Uncovering Hidden Spin of Scalar Fields with Higher-Order Derivative Lagrangian: On the Wave Spin in Drifted and Dissipative Fields

Shuo Xin[1,2†], Mengxiang Xie[1†], and Jie Ren[1*]

[1] *Center for Phononics and Thermal Energy Science, China-EU Joint Lab on Nanophononics, School of Physics Science and Engineering, Tongji University, Shanghai 200092, China*
[2] *Physics Department, Stanford University, Stanford, CA 94305, USA*
SLAC National Accelerator Laboratory, Menlo Park, CA 94025, USA

(Received 10 March 2025; accepted manuscript online 29 April 2025)

Scalar fields should have no spin angular momentum according to conventional textbook understandings in classical field theory. Yet, recent studies demonstrate the undoubted existence of wave spin endowed by acoustic and elastic longitudinal waves, which are of irrotational curl-free nature without vorticity and can be described by scalar fields. Moreover, the conventional theory cannot even answer the question of whether wave spin exists in dissipative fields, given the ubiquitous dissipation in reality. Here, to resolve the seeming paradox and answer the challenging question, we uncover the origin of wave spin in scalar fields beyond traditional formalism by clarifying that the presence of higher-order derivatives in scalar field Lagrangians can give rise to non-vanishing wave spin. For "spinless" scalar fields of only first-order derivatives, we can make the hidden wave spin emerge by revealing a latent field that leads to the original field through a time derivative, thus giving higher-order terms in Lagrangian. Based on the standard Noether theorem approach, we exemplify the wave spin for unconventional drifted acoustic fields, and even for dissipative media, in scalar fields with higher-order derivative Lagrangian. The results would prompt people to build more comprehensive and fundamental understandings of structural wave spin in classical fields.

DOI: 10.1088/0256-307X/42/6/064301 CSTR: 32039.14.0256-307X.42.6.064301

1. Introduction. Recently, there has been a rapid sprouting interest in the study of elastic spin[1–6] and acoustic spin,[7–16] after people recognized the existence of spin angular momentum (spin AM) in longitudinal waves.[1,7] Both simulations and experiments have demonstrated that the spin AM of longitudinal waves can be observed in various scenarios, such as metamaterials, evanescent waves, and wave interferences.[2,4,6,7,9–13,16–25] Its orbital counterpart has also attracted much attention.[26–30]

Great efforts to understand elastic spin[1] and acoustic spin[7] have been dedicated to both experimental and theoretical approaches. Numerous studies aim to establish a solid theoretical foundation for the spin of acoustic and elastic waves.[1,4,5,7,12,15,30,31] However, amidst the abundance of recent research lies the unresolved question of how longitudinal waves, which are scalar in nature, can possess spin AM. Some researchers attribute this phenomenon to the limitations of scalar descriptions.[7,31,32] Indeed, most studies employ a vector field framework when discussing elastic and acoustic waves.[4,5,15,31] Here, however, we demonstrate that a scalar description is not only concise but also capable of explaining the emergence of spin AM.

The assertion that scalar fields do not possess spin AM is a well-established result in classical field theory.[33–35] However, there's an implicit prerequisite: the Lagrangian of the scalar field is only of the first order, i.e., containing only first-order derivatives. While this is true for most fields that describe fundamental interactions in nature, as in the standard model,[36,37] it is not guaranteed when we redefine fields, e.g., for the effective scalar potential ψ of displacement ($\boldsymbol{x} = \nabla\psi$) in longitudinal elastic and acoustic waves. In these instances, we find that second and third derivatives are involved, which are crucial for the emergence of spin AM. Consequently, our discussions must be based on higher-order Lagrangians.

The study of higher-order Lagrangians has a long history. Notable results, such as Ostrogradsky instability,[38] continue to be discussed in modern studies.[39–42] Ostrogradsky's construction of higher-order field theory provides a paradigm for later studies like the Pais-Uhlenbeck oscillator,[43] and the Ostrogradsky instability was, in fact, shown not to be universal.[44,45] Problems with ghost fields and unitarity were also clarified by Hawking *et al.*[46] Therefore, higher derivative field theories have always been an active subject of study. A more comprehensive review can be found in Ref. [45].

There has been a growing interest in applying the formal development of field theory to classical phenomena like acoustics and *vice versa*. For instance, Iyer *et*

†These authors contributed equally to this work.
*Corresponding author. Email: Xonics@tongji.edu.cn
© 2025 Chinese Physical Society and IOP Publishing Ltd. All rights, including for text and data mining, AI training, and similar technologies, are reserved.

al.[47,48] applied the traditional treatment of dispersive water waves[49] to classify string theories. Here, we find that the description of curl-free longitudinal waves in acoustic and elastic systems can be unified in a simple yet elegant class of higher-order scalar field theories. Remarkably, the spin AM follows simply from the standard Noether current derivations in these higher-order scalar field theories. The Noether currents associated with higher-order Lagrangians have been summarized in the literature from the last century.[50–54] Latent acoustic spin and elastic spin in scalar longitudinal wave fields, however, were not given attention there. Higher derivative field theories have also been studied for the purpose of quantum electrodynamics by Lee and Wick.[55,56] Recently, the Lee-Wick extension to the standard model has also been proposed as a solution to the hierarchy problem,[57,58] leading to revived discussions on Lee-Wick theory.[59–61] It would be interesting to further explore its value in classical field theory, and we find that the simple scalar case already implies a concise description for acoustic wave spin.

In this work, we explain the emergence of spin AM for scalar fields with higher-order derivative Lagrangian and apply the higher-order field theory to unconventional drifted acoustic field and dissipative acoustic field of irrotational curl-free nature. For scalar fields that have no spin AM, we can make the hidden wave spin emerge by constructing a latent field whose time derivative leads to the original field, so that is of higher order. Finally we give remarks on general higher-order scalar fields, discuss, and conclude in the end.

2. Spin Angular Momentum of Scalar Fields. Following Noether's theorem, the AM density of a field is the conserved current corresponding to rotational symmetry. Usually this AM separates into two parts: the orbital AM part with extrinsic dependence on the choice of reference origin point, and the intrinsic spin AM part without such dependence. For a typical first-order Lagrangian $\mathcal{L}(x_\mu, \phi, \partial_\mu\phi)$, the scalar field ϕ has only orbital AM while the vector counterpart can possess additional spin AM. We briefly review the reasoning behind this argument.

For the Lagrangian $\mathcal{L}(x_\mu, \phi, \partial_\mu\phi)$ dependent on up to the first-order derivative of the scalar field ϕ, the evolution of ϕ over spacetime is determined by imposing the variation $\delta S = 0$ on the action $S = \int \mathrm{d}^4x\mathcal{L}(x_\mu, \phi, \partial_\mu\phi)$, giving the usual Euler-Lagrange equation,

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0. \tag{1}$$

Following Noether's theorem, the energy-momentum density is the conserved current corresponding to spacetime translation. Namely, under translation $x_\mu \to x_\mu + \delta x_\mu$, we can find the variation of Lagrangian $\delta\mathcal{L} = \partial_\mu(T_{\mu\nu})\delta x_\nu$ with the energy-momentum tensor $T_{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + \delta_{\mu\nu}\mathcal{L}$, and the time-space translational symmetry $\partial_\mu(T_{\mu\nu}) = 0$ gives rise to the conservation of energy $\int_V T_{00}$ and canonical momentum $\int_V T_{0j}$, respectively.

We then consider the Noether current corresponding to spatial rotation along an infinitesimal angle $\boldsymbol{\theta}$: $x_j \to x_j + \theta_i\epsilon_{jik}x_k$, i.e., the AM. The variation of the Lagrangian is $\delta\mathcal{L} = \theta_i\partial_\mu(T_{\mu j}\epsilon_{jik}x_k)$. As expected, the conserved current only has an orbital part $J_i = \epsilon_{ikj}x_kT_{0j}$, dependent on the choice of reference point x_μ (so-called orbital). Additional intrinsic terms without x_k dependence are absent. This is the standard conclusion that a scalar field has no spin AM, conventional in classical textbooks.[33–35]

However, the above derivation is only true for the standard Lagrangian of first-order derivatives, but invalid for Lagrangians of higher-order derivatives. Suppose we have a second-order Lagrangian $\mathcal{L}(x_\mu, \psi, \partial_\mu\psi, \partial_\mu\partial_\nu\psi)$ for the scalar field ψ, we will show that the hidden wave spin emerges due to the higher-order derivatives. We will derive the spin and orbital AM following Noether's theorem: in the presence of second-order terms, the Euler-Lagrange equation from the least action principle is modified as:

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi)} + \partial_\mu\partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\psi)}\Big) = 0. \tag{2}$$

With higher derivatives involved in the dynamics, the quantities we were familiar with from the first-order Lagrangian, especially the Noether current, would be modified.

The modified Noether current $T_{\mu\nu}$ of translation invariance is derived in the Supplementary Material (SM). Then, if we consider an infinitesimal rotation by an angle of $\boldsymbol{\theta}$, i.e., the variation of x_j is $\delta x_j = \theta_i\epsilon_{jik}x_k$, after some algebra we will find the variation of the Lagrangian takes a different form:

$$\delta\mathcal{L} = \theta_i\partial_\mu\Big[T_{\mu j}x_k\epsilon_{jik} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_k\psi)}\partial_j\psi\epsilon_{jik}\Big], \tag{3}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. Compared with the classical first-order case, an additional second term emerges in the parentheses of Eq. (3). As such, the conserved current corresponding to AM in rotational symmetry separates into two parts: the orbital AM part, the first term with extrinsic dependence on the choice of reference points, and the spin AM part, the additional intrinsic term independent of the choice of reference points. This is also consistent with $\delta\mathcal{L} = \theta_i\partial_\mu(T_{\mu j}\epsilon_{jik}x_k)$ for the first-order Lagrangian, where the second term vanishes $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_k\psi)} = 0$. In other words, the appearance of higher-order derivatives creates an intrinsic AM contribution from additional variations of the Lagrangian.

From Eq. (3), we can read off the expression for the spin AM density of scalar fields of second-order derivative Lagrangian. Rewriting into a more compact form, we have (an overdot in the expression means time derivative ∂_t)

$$s_i = \frac{\partial\mathcal{L}}{\partial(\partial_k\dot{\psi})}\partial_j\psi\epsilon_{ijk}, \quad \boldsymbol{s} = \nabla\psi \times \frac{\partial\mathcal{L}}{\partial(\nabla\dot{\psi})}. \tag{4}$$

One would immediately find this expression analogous to the spin AM density of a vector field $\boldsymbol{u}$, i.e., if we start with first-order Lagrangian $\mathcal{L}_{\mathrm{vec}}(\boldsymbol{u}, \partial_\nu\boldsymbol{u})$, we would find the spin AM density to be $\boldsymbol{s}_{\mathrm{vec}} = \boldsymbol{u} \times \frac{\partial\mathcal{L}_{\mathrm{vec}}}{\partial(\partial_t\boldsymbol{u})}$. This observation guarantees consistency when $\nabla\psi = \boldsymbol{u}$, which is the case for longitudinal waves of irrotational curl-free nature in shearless media.

If we have a third-order Lagrangian, following the similar derivation in SM, we would find the spin AM density

emerging out of the scalar fields as well,

$$s_i = 2\frac{\partial\mathcal{L}}{\partial\psi_{,t\mu k}}\partial_\mu\partial_j\psi\epsilon_{ijk} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\psi_{,\mu tk}}\partial_j\psi\epsilon_{ijk} + \frac{\partial\mathcal{L}}{\partial(\partial_t\partial_k\psi)}\partial_j\psi\epsilon_{ijk}, \tag{5}$$

where the last term is the same as the hidden spin for the second-order scalar field Lagrangian. Other terms involve third-order derivatives and generally have no compact notations to represent how indexes are contracted. Though immediate intuitive interpretation of them is not clear, they are necessary components in, e.g., canceling the dissipative part in the Lagrangian, as we would see in the example of a viscous acoustic field with dissipations.

Origin of the additional emergence of spin AM in higher-order scalar field is explained in a simplified way in Fig. 1. The gradient of the scalar field has nonzero variation under rotation transformation. Although the first-order Lagrangian depends on $\partial_\mu\psi$, the field configuration has to respect the Euler-Lagrange equation; thus, its contribution can be absorbed into a total derivative of a current that does not depend on $\delta\partial_\mu\psi$. Thus, the Noether current of rotation does not have a spin part. However, when the Lagrangian is a general higher-order one, this Noether current does depend on the variation of the gradient $\delta\partial_\mu\psi$ so that the first-order derivative, as a kind of polarization, becomes intrinsic with the assistance of higher-order derivatives. The rotation of the emergent intrinsic polarization leads to the nontrivial spin AM (see SM).

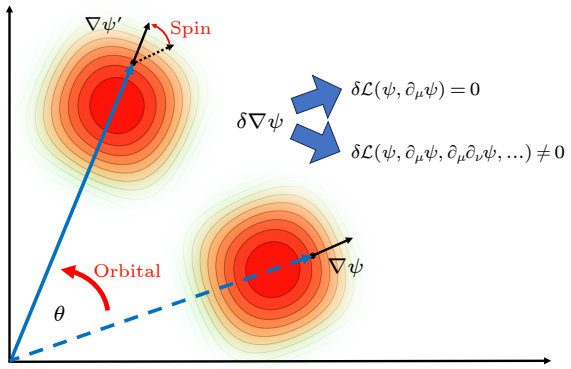

Fig. 1. Origin of the emergence of spin in higher order scalar field. For arbitrary scalar field ψ under a rotational transformations θ, the global coordinate transformation corresponds to the orbital contribution, while the variation of the gradient in local polarization $\delta\nabla\psi = \nabla\psi' - \nabla\psi$ represents the spin contribution. In the first-order Lagrangian, this term can be be absorbed into the variation of scalar field itself, resulting $\delta\mathcal{L}(\psi,\partial_\mu\psi)=0$, indicating that the intrinsic AM part is absent. And for the higher-order Lagrangian, $\delta\mathcal{L}(\psi,\partial_\mu\psi,\partial_\mu\partial_\nu\psi,...)\neq 0$ uncovers the hidden spin.

In what follows, we exemplify the emergence of wave spin for acoustic fields in unconventional drifted media and even in dissipative media with higher-order derivative Lagrangians.

3. Wave Spin in Drifted Acoustic Field. Consider longitudinal waves in shearless fluid, described by Navier-Stokes equations[62] and continuity

$$\rho[\partial_t\boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}] = -\nabla P + \eta\nabla^2\boldsymbol{V} + \frac{\eta}{3}\nabla(\nabla\cdot\boldsymbol{V}),$$
$$\partial_t\rho + \nabla\cdot(\rho\boldsymbol{V}) = 0. \tag{6}$$

First we study the non-dissipative ($\eta = 0$) and adiabatic fluid ($\frac{\delta P}{\delta\rho} = c_0^2$ with c_0 being the speed of sound) in linear regime, i.e., $\rho = \rho_0 + p/c_0^2$, $P = p_0 + p$, $\boldsymbol{V} = \boldsymbol{v}_0 + \boldsymbol{v}$, with $p, \boldsymbol{v}$ being small perturbations around the background pressure p_0 and drifted velocity $\boldsymbol{v}_0$. The background quantities satisfy $\dot{\rho}_0 + \nabla\cdot(\rho_0\boldsymbol{v}_0) = 0$. Note that in the above formulation, the background quantities ρ_0, p_0, and $\boldsymbol{v}_0$ are allowed to vary in space and time.

Traditionally, one can introduce a scalar field ϕ such that $\boldsymbol{v} = \nabla\phi$ and $p = -\rho_0(\partial_t + \boldsymbol{v}_0\cdot\nabla)\phi$, which automatically satisfies the linearized Navier-Stokes equation. The linearized continuity equation gives the master equation governing the dynamics of the scalar field ϕ,[63]

$$\frac{1}{\rho_0}\nabla\cdot(\rho_0\nabla\phi) - (\partial_t + \boldsymbol{v}_0\cdot\nabla)\frac{1}{c_0^2}(\partial_t + \boldsymbol{v}_0\cdot\vec{\nabla})\phi = 0. \tag{7}$$

This differential equation can be reformulated using the least action principle, and one arrives at a typical Lagrangian density of a linearized scalar field,

$$\mathcal{L} = \frac{\rho_0}{2}[(\nabla\phi)^2 - c_0^{-2}(\partial_t\phi + \boldsymbol{v}_0\cdot\nabla\phi)^2], \tag{8}$$

which involves only first-order derivatives. Following the standard theory after Eq. (1), one will arrive at the conclusion that the spin AM is absent in this field, as in classical textbooks.[33–35] A recent work[31] took the particular case of static background $\boldsymbol{v}_0 = 0$ and explained latent acoustic spin by taking the analogy of Maxwell's electromagnetic equation and adopting bi-vector representations to reproduce the naturally existing spin AM in a vector field.[1,7,12] Here, we'd like to emphasize that the framework of high-order scalar field is capable of describing the full physics, including the spin AM, which even extends the wave spin to dissipative fields.

To uncover the hidden spin AM of the acoustic field, we need to construct a latent scalar field to make the Lagrangian contain higher-order derivatives, as suggested by the above discussions. Here, we give the Lagrangian density of a new scalar field ψ whose time derivative leads to the original scalar field through $\phi = \partial_t\psi$. Then the new Lagrangian density with higher-order derivatives for ψ is written as

$$\mathcal{L}(x_\mu,\partial_\mu\partial_\nu\psi) = \frac{\rho_0}{2}[(\nabla\dot{\psi})^2 - c_0^{-2}(\ddot{\psi} + \boldsymbol{v}_0\cdot\nabla\dot{\psi})^2]. \tag{9}$$

The latent scalar field ψ has a very clear physical meaning of potential for the displacement $\boldsymbol{u} = \nabla\psi$, because $\phi = \partial_t\psi$ is the potential for the velocity $\boldsymbol{v} = \dot{\boldsymbol{u}} = \nabla\phi$.

According to Eq. (4), the spin AM density of the longitudinal acoustic wave is

$$\boldsymbol{s} = \rho_0\boldsymbol{u}\times\boldsymbol{v} + \frac{p}{c_0^2}\,\boldsymbol{u}\times\boldsymbol{v}_0, \tag{10}$$

where $\boldsymbol{u} = \nabla\psi$ is the displacement field, $\boldsymbol{v} = \partial_t\boldsymbol{u}$ is the velocity field, and $p = -\rho_0(\partial_t + \boldsymbol{v}_0\cdot\nabla)\dot{\psi}$ is the pressure field. Considering the oscillating nature of the wave fields

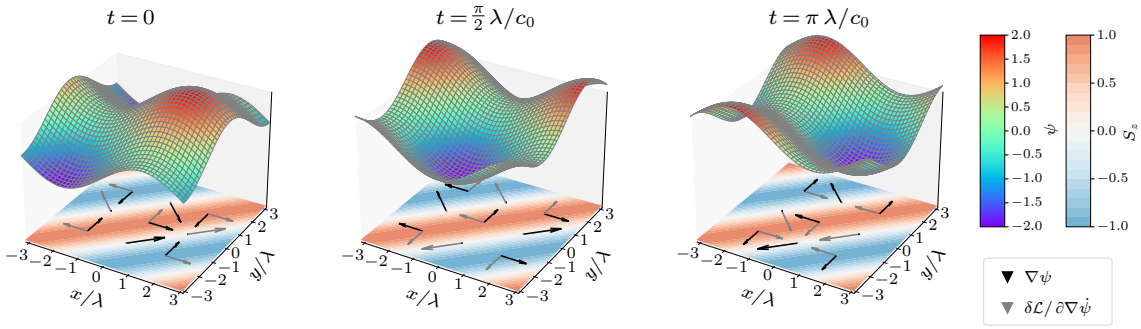

Fig. 2. Nonzero spin AM S_z (shown on the xy planes) endowed by interfering scalar waves ψ (shown on the 3D contours) traveling in the x and y directions with the same wavelength λ and unity amplitude. The distribution of the scalar wave ψ itself, with wavelength λ and speed c_0, is shown by the mesh surface on top for $t=0$, $\frac{\pi}{2}\lambda/c_0$, and $\pi\lambda/c_0$, respectively. The spin AM density S_z is given by the wedge product of the gradient $\nabla\psi$ (black arrow) and $\frac{\partial\mathcal{L}}{\partial\nabla\dot{\psi}}$ (grey arrow). From the Hamiltonian point of view, the gradient (black arrow) is the new degree of freedom of the dynamic coordinate with conjugate momentum $\frac{\partial\mathcal{L}}{\partial\nabla\dot{\psi}}$ (grey arrow). The momentum $\frac{\partial\mathcal{L}}{\partial\nabla\dot{\psi}}$ conjugate to $\nabla\psi$ is linked to the quantity of S_z, which cannot be obtained from the original first-order field $\phi=\dot{\psi}$.

$\boldsymbol{u}$ and $\boldsymbol{v}$, only the first term survives while the second term vanishes after time average on period. The result is consistent with the acoustic spin obtained previously,[1–4,7,10,12,13,25] which is not affected by drifting $\boldsymbol{v_0}$. The physical meaning of this spin AM density would be more evident if we rewrite

$$\boldsymbol{s}=\boldsymbol{u}\times\delta(\rho\boldsymbol{V}) \tag{11}$$

with the first-order perturbation of local acoustic momentum density $\delta(\rho\boldsymbol{V})=\rho_0\delta\boldsymbol{V}+\delta\rho\boldsymbol{v}_0=\rho_0\boldsymbol{v}+\frac{p}{c_0^2}\boldsymbol{v}_0$ spinning around the equilibrium position.

4. Wave Spin in Dissipative Acoustic Field. Next, we turn on dissipation in a static fluid background. The linearized Navier-Stokes equation, $\rho_0\frac{\partial\boldsymbol{v}}{\partial t}=-\nabla p+\eta\nabla^2\boldsymbol{v}+\frac{\eta}{3}\nabla(\nabla\cdot\boldsymbol{v})$, can be satisfied by introducing a scalar field ϕ with $\boldsymbol{v}=\nabla\phi$ and $p=-\rho_0\frac{\partial\phi}{\partial t}+\frac{4}{3}\eta\nabla^2\phi$, and the continuity condition leaves us with the master equation governing the scalar field,

$$\nabla^2\phi-\frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2}+\frac{4}{3}\frac{\eta}{\rho_0c_0^2}\frac{\partial}{\partial t}\nabla^2\phi=0. \tag{12}$$

This differential equation can be reformulated using the least action principle with Lagrangian density

$$\mathcal{L}=\frac{1}{2}\rho_0(\nabla\phi)^2-\frac{1}{2c_0^2}\rho_0\dot{\phi}^2+\frac{2}{3}\frac{\eta}{c_0^2}\nabla\phi\cdot\nabla\dot{\phi}. \tag{13}$$

Although this is a Lagrangian of second-order derivatives, the Lagrangian explicitly depends on the gradient terms of ϕ, and the spin AM of the velocity potential ϕ vanishes; i.e., according to Eq. (4), one obtains $\boldsymbol{s}=\nabla\phi\times\frac{\partial\mathcal{L}}{\partial(\nabla\dot{\phi})}=\frac{2}{3}\frac{\eta}{c_0^2}\nabla\phi\times\nabla\phi=0$, which indicates that ϕ is not the latent field. This is also physically expected, since the additional term comes from the dissipative viscous force, merely depending on which shall not give rise to any finite conserved quantities.

Again, this forces us to consider the displacement potential underlying the velocity potential, by digging into the latent field ψ, whose time derivative leads to the original field $\phi=\dot{\psi}$. The Lagrangian density of the displacement potential ψ is

$$\mathcal{L}=\frac{1}{2}\rho_0(\nabla\dot{\psi})^2-\frac{1}{2c_0^2}\rho_0\ddot{\psi}^2+\frac{2}{3}\frac{\eta}{c_0^2}\nabla\dot{\psi}\cdot\nabla\ddot{\psi}. \tag{14}$$

This Lagrangian has third-order derivatives, which can have nontrivial spin AM, as expressed in Eq. (5). The terms involving the third-order derivatives for this Lagrangian are $\frac{\partial\mathcal{L}}{\partial\psi_{,tij}}=0$ and $\frac{\partial\mathcal{L}}{\partial\psi_{,tti}}=\frac{2}{3}\frac{\eta}{c_0^2}\partial_i\partial_t\psi$. Expanding all terms, we would also see the contributions from the dissipative parts cancel, $s_i=\frac{4\eta}{3c_0^2}\partial_j\partial_t\psi\partial_t\partial_k\psi\epsilon_{ijk}-\partial_t(\frac{2\eta}{3c_0^2}\partial_k\partial_t\psi)\partial_j\psi\epsilon_{ijk}+(\rho_0\partial_t\partial_k\psi+\frac{2\eta}{3c_0^2}\partial_k\partial_t\partial_t\psi)\partial_j\psi\epsilon_{ijk}=\rho_0\partial_t\partial_k\psi\partial_j\psi\epsilon_{ijk}$. Therefore, in a compact form, we arrive at

$$\boldsymbol{s}=\rho_0\nabla\psi\times\nabla\dot{\psi}=\rho_0\boldsymbol{u}\times\boldsymbol{v}. \tag{15}$$

This is again consistent with the spin AM of acoustic fields obtained previously,[1–4,7,10,12,13,25] which is not affected by viscous dissipation.

As an example, we illustrate nonzero spin angular momenta endowed by interfering scalar waves traveling in x and y directions in Fig. 2. We can clearly see the rotational evolution of the gradients when $\nabla\psi\times\nabla\dot{\psi}\neq0$, e.g., at point $x=y=0$. As a connection between field theory description and traditional understanding, this is how the latent field ψ, through its gradient, drives the apparent motion $\boldsymbol{u}=\nabla\psi$ of either displacement in solid objects or particles in fluids.

5. General Wave Spin from Higher-Order Scalar Fields. For completeness, we also give the general expression of spin AM for a scalar field with an N-th order derivative Lagrangian, following the same approach of as Noether's theorem

$$s_i=\epsilon_{ijk}\sum_{m,n=0}^{m+n\leq N-2}(-1)^n(m+1)\underbrace{\psi_{,j\kappa_1\ldots\kappa_m}}_{(m+1)\text{th derivative}}\times\underbrace{\partial_{\chi_1\cdots\chi_n}}_{n\text{th derivative}}\frac{\partial\mathcal{L}}{\partial\psi_{,tk\chi_1\cdots\chi_n\kappa_1\ldots\kappa_m}}. \tag{16}$$

This is similarly obtained in T. S. Chang's 1948 paper,[51] though in slightly different notations. Noether currents of arbitrary higher-order Lagrangian was mathematically discussed in notes from last century,[51–53] although without emphasis on the emergence of hidden acoustic spin and elastic spin from scalar fields.

From Eq. (16), the case $N < 2$ is trivially 0, i.e., the classical scalar Lagrangian of first-order derivative has no spin AM. The cases of $N = 2, 3$ will reduce to the Eqs. (4) and (5), respectively, which clearly unravel the hidden wave spin of scalar fields from higher-order derivative Lagrangian, explaining the origin of spin AM in irrotational curl-free longitudinal waves. Our representation makes it readily applicable to elastic and acoustic waves represented in modern studies, and the wave spin results are consistent with those from vector field representations.[1,4,12,15,31] Heuristically speaking, with higher-order derivatives, the $(N-1)$-th order gradient of ψ would become a dynamical coordinate, while the term $\partial\mathcal{L}/\partial\psi_{,tk\chi_1\cdots\chi_n\kappa_1\cdots\kappa_m}$ in Eq. (16) plays the role of conjugate momentum, and their antisymmetric contraction with ϵ_{ijk} would contribute to spin AM density. This treatment follows from Ostrogradsky's construction,[38,43] if one were to find the corresponding Hamiltonian system and further quantize the higher-order field theory. One may draw on this choice of dynamical coordinates and conjugate momenta as an understanding of extra terms in Eq. (16).

As a consistency check, we can also describe the longitudinal elastic wave by a scalar field consistently without loss of spin AM. Consider the elastic wave propagating in a solid object. The local displacement in the solid is a vector field u_i. The equation of motion is $\rho\ddot{u}_i = c_{ijkl}\partial_j\partial_l u_k$, where ρ is the density and c_{ijkl} is the Christoffel matrix of this material. The Lagrangian density of this system is[34,64] $\mathcal{L} = \frac{1}{2}(\rho\dot{u}_i\dot{u}_i - c_{ijkl}u_{i,j}u_{k,l})$, a typical Lagrangian for a vector field with only first-order derivatives. The spin AM corresponding to this Lagrangian is $\boldsymbol{s} = \boldsymbol{u}\times\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{u}}} = \rho\boldsymbol{u}\times\dot{\boldsymbol{u}}$.[4] Yet, there is a myth when the displacement field is separated into longitudinal (non-curling, irrotational) and transverse (non-diverging, solenoidal) parts, according to the Helmholtz decomposition theorem. For the longitudinal (irrotational curl-free) part, the displacement field is given by a scalar potential $\boldsymbol{u} = \nabla\psi$. The longitudinal wave was wrongly regarded as spinless since it is a scalar field and $\nabla\times\nabla\phi = 0$. However, the Lagrangian for the scalar potential, $\mathcal{L}(x_\mu, \partial_\mu\partial_\nu\psi) = \frac{1}{2}(\rho\partial_i\dot{\psi}\partial_i\dot{\psi} - c_{ijkl}\partial_i\partial_j\psi\partial_k\partial_l\psi)$, involves second-order derivatives of ψ. According to Eq. (4), the spin AM density of the field is $\boldsymbol{s} = \nabla\psi\times\frac{\partial\mathcal{L}}{\partial(\nabla\dot{\psi})} = \rho\nabla\psi\times\nabla\dot{\psi}$, consistent with the vector description.[4] This exemplifies our understanding of Eq. (16) for the $N = 2$ case. The gradient of the scalar field $\nabla\psi$ becomes dynamical and can be interpreted as local displacement. Both the second-order scalar Lagrangian and the first-order vector Lagrangian give the same spin AM.

6. Discussion and Conclusions. As we have seen in this work, the spin AM simply originates from Lagrangian with higher-order derivatives, which invalidates the argument that the scalar field has no spin, a result specific to first-order Lagrangians. With the help of this insight, we have uncovered the spin AM of longitudinal elastic and acoustic waves, as well as in dissipative media.

This study further reminds us to reconsider typical scalar fields with only first-order Lagrangian. We can always construct higher-order Lagrangian by introducing a latent field describing the same system after fixing a gauge. The acoustic field is an instance. The Lagrangian for velocity potential ϕ is of first order, whereas the Lagrangian for the displacement potential ψ (where $\phi = \partial_t\psi$) is of second order. By introducing this latent field ψ, the Lagrangian acquires a gauge symmetry: $\mathcal{L}$ is invariant under transformations of ψ that leave ϕ invariant. In contrast to the $U(1)$ gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda(\boldsymbol{x})$ in electromagnetism, the gauge transformation here takes a simpler form $\psi \to \psi + f(\boldsymbol{x})$ with an arbitrary time-independent function $f(\boldsymbol{x})$. This freedom of gauge choice reflects the fact that the initial displacement reference can be arbitrarily selected without affecting observable dynamics. Namely, if we interpret the gradient of the latent scalar field ψ as physical displacement, the transformation represents a diffeomorphism invariance, i.e., freedom to choose local coordinate systems. By introducing this latent field, we uncover the hidden spin AM of acoustic waves. The same logic shall apply to other scalar fields.

This classical picture of Lagrangian mechanics is already enough to see the emergence of spin out of scalar fields. To further understand the spin and other physics of elastic and acoustic waves in this higher-order Lagrangian description, one may want to do field quantization to see the behavior of phonons and other elementary excitations in this description. Quantization of higher-order Lagrangian can be significantly different from that of first order.[65,66] This will deserve further exploration in the future.

Acknowledgments. This work was supported by the National Key R&D Program of China (Grant Nos. 2022YFA1404400 and 2023YFA1406900), the Natural Science Foundation of Shanghai (Grant No. 23ZR1481200), and the Program of Shanghai Academic Research Leader (Grant No. 23XD1423800).

References

[1] Long Y, Ren J, and Chen H 2018 *Proc. Natl. Acad. Sci.* **115** 9951
[2] Yuan W, Yang C, Zhang D, Long Y, Pan Y, Zhong Z, Chen H, Zhao J, and Ren J 2021 *Nat. Commun.* **12** 6954
[3] Zhao J, Yang C, Yuan W, Zhang D, Long Y, Pan Y, Chen H, Zhong Z, and Ren J 2022 *Phys. Rev. Lett.* **129** 275501
[4] Ren J 2022 *Chin. Phys. Lett.* **39** 126301
[5] Bliokh K Y 2022 *Phys. Rev. Lett.* **129** 204303
[6] Cao L, Wan S, Zeng Y, Zhu Y, and Assouar B 2023 *Sci. Adv.* **9** eadf3652
[7] Shi C, Zhao R, Long Y, Yang S, Wang Y, Chen H, Ren J, and Zhang X 2019 *Natl. Sci. Rev.* **6** 707
[8] Bliokh K Y and Nori F 2019 *Phys. Rev. B* **99** 174310
[9] Bliokh K Y and Nori F 2019 *Phys. Rev. B* **99** 020301
[10] Long Y, Zhang D, Yang C, Ge J, Chen H, and Ren J 2020 *Nat. Commun.* **11** 4716

[11] Sonner M M, Khosravi F, Janker L, Rudolph D, Koblmüller G, Jacob Z, and Krenner H J 2021 *Sci. Adv.* **7** eabf7414
[12] Long Y, Ge H, Zhang D, Xu X, Ren J, Lu M H, Bao M, Chen H, and Chen Y F 2020 *Natl. Sci. Rev.* **7** 1024
[13] Yang C, Tan Y T, Chen H, and Ren J 2021 *J. Appl. Phys.* **129** 135106
[14] Sasaki R, Nii Y, and Onose Y 2021 *Nat. Commun.* **12** 2599
[15] Bliokh K. Y, Punzmann H, Xia H, Nori F, and Shats M 2022 *Sci. Adv.* **8** eabm1295
[16] Alhaïtz L, Brunet T, Aristégui C, Poncelet O, and Baresch D 2023 *Phys. Rev. Lett.* **131** 114001
[17] Wang S, Zhang G, Wang X, Tong Q, Li J, and Ma G 2021 *Nat. Commun.* **12** 6125
[18] Toftul I, Bliokh K, Petrov M. I, and Nori F 2019 *Phys. Rev. Lett.* **123** 183901
[19] Calderin L, Hasan M A, Jenkins N G, Lata T, Lucas P, Runge K, and Deymier P A 2019 *Sci. Rep.* **9** 1
[20] Bliokh K Y and Bliokh Y P 2022 *Phys. Rev. E* **105** 065208
[21] Zhang X, Bauer G E, and Yu T 2020 *Phys. Rev. Lett.* **125** 077203
[22] Sato T, Yu W, Streib S, Bauer G E *et al.* 2021 *Phys. Rev. B* **104** 014403
[23] Zhao C, Li Y, Zhang Z, Vogel M, Pearson J E, Wang J, Zhang W, Novosad V, Liu Q, and Hoffmann A 2020 *Phys. Rev. Appl.* **13** 054032
[24] Rondón I and Leykam D 2019 *J. Phys.: Condens. Matter* **32** 104001
[25] Yang C, Zhang D, Zhao J, Gao W, Yuan W, Long Y, Pan Y, Chen H, Nori F, Bliokh K. Y, Zhong Z, and Ren J 2023 *Phys. Rev. Lett.* **131** 136102
[26] Deymier P, Runge K, Vasseur J O, Hladky A C, and Lucas P 2018 *J. Phys. B: At. Mol. Opt. Phys.* **51** 135301
[27] Nieves M, Carta G, Pagneux V, and Brun M 2020 *Int. J. Eng. Sci.* **156** 103365
[28] Li S, Kim I, Iwamoto S, Zang J, and Yang J 2019 *Phys. Rev. B* **100** 195102
[29] Geilen M, Kohl F, Nicoloiu A, Müller A, Hillebrands B, and Pirro P 2020 *Appl. Phys. Lett.* **117** 213501
[30] Chaplain G, De Ponti J, and Craster R 2022 *Phys. Rev. Lett.* **128** 064301
[31] Burns L, Bliokh K Y, Nori F, and Dressel J 2020 *New J. Phys.* **22** 053050
[32] Du H, Ma C, and Fang N X 2020 *Natl. Sci. Rev.* **7** 2
[33] Soper D E 2008 *Classical field theory* (Courier Dover Publications)
[34] Landau L D and Lifshitz E M 1986 *Theory of elasticity* (Elsevier)
[35] Landau L D and Lifschits E M 1975 *The Classical Theory of Fields* (Pergamon Press, Oxford)
[36] Donoghue J F, Golowich E, and Holstein B R 2014 *Dynamics of the standard model* (Cambridge University Press)
[37] Schwartz M D 2014 *Quantum field theory and the standard model* (Cambridge University Press)
[38] Ostrogradsky M 1850 *Mem. Acad. St. Petersbourg* **6** 385
[39] Motohashi H and Suyama T 2015 *Phys. Rev. D* **91** 085009
[40] Langlois D and Noui K 2016 *J. Cosmol. Astropart. Phys.* **2016** 034
[41] Dean D S, Miao B, and Podgornik R 2019 *J. Phys. A: Math. Theor.* **52** 505003
[42] Dean D S, Miao B, and Podgornik R 2020 *J. Phys. A: Math. Theor.* **53** 355005
[43] Pais A and Uhlenbeck G 1950 *Phys. Rev.* **79** 145
[44] Smilga A V 2006 *Phys. Lett. B* **632** 433
[45] Gibbons G, Pope C, and Solodukhin S 2019 *Phys. Rev. D* **100** 105008
[46] Hawking S W and Hertog T 2002 *Phys. Rev. D* **65** 103515
[47] Iyer R, Johnson C V, and Pennington J S 2010 *J. Phys. A: Math. Theor.* **44** 015403
[48] Iyer R, Johnson C V, and Pennington J S 2011 *J. Phys. A: Math. Theor.* **44** 375401
[49] Broer L 1975 *Appl. Sci. Res.* **31** 377
[50] Belinfante F J 1940 *Physica* **7** 449
[51] Chang T S 1948 *Math. Proc. Camb. Philos. Soc.* **44** 76
[52] Thielheim K O 1967 *Proc. Phys. Soc.* **91** 798
[53] Dürr H 1974 *Nuov. Cim. A* **22** 386
[54] Günther H 1982 *Ann. Phys.* **494** 445
[55] Lee T D and Wick G C 1969 *Nucl. Phys. B* **9** 209
[56] Lee T D and Wick G C 1970 *Phys. Rev. D* **2** 1033
[57] Grinstein B, O'Connell D, and Wise M B 2008 *Phys. Rev. D* **77** 025012
[58] Rizzo T G 2007 *JHEP* **06** 070
[59] Grinstein B, O'Connell D, and Wise M B 2009 *Phys. Rev. D* **79** 105019
[60] Cho I and Kwon O K 2010 *Phys. Rev. D* **82** 025013
[61] Chivukula R S, Farzinnia A, Foadi R, and Simmons E H 2010 *Phys. Rev. D* **82** 035015
[62] Landau L D and Lifshitz E M 2013 *Fluid Mechanics* (Elsevier)
[63] Campos L 2008 *Progress in Industrial Mathematics at ECMI 2006* (Springer) p. 73
[64] Laude V and Beugnot J C 2015 *New J. Phys.* **17** 125003
[65] de Wet J S 1948 *Math. Proc. Camb. Philos. Soc.* **44** 546
[66] Chang T S 1958 *Acta Phys. Sin.* **14** 300 (in Chinese)