Residual Symmetry Reductions and Painlevé Solitons

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This letter introduces the novel concept of Painlevé solitons—waves arising from the interaction between Painlevé waves and solitons in integrable systems. Painlevé solitons can also be viewed as solitons propagating against a Painlevé wave background, in analogy to the established notion of elliptic solitons, which refers to solitons on an elliptic wave background. By employing a novel symmetry decomposition method aided by nonlocal residual symmetries, we explicitly construct (extended) Painlevé II solitons for the Korteweg–de Vries equation and (extended) Painlevé IV solitons for the Boussinesq equation.

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The study of integrable nonlinear evolution equations has been a cornerstone of mathematical physics for decades, yielding profound insights into the behavior of wave phenomena across diverse fields. Among the most celebrated discoveries in this area is the soliton, a wave packet that maintains its shape and speed upon collision with other solitons, exhibiting particle-like properties.[1,2] The robustness and universality of solitons have made them a central object of study in fields as varied as fluid dynamics, nonlinear optics, plasma physics, and Bose–Einstein condensates, among others.

Parallel to the theory of solitons, Painlevé analysis has served as a powerful integrability detector and solution-generating technique. The Painlevé property, which requires that all movable singularities of the solution are poles, provides a stringent test for integrability.[3,4] Beyond mere classification, the Painlevé expansion method, particularly in its truncated form, offers a systematic algorithm for deriving exact solutions, including solitary waves and Painlevé waves. These Painlevé waves are solutions expressible in terms of the Painlevé transcendents, which are nonlinear special functions satisfying the six Painlevé equations, Painlevé I–Painlevé VI (PI–PVI).[5] They often describe rich, non-periodic oscillatory structures and act as important asymptotic backgrounds in many physical contexts.

A natural and fruitful generalization of the classical soliton concept involves placing it on a non-trivial background. A quintessential example is the elliptic soliton, which is defined as a soliton propagating on a background of elliptic (cnoidal) waves. These solutions capture the intricate interaction between a localized pulse and a periodic structure, leading to a more complex and physically relevant wave profile than a soliton in a vacuum.[6] The term "elliptic soliton" elegantly encapsulates this hybrid nature, distinguishing it from its simpler counterpart. In Ref. [7], Ling and Sun constructed multiple elliptic solitons and analyzed their asymptotic behaviors for the focusing modified Korteweg–de Vries (KdV) equation via Darboux–Bäcklund transformation. Recently, Nijhoff, Sun, and Zhang[8] introduced the concept of an elliptic Nth root of unity and established an elliptic direct linearization scheme (of Dutch version). Elliptic Nth root of unity can be used to defined plane wave factors for the Boussinesq equations and the equations in the Gelfand–Dickey hierarchy.[9] Meanwhile, a bilinear framework serving for elliptic solitons (based on the Lamé function) was established by Li and Zhang for both continuous[10] and discrete systems. Vertex operators that generate τ functions of elliptic solitons were found, and bilinear identities for the related τ functions were given. More recently, an elliptic direct linearization scheme (Fokas–Ablowitz's version) was established by Li, Sun and Zhang[11] for the Kadomtsev–Petviashvili (KP), KdV and Boussinesq equations.

In recent years, the role of nonlocal symmetries has emerged as a pivotal tool for understanding and solving integrable systems. Unlike local symmetries, which depend on a finite number of derivatives of the field variables, nonlocal symmetries involve integrals of the dependent variables and are often connected to the recursion operator as well as Bäcklund transformations and Darboux transformations.[12–15] The concept of residual symmetry, which arises from the Möbius invariance of the Painlevé expansion and remains after the truncation process, is a specific type of nonlocal symmetry that has proven particularly effective. This residual symmetry can be localized by introducing a suitable auxiliary system, thereby enabling

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the application of standard Lie group methods to generate new and complex solutions from known ones.[16–18]

Building upon these foundations, this Letter introduces and develops the novel concept of Painlevé solitons. We define these as sophisticated wave structures that represent the nonlinear interaction between a classical soliton and a Painlevé wave. In essence, a Painlevé soliton can be conceptualized as a soliton riding upon a dynamic, nonperiodic background governed by a Painlevé transcendent. This definition is in direct analogy with the established notion of elliptic solitons, where the elliptic wave background is replaced by the more complex Painlevé wave background. The nomenclature Painlevé soliton is thus proposed to describe this new class of solutions, highlighting their composite nature.

The primary objective of this work is to formally establish the existence of Painlevé solitons and to provide a constructive method for obtaining them. To this end, we employ a novel symmetry decomposition method that leverages the power of nonlocal residual symmetries. This methodology allows us to systematically deconstruct the solution process, isolating the contributions from the soliton and the Painlevé background. Through this approach, we successfully derive explicit forms of two distinct types of Painlevé solitons:

- **Painlevé II solitons** for the KdV equation,
- **Painlevé IV solitons** for the Boussinesq equation.

The KdV equation, a paradigmatic model for shallow water waves, and the Boussinesq equation that describes waves in dispersive media, provide excellent testbeds for our theory. The derivation of Painlevé II and IV solitons for these systems not only validates our conceptual framework, but also demonstrates the efficacy of the symmetry decomposition method.

It is known that the KdV equation

$$u_t = u_{xxx} + 6uu_x, \tag{1}$$

possesses a symmetry[18]

$$\sigma = \frac{1}{3}c\lambda - 2cu - bh + \frac{1}{6}\delta - (cx - 2c\lambda t - \delta t + x_0)u_x - (3ct + t_0)u_t, \tag{2}$$

which is a solution of the linearized equation

$$\sigma_t = \sigma_{xxx} + 6\sigma u_x + 6u\sigma_x \tag{3}$$

of the KdV Eq. (1), where

$$h = f_{xx} \tag{4}$$

and f is a solution of the Schwartz KdV equation

$$f_t = h_x - \frac{3}{2}h^2 g^{-1} + \lambda g \tag{5}$$

with

$$g = f_x, \tag{6}$$

while the KdV Eq. (1) and the Schwartz KdV Eq. (5) are related by nonauto-Bäcklund transformation

$$u = -\frac{2}{3}\frac{h_x}{g} + \frac{1}{6}\frac{f_t}{g} + \frac{1}{2}\frac{h^2}{g^2} \tag{7}$$

and/or the truncated Painlevé expansion $u = -2\frac{g^2}{f^2} + 2\frac{h}{f} - \frac{2}{3}\frac{h_x}{g} + \frac{1}{6}\frac{f_t}{g} + \frac{1}{2}\frac{h^2}{g^2}$.

In the symmetry expression (2), the constant c related parts constitute the local scaling symmetry, x_0 and t_0 parts are related to the local space-time translation invariance, δ parts correspond to the local Galileo transformation, λ is same as in the Schwartz KdV Eq. (5) and also related to the spectral parameter. The term bh of Eq. (2) is a nonlocal symmetry of the single KdV Eq. (1). However, equation (2) is a local symmetry of the prolonged system (1), (4), (6), and (7) for the fields u, f, g, and h.

The full symmetry equation system of Eqs. (1), (4), (6), and (7) can be written as that of Eq. (3),

$$\sigma_3 = \sigma_{1xx}, \tag{8}$$

$$\sigma_2 = \sigma_{1x}, \tag{9}$$

and

$$\sigma = -\frac{2}{3}\frac{\sigma_{3x}}{g} + \frac{2}{3}\frac{h_x\sigma_2}{g^2} + \frac{1}{6}\frac{\sigma_{1t}}{g} - \frac{1}{6}\frac{f_t\sigma_2}{g^2} + \frac{h\sigma_3}{g^2} - \frac{h^2\sigma_2}{g^3}. \tag{10}$$

Related to the symmetry (2), the corresponding symmetry solutions of Eqs. (8), (9), and (10) read

$$\sigma_1 = \frac{1}{2}bf^2 + df + e - (cx - 2c\lambda t - \delta t + x_0)f_x - (3ct + t_0)f_t, \tag{11}$$

$$\sigma_2 = g(bf + d - c) - (cx - 2c\lambda t - \delta t + x_0)g_x - (3ct + t_0)g_t, \tag{12}$$

$$\sigma_3 = bg^2 + h(bf + d - 2c) - (cx - 2c\lambda t - \delta t + x_0)h_x - (3ct + t_0)h_t. \tag{13}$$

To solve the group invariant solutions related to the symmetries (2), (11), (12), and (13), one can discuss two subcases $c \neq 0$ and $c = 0$.

Case 1: $c = 0$. In this case, without loss of generality, one can take $t_0 = -1$ and the group invariant conditions $\sigma = \sigma_1 = \sigma_2 = \sigma_3 = 0$ lead to the solutions of Eqs. (2), (11), (12), and (13) in the forms

$$u = U(\xi, t), \quad f = F(\xi, t), \quad g = G(\xi, t), \quad h = H(\xi, t), \quad \xi = x + x_0 t, \tag{14}$$

and

$$U_t = bH, \tag{15}$$

$$F_t = -\frac{1}{2}bF^2 - dF - e, \tag{16}$$

$$G_t = -G(bF + d), \tag{17}$$

$$H_t = -H(bF + d) - G^2. \tag{18}$$

Substituting the relations (14) and (15)–(18) into the prolonged KdV system (1), (4), (6), and (7), we have

$$F_\xi = G, \tag{19}$$
$$G_\xi = H, \tag{20}$$
$$H_\xi = -2UG + \frac{1}{3}\lambda G + \frac{H^2}{2G}, \tag{21}$$
$$U_{\xi\xi\xi} = (x_0 - 6U)U_\xi + bH. \tag{22}$$

One can directly verify that the finite dimensional dynamic systems (15)–(18) and (19)–(22) are consistent because $F_{\xi t} = F_{t\xi}$, $G_{\xi t} = G_{t\xi}$, $H_{\xi t} = H_{t\xi}$ and $U_{\xi\xi\xi t} = U_{t\xi\xi\xi}$ are naturally satisfied.

From the consistent dynamic systems (15)–(18) and (19)–(22), we find that the (1+1)-dimensional integrable system (1), (4), (6), and (7) can be solved by a t-dynamic system (15)–(18) and a ξ-dynamic system (19)–(22). This kind of decomposition method appeared in two different situations. The first one is the so-called nonlinearization approach[19,20] for the Lax pair. The second one is the so-called formal variable separation approach.[21,22] The decomposition method proposed in Ref. [21] can solve high dimensional systems via consistent lower dimensional ones. On the other hand, lower dimensional integrable systems can be combined to higher dimensional integrable ones. Furthermore, the decomposition idea yields a deformation algorithm[23] to find new higher dimensional integrable systems and new types of solitons—twisted nontravelling solitary waves.[24]

The consistent dynamical systems can be simply solved with the final results ($\xi \equiv x + x_0 t$),

$$u = U(\xi) + \frac{b}{a}H(\xi)\tanh(a\phi) + \frac{b^2}{2a^2}G(\xi)^2\mathrm{sech}^2(a\phi),$$
$$\phi \equiv t + F(\xi), \tag{23}$$
$$f = -\frac{d}{b} + \frac{2a}{b}\tanh(a\phi), \tag{24}$$
$$g = G(\xi)\mathrm{sech}^2(a\phi), \tag{25}$$
$$h = \frac{1}{a}\mathrm{sech}^2(a\phi)[aH(\xi) - bG(\xi)^2\tanh(a\phi)], \tag{26}$$

where $a \neq 0$ is related to nonlocal symmetry parameters by

$$a^2 = \frac{1}{4}(d^2 - 2be), \tag{27}$$

while the group invariant functions $F = F(\xi)$, $G = G(\xi)$, $H = H(\xi)$, and $U = U(\xi)$ are determined by

$$U = -\frac{b^2}{2a^2}G^2 + \frac{2}{3}\lambda - \frac{x_0}{2} - \frac{a^2}{bG} - \frac{H^2}{2G^2}, \tag{28}$$
$$H = G_\xi, \tag{29}$$
$$F = \frac{b}{2a^2}\int^\xi G(y)\mathrm{d}y, \tag{30}$$

and

$$G_\xi^2 = \frac{b^2}{a^2}G^4 + CG^3 + 2(\lambda - x_0)G^2 - \frac{2a^2G}{b}. \tag{31}$$

It is clear that the G function of Eq. (31) is an elliptic integral which can be explicitly expressed by Jacobi or Weierstrass elliptic functions. Thus, the solution (23)–(26) is just the well known elliptic soliton of the KdV Eq. (1).

Case 2: $c \neq 0$. In this case, without loss of generality, we can take

$$c = 1, \quad x_0 = t_0 = 0, \quad \delta = -2\lambda - c_0 = 0. \tag{32}$$

Under the constant conditions of Eq. (32), the constraint conditions $\sigma = \sigma_1 = \sigma_2 = \sigma_3 = 0$ become

$$u = U(\eta, t), \quad f = F(\eta, t), \quad g = G(\eta, t),$$
$$h = H(\eta, t), \quad \eta \equiv \frac{x + \lambda t}{t^{1/3}}, \tag{33}$$

and

$$tU_t = \frac{1}{9}(3cH - 6U + \lambda), \tag{34}$$
$$tF_t = -\frac{1}{6c}(cF - 6a - c_1)(cF + 6a - c_1), \tag{35}$$
$$tG_t = -\frac{1}{3}G(cF - c_1 + 1), \tag{36}$$
$$tH_t = -\frac{c}{3}G^2 - \frac{1}{3}H(cF - c_1 + 2). \tag{37}$$

Substituting the expressions (33) and (34)–(37) into the equation system (1), (4), (6), and (7), we get an η-dynamic system,

$$U = \frac{1}{6}\eta t^{-2/3} + \frac{1}{6}\lambda + \frac{(cF - 6a - c_1)(cF + 6a - c_1)}{12ctG} - \frac{H^2}{2G^2}, \tag{38}$$
$$F_\eta = Gt^{1/3}, \tag{39}$$
$$G_\eta = Ht^{1/3}, \tag{40}$$
$$H_\eta = \frac{3H^2t^{1/3}}{2G} - \frac{\eta G}{3t^{1/3}} - \frac{(cF - 6a - c_1)(cF + 6a - c_1)}{6ct^{2/3}}. \tag{41}$$

The group invariant solutions of the consistent symmetry decomposition systems (34)–(37) and (38)–(41) possess the form

$$f = \frac{c_1}{c} + \frac{6a}{c}\tanh(\psi) = \frac{c_1}{c} + \frac{6a}{c}\frac{t^{2a}\exp[2aF(\eta)] - 1}{t^{2a}\exp[2aF(\eta)] + 1},$$
$$\psi \equiv aF(\eta) + \ln(t), \tag{42}$$
$$g = \frac{6a^2}{ct^{1/3}}G(\eta)\mathrm{sech}^2(\psi), \tag{43}$$
$$h = \frac{6a^2}{ct^{2/3}}\mathrm{sech}^2(\psi)[H(\eta) - 2aG(\eta)^2\tanh(\psi)], \tag{44}$$
$$u = \frac{\lambda}{6} + \frac{\eta}{6t^{2/3}} - \frac{1}{2t^{2/3}G(\eta)} - \frac{H(\eta)^2}{2t^{2/3}G(\eta)^2} + \frac{2a}{t^{2/3}}\tanh(\psi)[H(\eta) - aG(\eta)^2\tanh(\psi)] \tag{45}$$

with $G(\eta) = F_\eta(\eta)$ and $H(\eta) = F_{\eta\eta}(\eta)$, while the $F(\eta) = F$ satisfy

$$6F_\eta F_{\eta\eta\eta} - 9F_{\eta\eta}^2 + 2\eta F_\eta^2 - 6F_\eta - 12a^2F_\eta^4 = 0. \tag{46}$$

Thus, whence the solution of Eq. (46) is fixed, the corresponding solution (45) of the KdV Eq. (1) is obtained. The solution (45) is a soliton solution under the F background.

To see the property of Eq. (46), we make the following transformation $[P(\zeta) = P]$:

$$F(\eta) = F_1(b\eta) = F_1(\zeta) = F_1, \quad F_{1\zeta} = \frac{3}{\zeta + 2P^2 + 2\delta_1 P_\zeta},$$
$$\delta_1^2 = 1, \quad \zeta = -\frac{x + \lambda t}{(3t)^{1/3}}, \tag{47}$$

where b serves as a scale parameter connecting the original variable η and the new variable ζ. It realizes the linear scaling of the original variable η through $\zeta = b\eta$.

Substituting the transformation (47) into Eq. (46) yields

$$P_{\zeta\zeta} = 2P^3 + \zeta P + 3a\delta_2 - \frac{\delta_1}{2} - \frac{6a\delta_2}{1 + A\exp(-6a\delta_2 Q)}, \tag{48}$$
$$Q_\zeta = \frac{1}{\delta_1\zeta + 2\delta_1 P^2 + 2P_\zeta}, \quad \delta_1^2 = \delta_2^2 = 1 \tag{49}$$

with an arbitrary constant A.

It is clear that when $A = 0$, the P-Eq. (48) reduces to the standard Painlevé II equation. For $A \neq 0$, however, the general P-Eq. (48), hereafter referred to as the extended Painlevé II equation, incorporates an additional non-rational, non-polynomial exponential modulation term: $\frac{6a\delta_2}{1+A\exp(-6a\delta_2 Q)}$. This term fundamentally expands the solution space of the original Painlevé II equation, enabling more diverse dynamic behaviors.

It should be emphasized that the symmetry reduction leading to Eq. (46) or Eq. (48) is a new important Painlevé-type reduction. This is because Eq. (48) deviates from the canonical form of standard Painlevé equations,

$$P_{\zeta\zeta} = W(\zeta, P, P_\zeta), \tag{50}$$

where $W(\zeta, P, P_\zeta)$ is rational in P_ζ, algebraic in P and analytic in ζ.

Accordingly, the solution (45) is an extended Painlevé II soliton for $A \neq 0$ and a Painlevé II soliton for $A = 0$. In other words, it is a soliton ride on an extended Painlevé wave.

In summary, for the KdV Eq. (1), by means of the nonlocal residual symmetry and local symmetries, one can obtain some types of symmetry decompositions including the elliptic solitons, extended Painlevé II solitons, and Painlevé II solitons. Though the KdV is well known and studied by various experts, there are still many mysteries including the firstly discovered extended Painlevé II Eq. (48) in this paper.

Similarly, there are various uncovered mysteries for all well known integrable systems. To support this opinion, we study the residual symmetry reductions of the Boussinesq equation,

$$u_{tt} = (u_{xx} + u^2)_{xx}. \tag{51}$$

By using the truncated Painlevé expansion method, one can find that both the transformation

$$u = \frac{1}{2}\frac{f_t^2}{f_x^2} - 2\frac{f_{xxx}}{f_x} + \frac{3}{2}\frac{f_{xx}^2}{f_x^2}, \tag{52}$$

and the truncated expansion

$$u = -6\frac{f_x^2}{f^2} + 6\frac{f_{xx}}{f} + \frac{1}{2}\frac{f_t^2}{f_x^2} - 2\frac{f_{xxx}}{f_x} + \frac{3}{2}\frac{f_{xx}^2}{f_x^2} \tag{53}$$

can transform the Boussinesq Eq. (51) to its Schwartz Boussinesq equation,

$$R_t = S_x - RR_x, \quad R = \frac{f_t}{f_x}, \quad S = \frac{f_{xxx}}{f_x} - \frac{3}{2}\frac{f_{xx}^2}{f_x^2}. \tag{54}$$

One can directly verify that if u and f are related by Eq. (52), the residual, f_{xx}, with respect to the arbitrary singular manifold f, is a nonlocal symmetry of the Boussinesq Eq. (51).

Similarly to the KdV equation, though the residual symmetry f_{xx} is nonlocal for the Boussinesq Eq. (51), it can become a local symmetry for the extended Boussinesq system (51), (4), (6), and (52).

Using the standard classical symmetry approach, it is not difficult to find that the Lie point symmetry of the extended Boussinesq system (51), (4), (6), and (52) possesses the form

$$\sigma = ah - 2cu - (cx + x_0)u_x - (2ct + t_0)u_t, \tag{55}$$
$$\sigma_1 = c_0 + c_1 f - \frac{1}{6}af^2 - (cx + x_0)f_x - (2ct + t_0)f_t, \tag{56}$$
$$\sigma_2 = g\Big(c_1 - c - \frac{1}{3}af\Big) - (cx + x_0)g_x - (2ct + t_0)g_t, \tag{57}$$
$$\sigma_3 = h\Big(c_1 - \frac{1}{3}af\Big) - \frac{1}{3}ag^2 - (cx + x_0)h_x - (2ct + t_0)h_t. \tag{58}$$

To find the group invariant solution related to the symmetry (55)–(58), we discuss two subcases.

Case I: $c = 0$, $t_0 = -1$, $c_0 = \frac{3(a_1^2 - c_1^2)}{2a}$. In this case, the group invariant condition $\sigma = \sigma_1 = \sigma_2 = \sigma_3 = 0$ yields the t-dynamic system,

$$U_t = -aH, \quad u = U(x + x_0 t, t) = U(\xi, t), \quad h = H(\xi, t),$$
$$g = G(\xi, t), \quad f = F(\xi, t), \tag{59}$$
$$F_t = \frac{a}{6}F^2 - c_1 F + \frac{3}{2a}(c_1^2 - a_1^2), \tag{60}$$
$$G_t = \frac{a}{3}FG - c_1 G, \tag{61}$$
$$H_t = \frac{a}{3}G^2 + \frac{a}{3}FH - c_1 H. \tag{62}$$

Substituting the t-dynamic system (60)–(62) into the prolonged Boussinesq system (51), (4), (6), and (52), one can get the ξ-dynamic system,

$$F_\xi = G, \tag{63}$$
$$G_\xi = H, \tag{64}$$
$$H_\xi = \frac{1}{4}(x_0^2 - 2U)G + \Big[\frac{a}{12}F^2 - \frac{c_1}{2}F - \frac{3}{4a}(a_1^2 - c_1^2)\Big]x_0 + \frac{1}{G}\Big[\frac{3}{4}H^2 + \frac{a^2F^4}{144} - \frac{ac_1F^3}{12} + \frac{3c_1^2 - a_1^2}{8}F^2 + \frac{3c_1}{4a}(a_1^2 - c_1^2)F + \frac{(a_1^2 - c_1^2)^2}{16a^2}\Big], \tag{65}$$
$$H_\xi = \frac{U_{\xi\xi}}{a}\Big(\frac{x_0}{2} - \frac{U}{x_0}\Big) - \frac{U_{\xi\xi\xi\xi}}{2ax_0} + \frac{H}{6x_0}(3c_1 - aF) - \frac{aG^2}{6x_0} - \frac{U_\xi^2}{ax_0}. \tag{66}$$

The symmetry decomposition system (60)–(62) and (63)–(66) possesses the generalized solution

$$f=\frac{3c_1}{a}-\frac{3a_1}{a}\tanh(\theta),\quad \theta\equiv\frac{a_1t}{2}+F(\xi),$$
$$\xi=x+x_0t, \tag{67}$$
$$g=G(\xi)\mathrm{sech}^2(\theta), \tag{68}$$
$$h=\frac{1}{3a_1}\mathrm{sech}^2(\theta)[2aG(\xi)^2\tanh(\theta)+3a_1H(\xi)], \tag{69}$$
$$u=U(\xi)-\frac{2aH(\xi)}{a_1}\tanh(\theta)+\frac{2a^2G(\xi)^2}{3a_1^2}\mathrm{sech}^2(\theta), \tag{70}$$

where

$$H(\xi)=G_\xi(\xi), \tag{71}$$
$$G(\xi)=-\frac{3a_1}{a}F_\xi(\xi), \tag{72}$$
$$U(\xi)=\frac{3H(\xi)^2}{2G(\xi)^2}-\frac{2H_\xi(\xi)}{G(\xi)}-\frac{2a^2G(\xi)^2}{9a_1^2}+\frac{x_0^2}{2}-\frac{3a_1^2x_0}{2aG(\xi)}+\frac{9a_1^4}{a^2G(\xi)^2}, \tag{73}$$

while $G(\xi)$ is given by the following elliptic integral:

$$\int^{G(\xi)}\frac{\mathrm{d}\mu}{\sqrt{V(\mu)}}=\xi+\xi_0,$$
$$V(\mu)=\frac{4a^2\mu^4}{9a_1^2}-C_2\mu^3+C_1\mu^2+\frac{3a_1^2x_0}{a}\mu-\frac{3a_1^4}{4a^2}, \tag{74}$$

with three more arbitrary integral constants C_1, C_2 and ξ_0. It is obvious that the solution (70) is just the elliptic soliton solution of the Boussinesq Eq. (51).

Case II: $c\neq0$, $x_0=t_0=0$. In this case, we take the transformation,

$$u=U(\eta,t),\quad f=F(\eta,t),\quad g=G(\eta,t),$$
$$h=H(\eta,t),\quad \eta=\frac{x}{\sqrt{t}}. \tag{75}$$

Using the transformation (75), the second type of symmetry decomposition possesses the form,

$$2ctF_t=-\frac{1}{6}aF^2+c_1F+c_0, \tag{76}$$
$$2ctG_t=-\frac{1}{3}aFG-cG+c_1G, \tag{77}$$
$$2ctH_t=-\frac{1}{3}aFH-\frac{1}{3}aG^2+c_1H-2cH, \tag{78}$$
$$2ctU_t=-2cU+aH, \tag{79}$$

and

$$F_\eta=\sqrt{t}G, \tag{80}$$
$$G_\eta=\sqrt{t}H, \tag{81}$$
$$H_\eta=\frac{(\eta^2-8U)G}{16\sqrt{t}}+\frac{\eta}{48tc}[aF^2-6c_1F-9(a_1^2-c_1^2)a^{-1}]+\frac{1}{t^{3/2}a^2c^2G}[a^4F^4-12c_1a^3F^3-18a^2(a_1^2-3c_1^2)F^2+108ac_1(a_1^2-c_1^2)F+432a^2c^2t^2H^2+81(a_1^2-c_1^2)^2], \tag{82}$$

$$U_\eta=\frac{\eta H(F_1^2-9a_1^2)}{16catG^2}-\frac{\eta(5c+F_1)}{4ct}+\frac{3H(8tU-\eta^2)}{16\sqrt{t}G}-\frac{(F_1^2-9a_1^2)(F_1+9c)}{72ac^2t^{3/2}G}+\frac{3\sqrt{t}H^3}{4G^3}+\frac{(F_1^2-9a_1^2)^2H}{576a^2c^2t^{3/2}},\quad F_1\equiv aF-3c_1. \tag{83}$$

The general solution of the symmetry decomposition (76)–(79) and (80)–(83) reads

$$f=\frac{3c_1}{a}+\frac{3a_1}{a}\tanh(\Phi),\quad \Phi=\frac{a_1}{4c}\ln(t)+F(\eta), \tag{84}$$
$$g=\frac{G(\eta)}{\sqrt{t}}\mathrm{sech}^2(\Phi), \tag{85}$$
$$h=\frac{H(\eta)}{t}\mathrm{sech}^2(\Phi)-\frac{2aG(\eta)^2}{3a_1t}\mathrm{sech}^2(\Phi)\tanh(\Phi), \tag{86}$$
$$u=\frac{U(\eta)}{t}+\frac{2a^2G(\eta)^2}{3a_1^2t}\mathrm{sech}^2(\Phi)+\frac{2aH(\eta)}{a_1t}\tanh(\Phi), \tag{87}$$

where the group invariant functions $F(\eta)=F$, $G(\eta)=G$, $H(\eta)=H$, and $U(\eta)=U$ are related by

$$F=\frac{a}{3a_1}\int G\mathrm{d}\eta, \tag{88}$$
$$H=G_\eta, \tag{89}$$
$$U=\frac{3G_\eta^2}{2G^2}-\frac{2G_{\eta\eta}}{G}-\frac{2a^2G^2}{9a_1^2}+\frac{\eta^2}{8}-\frac{3a_1^2\eta}{8acG}+\frac{9a_1^4}{32a^2c^2G^2}, \tag{90}$$

while G is determined by

$$432a^2c^2a_1^2G_\eta^3+(81a_1^6-64a^4c^2G^4-576a^2c^2a_1^2GG_{\eta\eta}-108aca_1^4\eta G)G_\eta+144a^2c^2a_1^2G^2G_{\eta\eta\eta}-108aca_1^2G^2(ac\eta G-a_1^2)=0. \tag{91}$$

By means of the transformation,

$$G(\eta)=\frac{A}{P(\zeta)},\quad \zeta=B\eta, \tag{92}$$

with the scale constants A and B being determined by

$$B=-2^{-3/4}3^{-1/4}c^{-1/2}\beta^{-1/4}a_1^{1/2},$$
$$A=-\frac{3a_1B\sqrt{-\beta}}{\sqrt{2}a}, \tag{93}$$

and denoting a_1 as

$$a_1=\frac{3c\sqrt{-\beta}}{\sqrt{2}b_1}, \tag{94}$$

the G-Eq. (91) can be solved by an extended Painlevé IV equation,

$$P_{\zeta\zeta}=\frac{1}{2}\frac{P_\zeta^2}{P}+\frac{3}{2}P^3+4b_1\zeta P^2+2(b_1^2\zeta^2-\alpha)P+\frac{\beta}{P}. \tag{95}$$

The extended Painlevé IV Eq. (95) becomes the standard Painlevé IV equation only when $b_1=1$; notably, b_1 cannot be eliminated via scaling transformations of ζ or P. This non-scalable parameter b_1 is the core of the "extension": unlike the fixed linear/quadratic coefficients in the standard Painlevé IV equation, the extended form incorporates b_1–parameterized terms [e.g., $4b_1\zeta P^2$ and $2(b_1^2\zeta^2-\alpha)P$] that modify both linear coefficients and

nonlinear coupling strength. Correspondingly, the solution (87) denotes an extended Painlevé IV soliton when $b_1 \neq 1$ and a conventional Painlevé IV soliton when $b_1 = 1$.

This work introduces and systematically constructs a new class of exact solutions termed Painlevé solitons, which emerge from the nonlinear interplay between classical solitons and Painlevé wave backgrounds in integrable systems. By leveraging nonlocal residual symmetries and a novel symmetry decomposition method, we have explicitly derived:

- **(Extended) Painlevé II solitons** for the KdV equation, and
- **(Extended) Painlevé IV solitons** for the Boussinesq equation.

These solutions represent a natural yet profound generalization of the well-known elliptic solitons, where the periodic cnoidal wave background is replaced by a non-periodic, transcendentally rich Painlevé wave. The concept of Painlevé solitons thus bridges two central themes in integrable systems: the robust, localized structure of solitons and the complex, asymptotic behavior of Painlevé transcendents.

The methodology employed here—residual symmetry reduction—proves to be a powerful and unifying framework. It allows for the decomposition of the full system into consistent dynamical subsystems in both temporal and spatial variables, facilitating the explicit construction of hybrid solutions. This approach not only recovers known elliptic solitons but also leads to new types of reductions, including the extended Painlevé II Eq. (48) and the extended Painlevé IV Eq. (95), which generalize their classical counterparts and admit richer solution structures.

From a physical perspective, Painlevé solitons are expected to describe wave phenomena in settings where a coherent solitary wave interacts with a non-uniform, non-periodic background—such as in turbulent or inhomogeneous media. Their mathematical structure, combining localization and transcendental complexity, may offer new insights into the long-time asymptotics of integrable systems and the transition from order to chaos.

Several open questions and future directions arise from this work:

- Can Painlevé solitons be observed experimentally or numerically in physical systems described by the KdV or Boussinesq equations?
- How do these solutions behave under perturbation or in nearly-integrable settings?
- Can the symmetry decomposition method be applied to other integrable systems (e.g., KP and nonlinear Schrödinger) to generate new Painlevé-type solitons?
- What is the role of Painlevé solitons in the context of inverse scattering, Riemann–Hilbert problems, Darboux transformation and Hirota's bilinear algorithm?

In conclusion, this study not only enriches the taxonomy of exact solutions in integrable systems but also demonstrates the continuing relevance of symmetry-based methods in uncovering new mathematical structures. The concept of Painlevé solitons opens a promising avenue for future research at the intersection of soliton theory, Painlevé analysis, applied mathematics, and physics.

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