framework, we show that a long-range strange correlator stems directly from a magnitude degeneracy in the transfer matrix's leading eigenvalues. To avoid false positives, a valid reference state must not only be trivial, but also: (i) preserve the full global symmetry, (ii) match

the 1D symmetry representation of the target state (e.g., carry the same parity), and (iii) be chosen to ensure that high-dimensional irreps in the entanglement space do not lead to the dominant eigenvalues of the transfer matrix.

This finding provides a practical guide for numerical studies like quantum Monte Carlo or exact diagonalization, where the ground state is not known in MPS form. For these cases, we propose an "inverse scanning" strategy. The key idea is that the strange correlator probes the relative SPT invariant between the target and reference states. For a true SPT phase, the virtual space carries a projective representation (like a half-integer spin), a property that cannot be lifted by any reference state transforming in a linear representation (like an integer spin). The resulting eigenvalue degeneracy of the transfer matrix is therefore robust. In contrast, the spurious signal in a trivial phase (like the spin-2 AKLT model) stems from high-dimensional linear representations that are not topologically protected and are therefore fragile. In a trivial phase, generic symmetric perturbations will mix the scalar representation (a 1D irrep) into the virtual space. This effect can be captured by a well-chosen reference state (see Remark 1). In practice, one can test the target state against a family of symmetric reference states, perhaps using coarse-graining (blocking sites) to explore a larger space of trivial states. If the long-range correlation is robust across the entire family of references, it is a strong indicator of genuine SPT order. Conversely, if the signal is fragile—meaning a specific reference state can be found that results in short-range decay—the target state is identified as trivial.

We also comment that the spurious signal in Mechanism 1 has a direct physical interpretation in terms of "representation SPT" (RSPT) phases.[70] An RSPT phase has an MPS whose virtual bonds carry a high-dimensional linear irrep of the symmetry group (e.g., the 3D irrep of $SO(3)$ in Example 1). This makes it distinct from a simple product state, even though they belong to the same trivial group cohomology class. A deeper study of this connection is left for future work.

We can also relate these findings to the entanglement spectrum (ES). For a nontrivial SPT phase, the degeneracies in both the ES and the strange correlator are protected. For a trivial phase with a non-Abelian symmetry, the ES can still be degenerate, but this degeneracy can be lifted by a boundary phase transition without closing the bulk gap. The same is true for the strange correlator. However, a non-vanishing strange correlator does not necessarily imply an ES degeneracy of the maximal entanglement weight. For instance, in Mechanism 2, a trivial phase with an Abelian symmetry can have a non-degenerate ES, but still produce a long-range strange correlator if the reference state is chosen poorly.

Finally, our rigorous results are based on the exact contractibility of 1D bosonic MPS. Extending this analysis to higher dimensions (e.g., using PEPS) requires care, as approximate contraction schemes can obscure the clean spectral properties of the transfer matrix. For fermionic systems, one must use graded tensor networks and ensure the reference state correctly matches the target's fermion parity structure to avoid spurious cancellations. These extensions present challenges for future work.

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