Superadiabatic Approach to Quantum Geometric Transport

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We introduce a nonadiabatic approach to study nonlinear transport phenomena by combining quantum evolutions of superadiabatic states and semiclassical transport theories. Different from conventional adiabatic-switching methods, our approach is capable of avoiding numerical divergence and unveiling underlying quantum geometric contributions simultaneously. Based on the superadiabatic approach, we find that the intrinsic nonlinear conductance is determined by the Berry connection polarizability on the Fermi surface. Incorporation of numerical results shows that our approach reproduces qualitative nonlinear transport in $\mathcal{PT}$-symmetric antiferromagnets.

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Quantum geometry, the relative phase and amplitude distance between eigenstates in a given parameter space, has received continuous attention over the past decades, not only for its role in topological descriptions in band theory but also for its crucial role in determining novel interactions and quantum transports.[1–3] Among novel quantum transport phenomena, the quantum Hall effect has been considered a paradigmatic example. The quantized Hall conductance is protected by a topological invariant (Chern number) derived from the Berry curvature—the imaginary part of the quantum geometric tensor. As for the real part of the quantum geometric tensor, coined the quantum metric, it has been recognized as a key descriptor in a variety of phenomena, such as superfluid weight,[4,5] orbital magnetism,[6,7] and quantum noise.[8]

In recent years, nonlinear transport phenomena have arisen as a central topic in quantum geometry.[9] In several noncentrosymmetric materials, a transverse current quadratic in an applied electric field can emerge even in the absence of magnetic fields, known as the nonlinear anomalous Hall effect (NAHE). Early works have identified several competing extrinsic mechanisms in the NAHE, including the classical nonlinear Drude weight, the Berry curvature dipole (BCD), and other extrinsic processes.[9–12] The distinct intrinsic NAHE mechanism has also been recognized, arising not from the Berry curvature but from the Berry connection polarizability (BCP), which is the band-normalized quantum metric.[6,7] The theories have been verified in $\mathcal{PT}$-symmetric antiferromagnets, where the BCD is symmetry-forbidden, offering a clear experimental window to access the BCP contribution to the NAHE.

While quantum geometry plays a critical role in the NAHE, conventional frameworks such as the Kubo formalism[13] and adiabatic Thouless pumping[14,15] fail to capture essential physics due to the limitations of the adiabatic switching assumption. This oversight hinders the accurate description of nonadiabatic phenomena. Thus, there is an urgent need for conceptually intuitive and computationally efficient methods. In this work, we develop a superadiabatic approach inspired by Berry's pioneering works.[16,17] The superadiabatic formalism is fully compatible with semiclassical transport theories and incorporates nonadiabatic effects in a self-consistent manner, providing an improved basis for studying quantum geometric transport.

The superadiabatic approach is introduced as a nonadiabatic extension of the adiabatic theorem by constructing a time-dependent basis in which the propagator becomes asymptotically block-diagonal. The superadiabatic states are obtained through systematic iterations and remain invariant under time evolution up to beyond-all-orders corrections, allowing the superadiabatic states to naturally retain the properties of the instantaneous adiabatic eigenstates. This framework is particularly useful for evaluating observables such as intrinsic currents.

For simplicity, we consider a system with a given Hamiltonian $\hat{\mathcal{H}}(t)$, which has a finite matrix representation and is time-dependent under external drive. This Hamiltonian admits a spectral decomposition

$$\hat{\mathcal{H}}(t) = \sum_n \mathcal{E}_n(t)\hat{P}_n(t), \tag{1}$$

where $\mathcal{E}_n(t)$ are the instantaneous band energies and $\hat{P}_n(t) = |u_n(t)\rangle\langle u_n(t)|$ is the projector onto the n-th nondegenerate eigenstate $|u_n(t)\rangle$ (for degenerate bands, $\hat{P}_n$ projects onto the corresponding eigensubspace). To suppress nonadiabatic transitions, we assume a sufficiently

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smooth time dependence and a lower bound of the energy differences

$$\Delta\mathcal{E} = \min_{\substack{t_0\le t\le t_1\\ m\neq n}} |\mathcal{E}_m(t) - \mathcal{E}_n(t)| > 0 \tag{2}$$

over the time interval of interest. In this system, the evolution of a quantum state is governed by the time-dependent Schrödinger equation, in which the Hamiltonian $\hat{\mathcal{H}}$, the reduced Planck constant $\hbar$, and time t possess physical dimensions. By nondimensionalizing these quantities, we introduce the adiabatic parameter ϵ given by $\epsilon = \hbar\omega_{\rm ex}/\Delta\mathcal{E}$, where $\omega_{\rm ex}$ is the external driving frequency (or the Bloch oscillation frequency in crystals). Then one can recast the equation as

$$\mathrm{i}\epsilon\frac{\partial\hat{\mathcal{U}}(t,t_0)}{\partial t} = \hat{\mathcal{H}}(t)\hat{\mathcal{U}}(t,t_0), \tag{3}$$

where both t and $\hat{\mathcal{H}}(t)$ are dimensionless. The propagator $\hat{\mathcal{U}}(t,t_0)$ depends on the adiabatic parameter ϵ, the initial time t_0, and the final time t. In the diabatic limit $\epsilon\to\infty$, it admits an analytic expansion given by the Dyson series.[18] In contrast, the expansion for $\epsilon\to 0$ is non-analytic due to rapidly oscillating phases. Within the superadiabatic framework, we derive that $\hat{\mathcal{U}}(t,t_0)$ admits an asymptotic expansion,

$$\hat{\mathcal{U}}(t,t_0) = \sum_n \mathrm{e}^{\int_{t_0}^{t}\frac{\mathrm{d}t'}{\mathrm{i}\epsilon}\mathcal{E}_n(t')}\hat{U}_{n\epsilon}(t,t_0). \tag{4}$$

Here $\hat{U}_{n\epsilon}(t,t_0) = \sum_l \epsilon^l \hat{U}_n^{(l)}(t,t_0)$ is the related subband propagator. The zeroth-order term

$$\hat{U}_{n,\epsilon=0} = \hat{U}_n^{(0)} = |u_n(t)\rangle\langle u_n(t_0)| \tag{5}$$

denotes the two-time adiabatic propagator obtained from the Berry phase of the n-th band, which leaves the eigenstate invariant during evolution, as a conclusion of the adiabatic theorem. Higher-order terms $\hat{U}_n^{(l)}$ account for nonadiabatic corrections due to interband transitions. We can construct a systematic iteration to obtain the superadiabatic propagator at each order l. The detailed treatment of this systematic iteration is given in the Supplementary Materials (SM). The iterative relation reads

$$\begin{aligned}\hat{U}_n^{(l)}(t,t_0) &= -\mathrm{i}\hat{G}_n(t)\partial_t\hat{U}_n^{(l-1)}(t,t_0)\\ &+ \mathrm{i}\hat{U}_n^{(0)}(t,t_0)\sum_m \hat{G}_m(t_0)\frac{\partial\hat{U}_n^{(l-1)}(t=t_0,t_0)}{\partial t}\\ &- \mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\hat{U}_n^{(0)}(t,t')\frac{\partial\hat{P}_n(t')}{\partial t'}\hat{G}_n(t')\frac{\partial\hat{U}_n^{(l-1)}(t',t_0)}{\partial t'}.\end{aligned} \tag{6}$$

Here, $\hat{G}_n = \sum_{m\neq n}\hat{P}_m/(\mathcal{E}_n - \mathcal{E}_m)$ denotes the principal value of the matrix Green function.

Based on the group property of the propagator $\hat{\mathcal{U}}(t,t_0)$, one can define an orthonormal set of projectors $\hat{P}_{n\epsilon}(t) = \hat{U}_{n\epsilon}(t,t)$, which makes the full propagator $\hat{\mathcal{U}}(t,t')$ diagonal in the superadiabatic basis:

$$\hat{P}_{m\epsilon}(t)\hat{\mathcal{U}}(t,t')\hat{P}_{n\epsilon}(t') = \delta_{mn}\mathrm{e}^{\int_{t_0}^{t}\frac{\mathrm{d}t'}{\mathrm{i}\epsilon}\mathcal{E}_n(t')}\hat{U}_{n\epsilon}(t,t'). \tag{7}$$

This asymptotic relation generalizes the adiabatic theorem. In the nondegenerate case, the associated projector $\hat{P}_{n\epsilon}(t)$ is rank-one and takes the form $\hat{P}_{n\epsilon}(t) = |u_{n\epsilon}(t)\rangle\langle u_{n\epsilon}(t)|$, where $|u_{n\epsilon}(t)\rangle$ denotes the superadiabatic states. These states generalize the instantaneous adiabatic eigenstates by incorporating finite-ϵ corrections and are asymptotically invariant under the evolution, in the sense that the propagator satisfies

$$\hat{U}_{n\epsilon}(t,t_0) = |u_{n\epsilon}(t)\rangle\langle u_{n\epsilon}(t_0)|. \tag{8}$$

Next, we apply our superadiabatic framework to describe the dynamics of electrons in crystalline solids driven by external electric fields. This framework enables a systematic treatment of nonadiabatic effects beyond the conventional adiabatic approximation, which is crucial for accurately capturing nonlinear transport phenomena such as the NAHE.

We start from a crystalline system described by a tight-binding Hamiltonian

$$(\hat{H}_{\boldsymbol{k}})_{ab} = \sum_{\boldsymbol{R}} h_{ab}(\boldsymbol{R})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{R}+\boldsymbol{r}_b-\boldsymbol{r}_a)}, \tag{9}$$

where $\boldsymbol{k}$ is the crystal momentum within the Brillouin zone, a and b label the combined orbital and spin degrees of freedom, and $\boldsymbol{R}$ runs over all Bravais lattice vectors. The diagonal elements of $h_{ab}(\boldsymbol{R})$ represent corresponding onsite energies, while the off-diagonal elements are hopping amplitudes between orbitals. In the context of electric transport, an applied electric field in the velocity gauge induces a shift of crystal momentum $\boldsymbol{k}(t) = \boldsymbol{k} + \frac{e}{\hbar}\boldsymbol{A}(t)$, leading to a time-dependent Hamiltonian $\hat{\mathcal{H}}(t) = \hat{H}_{\boldsymbol{k}(t)}$.

In conventional treatments, handling a dc field can be problematic. A common treatment is adiabatic switching,[19,20] which replaces the static field by a slowly turned-on drive,

$$\boldsymbol{E}(t) = \lim_{\eta\to 0^+}\mathrm{Re}[\boldsymbol{E}\mathrm{e}^{\eta t}]. \tag{10}$$

The infinitesimal parameter η ensures a well-defined interaction picture from $t\to-\infty$ to finite t, making causality explicit. However, this scheme produces an unphysical divergence in the dc conductivity,

$$\sigma = -\frac{e^2}{\hbar}\int_{\boldsymbol{k}}\frac{\nabla_{\boldsymbol{k}}f_{\boldsymbol{k}}}{\eta}\nabla_{\boldsymbol{k}}\mathcal{E}_{\boldsymbol{k}} \propto \eta^{-1}\to\infty. \tag{11}$$

This divergence is commonly regularized by identifying η^{-1} with the relaxation time τ. Such a prescription is purely phenomenological and contradicts the intrinsic transport limit, in which τ is expected to approach 0 rather than ∞. This divergence stems from the semi-infinite evolution time interval, leading to an unphysical dependence on the specification of the initial momentum $\boldsymbol{k}_{-\infty}$ versus the final momentum $\boldsymbol{k}_0$.

In the superadiabatic framework, the evolution is local in time and tied to the instantaneous Hamiltonian, so one works in the neighborhood of time t rather than over a semi-infinite time interval. In our approach, we cast the band Hamiltonian into the general time-dependent framework discussed above, which admits superadiabatic states

$|u_{n\epsilon}\rangle$ that depend on t, $\boldsymbol{k}$, and $\boldsymbol{E}$. For dc transport, the explicit time dependence enters only through $\boldsymbol{k}(t)$, so the superadiabatic states can be written as $|u_{n\epsilon}(\boldsymbol{k}(t),\boldsymbol{E})\rangle$, emphasizing locality with respect to instantaneous data rather than causality conditions.

The semiclassical dynamics of electrons are governed by Hamilton's equations, which require explicit forms of the velocity and force. In our approach, the force is provided by the external field, while the velocity is obtained as the expectation value of the velocity operator in the superadiabatic states. The semiclassical velocity in the n-th band is given by

$$\boldsymbol{v}_{n\epsilon} = \mathrm{tr}\left[\hat{P}_{n\epsilon}(\boldsymbol{k},\boldsymbol{E})\frac{\nabla_{\boldsymbol{k}}\hat{H}_{\boldsymbol{k}}}{\hbar}\right], \tag{12}$$

where $\hat{P}_{n\epsilon}$ depends on the instantaneous crystal momentum $\boldsymbol{k} = \boldsymbol{k}(t)$ and the electric field $\boldsymbol{E}$. This expression constitutes an effective theory in which Zitterbewegung contributions are suppressed in superadiabatic states, removing rapid quantum oscillations and leading to semiclassical dynamics governed by quantum geometry. Using the iterative relation in Eq. (6), we can compute the semiclassical velocity to arbitrary order in $\boldsymbol{E}$. We introduce the BCP tensor

$$G^n_{\alpha\beta} = \sum_{m\neq n}\frac{\mathcal{A}^{mn}_{\alpha}\mathcal{A}^{nm}_{\beta} + (\beta\leftrightarrow\alpha)}{\omega_{nm}}, \tag{13}$$

where $\mathcal{A}^{mn}_{\alpha} = \mathrm{i}\langle u_{m\boldsymbol{k}}|\partial_\alpha|u_{n\boldsymbol{k}}\rangle$ is the interband Berry connection of the eigenstates $|u_{n\boldsymbol{k}}\rangle$, and $\omega_{nm} = \mathcal{E}_n - \mathcal{E}_m$ is the interband frequency. We find that the velocity up to second order is governed by the dispersion, the Berry curvature, and the BCP, written as (see SM for details)

$$\begin{aligned} v_{n\mu} = {} & \frac{\mathcal{E}_{n,\mu}}{\hbar} - \frac{e}{\hbar}\Omega^n_{\mu\alpha}E^\alpha \\ & + \frac{e^2}{\hbar}\frac{G^n_{\alpha\beta,\mu} - G^n_{\alpha\mu,\beta} - G^n_{\beta\mu,\alpha}}{2}E^\alpha E^\beta + o(E^2), \end{aligned} \tag{14}$$

where $G^n_{\alpha\beta,\mu} = \partial_\mu G^n_{\alpha\beta}$ denotes the BCP dipole, and $\Omega^n_{\mu\alpha}$ is the Berry curvature.

A complete transport theory requires not only a semiclassical description of single-particle dynamics but also an underlying statistical framework, which in electronic systems is provided by the Boltzmann equation. Under the relaxation time approximation, it takes the form

$$\frac{\partial f_n}{\partial t} - \frac{e\boldsymbol{E}}{\hbar}\cdot\nabla_{\boldsymbol{k}} f_n = \frac{f^{(0)}_n - f_n}{\tau}, \tag{15}$$

where τ is the relaxation time, f_n is the distribution function of the n-th band, and $f^{(0)}_n = (1+\mathrm{e}^{\frac{\mathcal{E}_n-\mu}{k_\mathrm{B}T}})^{-1}$ is the equilibrium Fermi–Dirac distribution. For intrinsic transport, we focus on the regime where the steady-state distribution is arbitrarily close to equilibrium, so that $f_n \to f^{(0)}_n$, and the nonlinear response arises from the field-induced polarization of observables rather than from distortions of the distribution. With Eq. (14), we can compute the intrinsic electric current density in the steady state, written as

$$\boldsymbol{J} = -e\sum_{n\boldsymbol{k}} f^{(0)}_n(\boldsymbol{k})\boldsymbol{v}_{n\epsilon}(\boldsymbol{k},\boldsymbol{E}), \tag{16}$$

where $\sum_{\boldsymbol{k}} = \int\frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}$ is the summation over the Brillouin zone. We mainly focus on the intrinsic second-order nonlinear conductivities, which are given by

$$\sigma^{(2,\mathrm{int})}_{\mu\alpha\beta} = -\frac{e^3}{\hbar}\sum_{n\boldsymbol{k}} f^{(0)}_n\frac{G^n_{\alpha\beta,\mu} - G^n_{\alpha\mu,\beta} - G^n_{\beta\mu,\alpha}}{2}. \tag{17}$$

The Ohmic and Hall components can be obtained via index symmetrization as

$$\sigma^{(\mathrm{INO})}_{\mu\alpha\beta} = \frac{e^3}{6\hbar}\sum_{n\boldsymbol{k}} f^{(0)}_n(G^n_{\alpha\mu,\beta} + G^n_{\beta\mu,\alpha} + G^n_{\alpha\beta,\mu}), \tag{18}$$

$$\sigma^{(\mathrm{INH})}_{\mu\alpha\beta} = \frac{e^3}{3\hbar}\sum_{n\boldsymbol{k}} f^{(0)}_n(G^n_{\alpha\mu,\beta} + G^n_{\beta\mu,\alpha} - 2G^n_{\alpha\beta,\mu}). \tag{19}$$

These results are consistent with previous studies,[6,10,19] albeit with different coefficients. Unlike in the linear-response regime, the nonlinear conductivity is highly sensitive to the choice of derivation pathway and theoretical framework, a feature that remains conceptually intriguing. As a recently developed framework, the superadiabatic approach offers an alternative method to study nonlinear conductivity in the presence of nonadiabatic effects. For extrinsic transport, we can likewise obtain the corresponding contributions, namely the BCD and the nonlinear Drude weight, which can be written as

$$\sigma^{(2,\mathrm{BCD})}_{\mu\alpha\beta} = -\tau\frac{e^3}{2\hbar}\sum_{n\boldsymbol{k}} f^{(0)}_n(\Omega^n_{\alpha\mu,\beta} + \Omega^n_{\beta\mu,\alpha}) \tag{20}$$

and

$$\sigma^{(2,\mathrm{Drude})}_{\mu\alpha\beta} = -\tau^2\frac{e^3}{\hbar^3}\sum_{n\boldsymbol{k}} f^{(0)}_n\mathcal{E}_{n,\mu\alpha\beta}. \tag{21}$$

The full derivation is given in the SM. To show the generality of the superadiabatic approach, we manage to obtain the intrinsic third-order current for a two-band model as follows:

$$\begin{aligned} J^{(3)}_\mu = {} & \frac{e^4}{\hbar}\sum_{\boldsymbol{k}}(f^{(0)}_0 - f^{(0)}_1)\Big\{\frac{\ddot{N}\Omega_{\mu t} - N\ddot{\Omega}_{\mu t}}{2} \\ & + \frac{3}{2}\partial_t(N\mathcal{C}_{\mu tt}) - \frac{1}{2}\partial_\mu(N\mathcal{C}_{ttt}) \\ & + \frac{5}{2}\mathrm{i}N\mathrm{tr}\Big(4\hat{P}\Big[\partial_\mu\hat{P},\dot{P}^3\Big] - \dot{\hat{P}}\Big[\partial_\mu\hat{P},\ddot{\hat{P}}\Big]\Big)\Big\}, \end{aligned} \tag{22}$$

where $\hat{P} = \hat{P}_0$ is the projector onto the lower band, $N = (\mathcal{E}_1 - \mathcal{E}_0)^{-2}$, and the tensor $\mathcal{C}_{\mu\alpha\beta} = \mathrm{i}\,\mathrm{tr}(\hat{P}[\partial_\mu\hat{P},\partial_\alpha\partial_\beta\hat{P}])$ denotes the quantum geometric connection.[21] Here, $\partial_t = -E^\alpha\partial_\alpha$ is the directional derivative along the electric field, and the index t indicates contractions with $\boldsymbol{E}$ accordingly.

Since NAHE arises only in specific materials, it is essential to delineate the constraints. For intrinsic contributions, the material must be metallic, noncentrosymmetric, and magnetic, while the $\mathcal{PT}$ symmetry ensures the suppression of BCD contributions. Magnetic point group (MPG) symmetries must also be considered, as they impose additional constraints on the conductivity tensor. The MPG symmetry classification is vast and leads to many distinct tensor patterns; therefore, instead of enumerating all possibilities, we select a few representative

cases that capture the salient mechanisms of nonlinear conductivity. For each case, we employ a minimal model that retains the essential features of the band geometry.

We focus on symmetry breaking at Dirac cones, in particular inversion-symmetry breaking, which can arise from two generic mechanisms: band tilting and threefold-rotational anisotropy. The former has been analyzed in Ref. [9], where tilt generates a finite BCD. In three-dimensional antiferromagnetic metals, we adopt a model for the Dirac point of tetragonal CuMnAs, a collinear antiferromagnet belonging to the MPG $2'/m$. This symmetry permits the nonzero intrinsic second-order conductivity components σ_{xxx}, σ_{xxy}, σ_{xyy}, σ_{xzz}, σ_{yxx}, σ_{yxy}, σ_{yyy}, σ_{yzz}, σ_{zxz}, and σ_{zyz}. To capture the avoided crossings along the Γ–X path in CuMnAs, we use a minimal tilted-Dirac model defined by the Hamiltonian [22]

$$\hat{H}_{\mathrm{D}}(\boldsymbol{k}) = k_x\hat{\tau}_1 + k_y(\hat{\tau}_2\hat{\sigma}_2 - t) + k_z\hat{\tau}_3 + m\hat{\tau}_2\hat{\sigma}_3, \quad (23)$$

where $\hat{\tau}_i$ and $\hat{\sigma}_i$ denote Pauli matrices acting on the pseudospin and real spin subspaces, m is the effective mass, and t represents the tilting parameter. This effective model exhibits an $\mathrm{O}(2) \times \mathcal{PT}$ symmetry, preserving the combined $\mathcal{PT}$ operation while inheriting the spatial symmetries of the tilt term. Within this framework, a finite σ_{yxx} emerges, reflecting the characteristic behavior of the BCP dipole in CuMnAs.

Another route to inversion-symmetry breaking is threefold-rotational anisotropy. Among materials with these constraints, $MnBi_2Te_4$ is a well-studied layered antiferromagnetic topological insulator with A-type antiferromagnetic order, known for its axion insulator phase and associated magnetoelectric effects. [23] Even-layer thin films are characterized by the MPG $\bar{3}'m'$, [24] where the simultaneous breaking of inversion and time-reversal symmetries provides a natural platform for intrinsic nonlinear transport. Few-layer thin films of $MnBi_2Te_4$ are gapped, [25] which suppresses the Fermi-surface contributions. To overcome this, we consider doped even-layer $MnBi_2Te_4$ thin films, where the Fermi level can be shifted into the conduction or valence band, enabling access to the BCP dipole response. We adopt the effective model proposed in previous studies, [26] with an additional anisotropic term introduced to restore the C_{3z} rotational symmetry. The Hamiltonian reads

$$\hat{H}_{\mathrm{MBT}}(\boldsymbol{k}) = \varepsilon_0(\boldsymbol{k}) + M(\boldsymbol{k})\hat{\tau}_3 + A\hat{\tau}_1\boldsymbol{k}\cdot\hat{\boldsymbol{\sigma}} + Z\hat{\tau}_1\hat{\sigma}_3, \quad (24)$$

where $\varepsilon_0(\boldsymbol{k}) = C_0 + C_2k^2 + C_3k_x(3k_y^2 - k_x^2)$ represents the band anisotropy, $M(\boldsymbol{k}) = M_0 + M_2k^2$ is the mass term, A denotes the spin-orbit coupling strength, and Z the antiferromagnetic exchange field. Its nonzero intrinsic $\sigma^{(2)}$ components are $\sigma_{xxx} = -\sigma_{xyy} = -\sigma_{yxy}$.

The intrinsic nonlinear conductivities of the models in Eqs. (23) and (24) are shown in Figs. 1 and 2. These results are consistent with previous studies. [22,27] In Figs. 1(b) and 2(a), the magnitude of $\sigma^{(2)}$ exhibits a strong dependence on the chemical potential, particularly when the interband gap is small. This behavior follows from Eq. (13), as the BCP scales inversely with the energy gap $\Delta\mathcal{E}$.

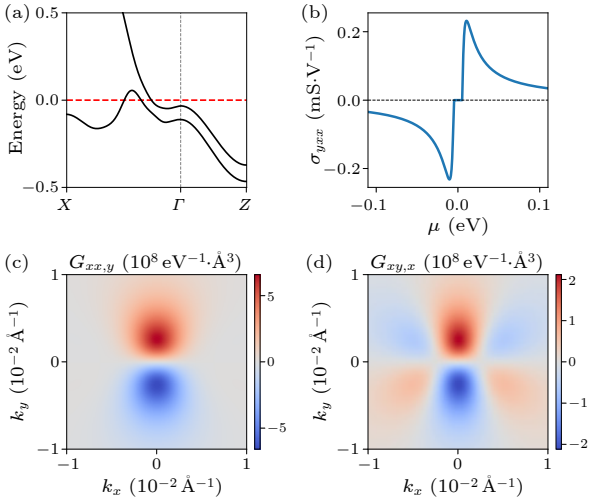

Fig. 1. Intrinsic nonlinear Hall conductivity in a tilted-Dirac model, as an analogue of tetragonal CuMnAs. (a) Band structure of CuMnAs near the avoided crossings. (b) Intrinsic σ_{yxx} as a function of the chemical potential μ for the tilted-Dirac model at zero temperature. Parameters: $t = 0.5\,\mathrm{eV}$, $m = 5\,\mathrm{meV}$. (c) and (d) The distribution of BCP dipole $G_{xx,y}$ and $G_{xy,x}$ at the $k_z = 0$ plane.

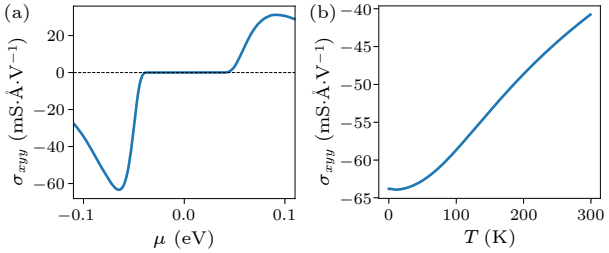

Fig. 2. Intrinsic nonlinear Hall conductivity of even-layer $MnBi_2Te_4$ model. Intrinsic σ_{xyy} as a function of (a) temperature T and (b) chemical potential μ for the doped even-layer $MnBi_2Te_4$ model. The chemical potential μ is set to $-64\,\mathrm{meV}$ in (b). Parameters: $C_0 = 0$, $C_2 = 0.3\,\mathrm{eV{\cdot}\AA^{-2}}$, $C_3 = 1\,\mathrm{eV{\cdot}\AA^{-3}}$, $M_0 = 0.03\,\mathrm{eV}$, $M_2 = 1\,\mathrm{eV{\cdot}\AA^{-2}}$, $A = 0.03\,\mathrm{eV{\cdot}\AA^{-1}}$, $Z = 0.03\,\mathrm{eV}$.

The intrinsic nonlinear conductivity also varies with temperature. From the Sommerfeld expansion, [28] its temperature dependence can be approximately expressed as

$$\sigma^{(2)}(\mu, T) \approx \sigma^{(2)}(\mu, 0) + \frac{(\pi k_{\mathrm{B}}T)^2}{6}\frac{\partial^2\sigma^{(2)}(\mu, 0)}{\partial\mu^2}. \quad (25)$$

Consequently, when the curvature $\frac{\partial^2\sigma^{(2)}(\mu,0)}{\partial\mu^2}$ is positive, $\sigma^{(2)}(\mu, T)$ increases with temperature, as observed in Fig. 2(b).

In summary, we develop a general framework to investigate intrinsic nonlinear transport phenomena in crystalline solids, motivated by the previous exploration of nonadiabatic effects beyond conventional treatments. We derive an analytic formula for the intrinsic nonlinear conductivity by incorporating superadiabatic corrections, which identifies the BCP dipole as the dominant contribution and establishes it as a Fermi-surface effect. The superadiabatic approach is based on the asymptotic analysis of the Schrödinger equation, beyond phenomenological adiabatic switching models, and independent of extrinsic scattering.

This framework provides a systematic route to obtain intrinsic transport quantities and enables the exploration of nonperturbative and nonadiabatic quantum phenomena in real materials.

We verify the theory in representative $\mathcal{PT}$ antiferromagnets in which inversion symmetry is broken by distinct mechanisms. In such $\mathcal{PT}$-symmetric systems, the BCD channel is forbidden, whereas the BCP dipole and nonlinear Drude terms are allowed. The framework, however, is general and applicable to a wide variety of material platforms with different symmetries and electronic structures.

Our results highlight the superadiabatic method as a fundamental approach in intrinsic transport theory, which fills the gap in the urgent demand for a quantum geometric transport theory by providing a nonadiabatic, divergence-free route to nonlinear responses. Our approach provides an intuitive perspective for interpreting experiments and unveils the underlying physics of nonlinear quantum responses in materials. The superadiabatic approach serves as a systematic tool for incorporating nonadiabatic phenomena into quantum dynamics. It is extendable to applications in broad areas beyond the transport context in the near future.

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