Generalized (2+1)-Dimensional Sharma–Tasso–Olver–Burgers Equation: Dispersionless Decompositions and Twisted Solitons

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We presents a generalized (2+1)-dimensional Sharma–Tasso–Olver–Burgers (STOB) equation, unifying dissipative and dispersive wave dynamics. By introducing an auxiliary potential y as a new space variable and employing a simpler deformation algorithm, we deform the (1+1)-dimensional STOB model to higher dimensions. The resulting equation is proven Lax-integrable via introducing strong and weak Lax pairs. Traveling wave solutions of the (2+1)-dimensional STOB equation are derived through an ordinary differential equation reduction, with implicit solutions obtained for a special case. Crucially, we demonstrate that the system admits dispersionless decompositions into two types: Case 1 yields non-traveling twisted kink and bell solitons, while Case 2 involves complex implicit functions governed by cubic-algebraic constraints. Numerical visualizations reveal novel anisotropic soliton structures, and the decomposition methodology is shown to generalize broadly to other higher dimensional dispersionless decomposition solvable integrable systems.

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Nonlinear partial differential equations serve as fundamental models for wave phenomena across diverse physical systems. Three closely related equations—the Burgers equation, Sharma–Tasso–Olver (STO) equation, and their combined form, the Sharma–Tasso–Olver–Burgers (STOB) equation—exhibit rich mathematical structures with significant physical applications. The Burgers equation represents the prototypical model for dissipative wave propagation and shock formation,

$$u_t + uu_x - \nu u_{xx} = 0. \tag{1}$$

The Burgers Eq. (1) can be exactly linearized via the Cole–Hopf transformation,[1]

$$u = -2\nu(\ln\phi)_x.$$

The STO equation describes dispersive wave systems with cubic nonlinearity,

$$v_t + \alpha(v^3)_x + \frac{3\alpha}{2}vv_{xx} + \beta v_{xxx} = 0, \quad \alpha\beta \neq 0. \tag{2}$$

The integrability of the STO Eq. (2) can be guaranteed by its Lax pairs,[2] Darboux transformations[3] and linearizability.[2] The exact multiple soliton solutions of the Burgers equation and the STO equation have been studied by many authors. Especially, the soliton fission and fusion phenomena of the Burgers equation and the STO equation have been investigated in Ref. [4]. Infinitely many symmetries, exact solutions and recursion operators of Eqs. (1) and (2) can be obtained by using the repeated similarity reductions.[5,6]

The STOB equation synthesizes dissipative and dispersive effects,

$$u_t + \alpha(u^3)_x + \frac{3\alpha}{2}(u^2)_{xx} + \beta u_{xxx} + \nu u_{xx} = 0. \tag{3}$$

Soliton molecules have been identified for the STOB Eq. (3), but not for the Burgers Eq. (1) or the STO Eq. (2).[7] The STOB equation has been extensively studied through various approaches, including Lie symmetry analysis,[8] bilinearization,[9] Painlevé analysis,[10] and Bäcklund transformations.[11]

Despite substantial research on the (1+1)-dimensional STOB equation, significant limitations remain:

- No systematic nontrivial (2+1)-dimensional generalizations exist;
- Limited understanding of anisotropic soliton interactions;
- Inadequate methods for cross-dimensional wave coupling;
- Absence of a framework connecting (1+1)-dimensional dispersionless systems to (2+1)-dimensional dissipative-dispersive systems.

Recently, a deformation algorithm has been developed to construct higher-dimensional integrable models from lower-dimensional systems via conservation laws.[12] This approach has successfully extended several (1+1)-dimensional integrable systems to higher dimensions, including, the Korteweg–de Vries (KdV) equation,[12] the Ablowitz–Kaup–Newell–Segur system,[12] the nonlinear

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Schrödinger equation,[13] the Camassa–Holm equation,[14] the modified KdV equation,[15] the Burgers equation,[16] the Degasperis–Procesi equation,[17] the dark Korteweg–de Vries system,[18] the decoupled integrable models,[19] and the Kaup–Newell system.[20] Theoretical[21,22] and experimental[23] studies of higher-dimensional solitons have also been conducted.

In this Letter, we employ an alternative and simpler approach to deform the STOB Eq. (3) to (2+1)-dimensions by introducing a potential y defined by

$$y_x = u, \quad y_t = -\alpha u^3 - 3\alpha u u_x - \beta u_{xx} - \nu u_x, \tag{4}$$

where the compatibility condition $y_{xt} = y_{tx}$ yields the original STOB Eq. (3).

The deformation extension is implemented through the ansatz

$$u = U(x, y, t). \tag{5}$$

Substituting Eq. (5) with Eq. (4) into Eq. (3) yields the following (2+1)-dimensional STOB equation:

$$\begin{aligned} &u_t + \Big[\alpha\Big(3uu_{xy} + u_{xx} + \frac{3}{2}u^2u_{yy} + 3uu_x + u^3\Big) + \nu(u^2 + u_x)\Big]_x \\ &+ \Big[\frac{1}{2}\alpha u^2(u^2 + 6uu_y + 2uu_{yy} + 12u_x + 3u_{xy}) + \frac{1}{3}\nu u^2(3u_y + u)\Big]_y \\ &+ 2\nu u u_{xy} - 2u_y u(3\alpha u u_y + 3\alpha u_x + \nu u_y) = 0. \end{aligned} \tag{6}$$

One can verify that Eq. (6) admits the following weak Lax pair:

$$\begin{aligned} &\hat{L}_{\rm w}\psi \equiv (\partial_x + u\partial_y - u - \lambda)\psi = 0, \\ &\hat{M}_{\rm w}\psi \equiv \{\partial_t + \alpha\partial_x^3 + \alpha u^3\partial_y^3 + 3\alpha u^2\partial_x\partial_y^2 + 3\alpha u\partial_x^2\partial_y \\ &\quad + (\nu - 3\alpha\lambda)\partial_x^2 + [3\alpha u_x + 3\alpha u u_y + 2u(\nu - 3\alpha\lambda)]\partial_x\partial_y \\ &\quad + u(3\alpha u_x + 3\alpha u u_y + \nu u - 3\alpha\lambda u)\partial_y^2 \\ &\quad + [3\alpha(u + \lambda)(u_x + u u_y) + \alpha u^3 + u^2\nu \\ &\quad + (2\nu - 3\alpha\lambda)\lambda u]\partial_y + \lambda(2\nu - 3\alpha\lambda)\partial_x\}\psi = 0, \end{aligned} \tag{7}$$

and strong Lax pair

$$\begin{aligned} &\hat{L}_{\rm s}\psi \equiv (\partial_x + u\partial_y + u - \lambda)\psi = 0, \\ &\hat{M}_{\rm s}\psi \equiv \{\partial_t + \alpha\partial_x^3 + \alpha u^3\partial_y^3 + 3\alpha u^2\partial_x\partial_y^2 + 3\alpha u\partial_x^2\partial_y \\ &\quad + (\nu + 3\alpha\lambda)\partial_x^2 + [3\alpha u_x + 3\alpha u u_y + 2u(\nu + 3\alpha u)]\partial_x\partial_y \\ &\quad + u(3\alpha u_x + 3\alpha u u_y + \nu u + 3\alpha u^2)\partial_y^2 \\ &\quad + [3\alpha(u_x + u u_y) + 2\nu u + 3\alpha u^2]\partial_x \\ &\quad + u[3\alpha(u_x + u u_y) + \nu u + 2\alpha u^2]\partial_y\}\psi = 0. \end{aligned} \tag{8}$$

Indeed, both the strong consistency condition

$$[\hat{L}_{\rm s}, \hat{M}_{\rm s}] \equiv \hat{L}_{\rm s}\hat{M}_{\rm s} - \hat{M}_{\rm s}\hat{L}_{\rm s} = 0,$$

and the weak consistency condition

$$[\hat{L}_{\rm w}, \hat{M}_{\rm w}]\psi\Big|_{\text{Eq. (7)}} = 0$$

are equivalent to the (2+1)-dimensional STOB Eq. (6).

Although the (2+1)-dimensional STOB Eq. (6) is Lax-integrable, finding its exact solutions remains challenging. To obtain traveling wave solutions, we employ the ansatz

$$u = q(y + kx + \omega t) - k \equiv q(\xi) - k, \tag{9}$$

which, when substituted into Eq. (6), yields the ordinary differential equation for q as

$$\Big(qq_\xi + \frac{3}{2}q^2 - \nu_1\Big)_\xi + q^2 - \nu_1 q - \frac{\omega_1}{q} + \frac{1}{4}(1 + c^2)\nu_1^2 = 0. \tag{10}$$

Here, c is an integration constant, while ν_1 and ω_1 relate to the original parameters through

$$\nu = \alpha(3k - \nu_1), \quad \omega = \alpha(\omega_1 - k^3) + k^2\nu.$$

Obtaining the general solution of Eq. (10) remains challenging except when $\omega_1 = 0$. In this special case ($\omega_1 = 0$), the general solution of Eq. (10) takes the form

$$q = a\nu_1\cos(cf)\exp(-X - f), \tag{11}$$

where the function $f \equiv f(X)$ is implicitly defined by

$$\exp(X + f) = \frac{2a[\cos(cf) - c\sin(cf)]}{1 + c^2 + 4b\exp(-X)}. \tag{12}$$

Here, a and b denote two new integration constants.

The (2+1)-dimensional STOB Eq. (6) constitutes a coupled system exhibiting both dissipative and dispersive effects. Using the decomposition approach introduced in Ref. [24], we find that this dissipative-dispersive equation can be solved through dispersionless systems of the form

$$u_t = F(u, u_x), \tag{13}$$

$$u_y = G(u, u_x), \tag{14}$$

where $F \equiv F(u, u_x)$ and $G \equiv G(u, u_x)$ are functions to be determined, depending on u and its spatial derivative u_x.

Substituting Eqs. (13) and (14) into the (2+1)-dimensional STOB Eq. (6) for an arbitrary solution u yields two constraints on the functions F and G:

$$G = -\frac{u_x}{u} + g, \quad g \equiv g(u), \tag{15}$$

$$\begin{aligned} F = &-\alpha u_x[ugg_u + g^2 + g(\mu + 3u) + u(\mu + u)] \\ &- \alpha u^2 g[u(gg_u + 3g + u^2)_u + (3g + \mu)(g_u + 1)], \end{aligned} \tag{16}$$

where $\mu \equiv \nu/\alpha$. The function g must be determined by the compatibility condition $u_{ty} = u_{yt}$ between Eqs. (13) and (14), which gives

$$\begin{aligned} &g[6ug_u^2 + (3u^2g_{uu} + 6g + 2\mu + 12u)g_u \\ &+ u(6g + \mu + 3u)g_{uu} + gu^2g_{uuu} + 6g + 2\mu + 6u] = 0. \end{aligned} \tag{17}$$

For the g-equation (17), we distinguish two cases: $g = 0$ and $g \neq 0$.

Case 1. $g = 0$.

In this case, the dispersionless decomposition given by Eqs. (13) and (14) reduces to

$$u_t = -\alpha u u_x(u + \mu), \quad u_y = -\frac{u_x}{u}. \tag{18}$$

Solving the dispersionless system Eq. (18) yields a non-traveling wave solution of the (2+1)-dimensional STOB Eq. (6) in the implicit form

$$u = U[xu - y - tu^2(\alpha u + \nu)], \tag{19}$$

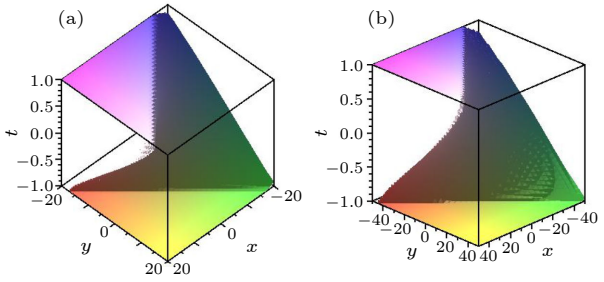

Fig. 1. (a) The twisted kink structure of Eq. (20) at time $t = 1$ with $\nu = \alpha = 1$. (b) The structure of the twisted kink Eq. (20) at time $t = 5$ with the same parameters as in (a).

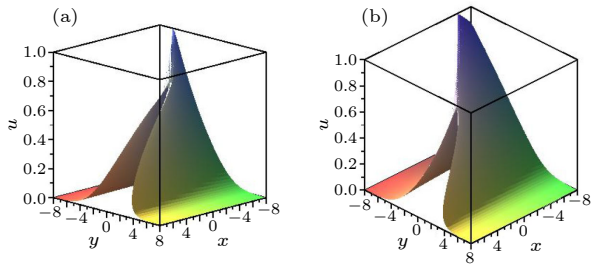

Fig. 2. Display the twisted bell soliton structure expressed by Eq. (21) at (a) time $t = -1$ and (b) $t = 1$ respectively for $\nu = \alpha = 1$.

where $U(z) \equiv U[xu - y - tu^2(\alpha u + \nu)]$ is an arbitrary function of its argument z. When U is specified as a hyperbolic tangent function,

$$u = \tanh[xu - y - tu^2(\alpha u + \nu)], \tag{20}$$

the solution corresponds to a non-traveling twisted kink. Figure 1(a) displays this twisted kink solution at time $t = 1$ with parameters $\nu = 1$ and $\alpha = 1$. Figure 1(b) illustrates the structure of the twisted kink at $t = 5$ using identical parameters.

Consider the function U of Eq. (19) defined as a hyperbolic secant function,

$$u = \operatorname{sech}[xu - y - tu^2(\alpha u + \nu)], \tag{21}$$

the solution corresponds to a non-traveling twisted soliton. Figure 2(a) displays the twisted soliton solution described by Eq. (21) at time $t = -1$, using the same parameters as in Fig. 1. Figure 2(b) exhibits the structure of the twisted soliton from Eq. (21) at time $t = 1$ with $\nu = \alpha = 1$.

Case 2. $g \neq 0$.

In this case, the general solution g in Eq. (17) takes the form

$$g = \frac{9(3b - \mu^2)(u^3 + \mu u^2 + bu + a)}{u[(2\mu^3 - 9b\mu + 27a)r + 2(\mu + 3u)(3b - \mu^2)]}, \tag{22}$$

where the function $r \equiv r(u)$ is defined implicitly by

$$\begin{aligned}&(u - c_1)^{C_1}(u - c_2)^{C_2}(u - c_3)^{C_3}\\ &= (r - z_1)^{Z_1}(r - z_2)^{Z_2}(r - z_3)^{Z_3}\end{aligned} \tag{23}$$

with c_i and z_i $(i = 1, 2, 3)$ being solutions to the cubic algebraic equations

$$\mu c_i^2 + c_i^3 + bc_i + a = 0,$$

and

$$a_1 z_i^3 + z_i - 1 = 0, \quad a_1 \equiv \frac{(2\mu^3 - 9b\mu + 27a)^2}{(9b - 3\mu^2)^3},$$

respectively. The exponents are given by

$$C_i \equiv (3b - \mu^2)(2\mu c_i + 3c_i^2 + b), \quad Z_i \equiv 9a_1 z_i^2 + 3,$$

where a, b, and μ are arbitrary constants.

In principle, exact relevant solutions of the (2+1)-dimensional STOB Eq. (6) can be derived by solving the dispersionless system

$$u_t = -\frac{u_x}{u} + g, \quad g \equiv g(u), \tag{24}$$

$$\begin{aligned}u_y = &-\alpha u_x[ugg_u + g^2 + g(\mu + 3u) + u(\mu + u)]\\ &-\alpha u^2 g[u(gg_u + 3g + u^2)_u + (3g + \mu)(g_u + 1)],\end{aligned} \tag{25}$$

where g is defined implicitly by Eqs. (22) and (23).

However, solving system (24) and (25) remains challenging due to the implicit definition of g. The integrability of this system follows from the existence of infinitely many symmetries. [25]

In summary, we extends the (1+1)-dimensional dissipative and dispersive STOB equation to an integrable (2+1)-dimensional version via a simplified deformation algorithm. Strong and weak Lax pairs are introduced to establish the Lax integrability of the (2+1)-dimensional STOB equation. Travelling wave solutions are derived by solving a second-order ordinary differential Eq. (10) containing three arbitrary constants. For the special case $\omega_1 = 0$, implicit forms of these solutions are obtained. Notably, this dissipative-dispersive system proves solvable through integrable dispersionless systems.

For the (2+1)-dimensional STOB Eq. (6), two distinct types of dispersionless decompositions exist. The first decomposition type (18) yields the general solution (19) in implicit form. This solution exhibits rich structural features, with explicit visualizations demonstrating non-traveling twisted kink and bell solitons. Crucially, the proposed dispersionless decomposition methodology applies broadly to higher-dimensional integrable systems, including models in Refs. [12–24]. We conjecture that twisted non-traveling kink and bell solitons constitute universal features of dispersionless decomposition solvable systems.

Based on the findings of this study, several critical directions merit further investigation:

1. What classes of higher-dimensional systems are solvable through dispersionless decompositions?
2. Which nonlinear systems admit non-traveling twisted solitons?
3. How do twisted kinks and bell solitons interact during collisions, particularly in numerical simulations?
4. What stability characteristics govern twisted soliton solutions?
5. How can exact solutions be constructed for implicit decomposition systems such as Eqs. (22)–(25)?

6. How might predicted twisted kinks and bell solitons be experimentally observed in physical systems (e.g., fluids, optics, Bose–Einstein condensates)?

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