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Reconciling Light Nuclei and Nuclear Matter: Relativistic *ab initio* Calculations

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An accurate and simultaneous *ab initio* prediction for both light nuclei and nuclear matter has been a long-standing challenge in nuclear physics, due to the significant uncertainties associated with the three-nucleon forces. In this Letter, we develop the relativistic quantum Monte Carlo methods for the nuclear *ab initio* problem, and calculate the ground-state energies of $A \leq 4$ nuclei using the two-nucleon Bonn force with an unprecedented high accuracy. The present relativistic results significantly outperform the nonrelativistic results with only two-nucleon forces. We demonstrate that both light nuclei and nuclear matter can be well described simultaneously in the relativistic *ab initio* calculations, even in the absence of three-nucleon forces, and a correlation between the properties of light $A \leq 4$ nuclei and the nuclear saturation is revealed. This provides a quantitative understanding of the connection between the light nuclei and nuclear matter saturation properties.

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Introduction. A major goal of nuclear theory is to explain the structure and properties of atomic nuclei in an *ab initio* approach, where the nuclei are solved by nuclear many-body method as individual nucleons interacting via nuclear forces fixed in free-space scattering. Already in the 1990s, various realistic models of two-nucleon (NN) forces had been constructed by fitting the NN scattering observables to high precision. Well-known and still widely used examples include the Argonne v_{18} (AV18)[1] and Bonn[2,3] potentials. Despite their high-quality description of NN scattering, these forces do not provide sufficient binding energies for ^{3}H and ^{4}He.[4,5] This leads to the discussion on the lack of a three-nucleon ($3N$) force, which is now widely accepted in nuclear *ab initio* calculations.

In the past two decades, many models of $3N$ forces have been constructed,[6–12] typically by adjusting to $A = 3$, 4 binding energies and/or other few-nucleon observables. The contribution of $3N$ force in the $A = 3$, 4 binding energies depends on the associated NN force and is around 10% in most cases. These $3N$ forces were then tested by calculating the bulk properties of heavier systems, among the most important ones being the saturation energy and density for infinite nuclear matter. However, the predicted saturation properties varied greatly depending on the specific model and fitting details of the $3N$ forces[13–15] (for a recent review, see Ref. [16]). Consequently, a simultaneous reproduction of the properties of light nuclei and nuclear saturation becomes a challenging problem in nuclear *ab initio* calculations.[16] This has led some groups to fit nuclear forces, besides the $A = 3$, 4 binding energies, to properties of medium-mass nuclei, e.g., in Ref. [17], or directly to the saturation properties of nuclear matter.[18]

Given the aforementioned background, it is crucial to investigate any potential aspects that may have been omitted in the current *ab initio* calculations. One such aspect is relativity, as most *ab initio* calculations are currently performed within the nonrelativistic framework. In fact, the typical momentum of the nucleons in nuclei is roughly 200 MeV, which is about 20% of the nucleon mass, so the relativistic correction is expected to be around $p^2/M^2 \simeq 4\%$. If this accounts for 4% of the binding energy, it is comparable to the contributions of $3N$ forces to the binding energies of $A = 3$, 4 nuclei. If it accounts for 4% of the kinetic or potential energies, then the relativistic effects may be even more significant, especially for nuclear saturation that results from delicate cancellations of large potential and kinetic energies.

The significance of the relativistic effects has been recognized in the saturation of nuclear matter with the relativistic Brueckner–Hartree–Fock (RBHF) calculations.[19–22] In contrast to the nonrelativistic results, the RBHF calculations can provide a satisfactory description of the saturation energy and density, even without introducing a $3N$ force. Such significant relativistic effects can be understood from the large scalar potential that affects the effective nucleon mass in the nuclear medium.[23] Similar calculations also provide reasonable results for the binding energies of several medium-mass nuclei with only NN forces.[23–25]

This raises a natural question of whether $A = 3$, 4 nuclei and nuclear matter saturation properties can be simultaneously described in the relativistic framework. While the RBHF theory is a good approximation for nuclear matter, it is less accurate for light nuclei due to the approximate treatment of the center-of-mass motion.[25] The existing relativistic three-body calculations[26,27] are specif-

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ically designed for ^{3}H using different scattering equations from the RBHF one and, thus, cannot match the nuclear matter results from the RBHF calculations. Moreover, their extensions to $A = 4$ or heavier systems are formidable. Therefore, a new relativistic *ab initio* method is highly demanded.

Among the variety of *ab initio* methods, the continuum Quantum Monte Carlo (QMC) methods [28–30] are notable for their high accuracy across various systems. The variational Monte Carlo and Green's function Monte Carlo (GFMC) can provide reliable solutions for nuclei up to $A = 12$. [4,11,31,32] The auxiliary-field diffusion Monte Carlo (AFDMC) additionally samples spin-isospin degrees of freedom, making possible the study of larger nuclei and nuclear matter. [15,33–35] Both the GFMC and AFDMC methods are based on the diffusion Monte Carlo algorithm that realizes the imaginary-time propagation stochastically and, thus, virtually exact. However, the existing QMC methods are only successful for nearly local forces in the coordinate space [28] and cannot be implemented in relativistic calculations, due to the nonlocality dictated by the covariance of nuclear forces.

In this Letter, we develop the relativistic QMC methods, including the relativistic variational Monte Carlo (RVMC) and relativistic diffusion Monte Carlo (RDMC), and the ground states of $A \leq 4$ nuclei are calculated using the Bonn forces with an unprecedentedly high accuracy. A quantitative understanding of the connection between the light nuclei and nuclear matter saturation properties is provided, and the crucial role of the relativistic effects is demonstrated.

The Bonn Force. The starting point is a covariant Lagrangian based on the meson-exchange picture, [2]

$$\mathcal{L}_{\rm int}^{\rm (s)} = g_{\rm s}\overline{\psi}\psi\varphi^{\rm (s)}, \quad \mathcal{L}_{\rm int}^{\rm (pv)} = -\frac{f_{\rm ps}}{m_{\rm ps}}\overline{\psi}\gamma^5\gamma^\mu\psi\partial_\mu\varphi^{\rm (ps)},$$
$$\mathcal{L}_{\rm int}^{\rm (v)} = -g_{\rm v}\overline{\psi}\gamma^\mu\psi\varphi_\mu^{\rm (v)} - \frac{f_{\rm v}}{4M}\overline{\psi}\sigma^{\mu\nu}\psi(\partial_\mu\varphi_\nu^{\rm (v)} - \partial_\nu\varphi_\mu^{\rm (v)}), \quad (1)$$

which includes the scalar (s) (σ, δ), the pseudovector (pv) (η, π), and the vector (v) (ω, ρ) meson-nucleon couplings, and in each channel it contains one isoscalar meson and one isovector meson. In the momentum space, the nucleon field is expanded in terms of the free Dirac spinors, neglecting the antinucleon degrees of freedom,

$$\psi(\boldsymbol{r}, t) = \int \frac{{\rm d}^3p}{(2\pi)^3}\sqrt{\frac{M}{E_p}}{\rm e}^{{\rm i}(\boldsymbol{p}\cdot\boldsymbol{r} - E_p t)}u(\boldsymbol{p})b_{\boldsymbol{p}}, \quad (2)$$

with $E_p = (\boldsymbol{p}^2 + M^2)^{1/2}$, $u(\boldsymbol{p})$ the positive-energy Dirac spinor (normalized as $\overline{u}u = 1$) and b_p the corresponding annihilation operator; the spin-isospin indices are suppressed for simplicity. We note that the antinucleon degrees of freedom have often been neglected in the relativistic description of low-energy nuclear systems, from the renowned relativistic mean-field model of Walecka [36] to the state-of-the-art relativistic chiral effective field theory, [37] because the virtual nucleon-antinucleon pairs are expected to play a minor role in low-energy nuclear systems. Then, the relativistic Hamiltonian with a relativistic kinetic part E_p and an interaction part V can be derived from the Dirac Hamiltonian. [38] The Bonn interaction is defined as the sum of one-meson-exchange amplitudes,

$$V(\boldsymbol{p}, \boldsymbol{p}') = \frac{M^2}{E_{p'}E_p}\sum_\varphi \overline{u}(\boldsymbol{p}')\Gamma_\phi u(\boldsymbol{p})\overline{u}(-\boldsymbol{p}')\Gamma_\phi u(-\boldsymbol{p}) \times \frac{[F_\phi(\boldsymbol{p}' - \boldsymbol{p})]^2}{m_\phi^2 + (\boldsymbol{p}' - \boldsymbol{p})^2}, \quad (3)$$

with $\boldsymbol{p}(\boldsymbol{p}')$ the initial (final) relative momentum of two nucleons, M the nucleon mass, and $E_p = (p^2 + M^2)^{1/2}$. Here, Γ_ϕ are the interaction vertices in the Lagrangian with F_ϕ the corresponding form factors, and m_ϕ the physical meson masses. The factor $M^2/E_{p'}E_p$ comes from the factor $\sqrt{M/E_p}$ in the expansion of nucleon fields [Eq. (2)]. The coupling strengths g_ϕ and the cutoff parameters in F_ϕ are determined by fitting NN scattering data and deuteron properties with the relativistic Thompson equation. Three parameter sets are available, denoted as A, B, and C. [2] The major difference between them is the cutoff value in the πNN form factor and, thus, the strength of tensor force.

The Relativistic QMC Methods. For the ground states of nuclei, we solve the many-body equation in the coordinate space $\{\boldsymbol{R} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_A)\}$

$$\hat{H}\Psi(\boldsymbol{R}) = \left(\sum_{i=1}^A \hat{K}_i + \sum_{i<j}^A \hat{V}_{ij}\right)\Psi(\boldsymbol{R}) = E\Psi(\boldsymbol{R}), \quad (4)$$

including the relativistic kinetic part $\hat{K}_i$ and the NN interaction part $\hat{V}_{ij}$,

$$\hat{K}_i\Psi(\boldsymbol{R}) = \frac{M^2}{2\pi^2}\int {\rm d}^3r_i' \frac{K_2(Mr_i')}{r_i'^2}[\Psi(\boldsymbol{R}) - \Psi(\boldsymbol{R} + \boldsymbol{r}_i')],$$
$$\hat{V}_{ij}\Psi(\boldsymbol{R}) = \int {\rm d}^3r_{ij}' V(\boldsymbol{r}_{ij}, \boldsymbol{r}_{ij}')\Psi(\boldsymbol{R} + \boldsymbol{r}_{ij}' - \boldsymbol{r}_{ij}). \quad (5)$$

Here, K_2 is the modified Bessel function of order 2 [39] and $V(\boldsymbol{r}_{ij}, \boldsymbol{r}_{ij}')$ is the coordinate-space representation of the Bonn potential with $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$. The Coulomb force between finite-size protons are also considered. [1] The present Hamiltonian differs from the nonrelativistic ones solved in the previous QMC methods, due to the strong nonlocality of the relativistic kinetic energy and the NN interactions in the coordinate space.

Within the RVMC method, the ground state is solved by minimizing the energy expectation for a given form of trial function $\Psi_{\rm T}$

$$E[\Psi_{\rm T}] = \frac{\int {\rm d}\boldsymbol{R}\Psi_{\rm T}^\dagger(\boldsymbol{R})\hat{H}\Psi_{\rm T}(\boldsymbol{R})}{\int {\rm d}\boldsymbol{R}\Psi_{\rm T}^\dagger(\boldsymbol{R})\Psi_{\rm T}(\boldsymbol{R})}. \quad (6)$$

The integral is evaluated with the standard Metropolis Monte Carlo sampling. [40] Different from the case with local interactions, the nonlocal integrals in Eq. (5) have to be evaluated for each sampled $\boldsymbol{R}$. This is carried out with a combination of the stochastic spherical designs for the angular part [41,42] and the Gaussian quadrature for the radial part.

The trial wave function is built with the neural-network ansatz,

$$\Psi_{\rm T}(\boldsymbol{R}) = \prod_{i<j} f_{\rm c}(r_{ij}) \left[1 + \sum_{i<j} \sum_{p=2}^{6} u^p(r_{ij}) O_{ij}^p\right] \Phi, \quad (7)$$

where $f_{\rm c}$ and $u^{p=2\text{-}6}$ are pair-correlation functions and O_{ij}^p are two-body spin-isospin operators

$$O_{ij}^{p=2\text{-}6} = \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j,\ \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j,\ \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j,\ S_{ij},\ S_{ij} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j, \quad (8)$$

with $S_{ij} = 3\boldsymbol{\sigma}_i \cdot \boldsymbol{r}_{ij} \boldsymbol{\sigma}_j \cdot \boldsymbol{r}_{ij} - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j r_{ij}^2$. The Slater determinant Φ is taken to be $\mathcal{A}(\uparrow_n \uparrow_p)$ for ^{2}H, $\mathcal{A}(\uparrow_n \downarrow_n \uparrow_p)$ for ^{3}H, and $\mathcal{A}(\uparrow_n \downarrow_n \uparrow_p \downarrow_p)$ for ^{4}He, with $\mathcal{A}$ being the antisymmetrization operator. Two separate feed-forward neural networks with 1 and 5 outputs are used to represent $f_{\rm c}$ and u^p, respectively. They both have two hidden layers, each with a width of 16 nodes. The bound-state boundary conditions are ensured via multiplying the trial wave function by a Gaussian function.[38,43] The parameters in the feed-forward neural networks are determined by minimizing the energy expectation iteratively.[44]

Figure 1 depicts the energy expectation of ^{2}H as a function of the iteration in the RVMC calculation with Bonn A. The converged results perfectly reproduce the experimental energy of ^{2}H within 1 keV. This is not surprising since the ^{2}H energy was used in the fit of the Bonn A potential[2] and the present trial wave function is, in principle, exact for two-body systems. Note that, due to the statistical fluctuations of the Monte Carlo method, some calculated energies may be lower than the ground state energy. For ^{3}H and ^{4}He, however, the RVMC calculations provide slightly underbound solutions, due to the missing three- and four-body correlations in the trial wave function. The missing energy can be evaluated by a subsequent RDMC calculation.

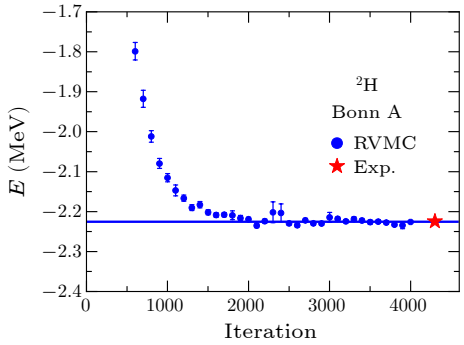

Fig. 1. The energy of ^{2}H as a function of the iteration in the RVMC calculation with Bonn A. The converged energy (solid line) is compared to the experimental value (star).

The RDMC method projects out the ground state $|\Psi_0\rangle$ via imaginary-time propagation starting from the trial wave function $|\Psi_{\rm T}\rangle$,

$$|\Psi_0\rangle \propto \lim_{\tau\to\infty} {\rm e}^{-\hat{H}\tau} |\Psi_{\rm T}\rangle. \quad (9)$$

It is realized by a sequence of short-time propagators $G_{\alpha\beta}^{\Delta\tau}(\boldsymbol{R}', \boldsymbol{R}) = \langle \boldsymbol{R}'\alpha | {\rm e}^{-\hat{H}\Delta\tau} | \boldsymbol{R}\beta \rangle$, where $\boldsymbol{R}$, $\boldsymbol{R}'$ are the coordinates before and after the propagation step, and α, β stand for spin-isospin states. The short-time propagator can be split into the relativistic free propagator $G_{\alpha\beta}^{\Delta\tau}$ and the interaction term $I_{\alpha\beta}^{\Delta\tau}$,[28]

$$G_{\alpha\beta}^{\Delta\tau}(\boldsymbol{R}', \boldsymbol{R}) = G_0^{\Delta\tau}(\boldsymbol{R}', \boldsymbol{R}) I_{\alpha\beta}^{\Delta\tau}(\boldsymbol{R}', \boldsymbol{R}), \quad (10)$$

where

$$G_0^{\Delta\tau}(\boldsymbol{R}', \boldsymbol{R}) = \prod_{i=1}^{A} \frac{M^2}{2\pi^2} \frac{\Delta\tau {\rm e}^{M\Delta\tau}}{(\boldsymbol{r}_i' - \boldsymbol{r}_i)^2 + \Delta\tau^2} \times K_2(M\sqrt{(\boldsymbol{r}_i' - \boldsymbol{r}_i)^2 + \Delta\tau^2}). \quad (11)$$

To evaluate the interaction term $I_{\alpha\beta}^{\Delta\tau}(\boldsymbol{R}', \boldsymbol{R})$, we define a local part of the NN interaction, $\bar{V}(\boldsymbol{r}) = \int {\rm d}^3\boldsymbol{r}' V(\boldsymbol{r}, \boldsymbol{r}')$, which dominates the potential energy in the RVMC calculations (about 95%). The nonlocality in the propagator is considered effectively by replacing $\hat{V} - \bar{V}$ with $a\bar{V}$, where a is fixed by $\langle \hat{V} - \bar{V} \rangle \simeq \langle a\bar{V} \rangle$ with $\langle \cdot \rangle$ denoting the average over the trial wave function. This replacement is equivalent to using an effective local interaction in the RDMC calculations and treating the remaining nonlocal part perturbatively. After the replacement, $I_{\alpha\beta}^{\Delta\tau}(\boldsymbol{R}', \boldsymbol{R})$ can be evaluated with the standard Trotter formula.[45] Note that a similar recipe has been adopted in the recent AFDMC calculations to treat chiral $3N$ forces.[35,46] Such a recipe is justified in the present calculations by the fact that the RDMC accounts for only a small part of the total energy. We also verified that the obtained ground-state energies are insensitive to small variations of parameter a.

In the present RDMC calculations, a set of walkers is initially sampled from the RVMC wave function, and then a branching random walk algorithm is carried out.[28] The imaginary-time step is taken to be $\Delta\tau = 5 \times 10^{-5}\,{\rm MeV}^{-1}$. For each step, the diffusion and branching of the walkers are carried out according to the free and interaction terms of the short-time propagator in Eq. (10), respectively. The importance sampling is carried out as in the GFMC calculations.[47,48] The mixed estimate of the energy is calculated every 20 propagation steps.

Figure 2 depicts the energy of ^{4}He as a function of the imaginary time in the RDMC calculation with Bonn A. The energy drops rapidly within $\tau = 2 \times 10^{-3}\,{\rm MeV}^{-1}$

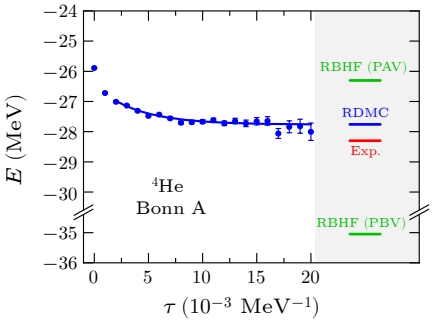

Fig. 2. The energy of ^{4}He as a function of the imaginary time in the RDMC calculations with Bonn A. The solid curve is an exponential fit to the Monte Carlo results. The statistical error of the RDMC ground-state energy, extracted from the exponential fit, is smaller than the symbol. Also shown are the RBHF results[25] obtained with the projection before/after variation (PBV/PAV) approximations.

by filtering out high excitations. Following the previous DMC calculations,[46,47] the ground-state energy is extracted from an exponential fit to the Monte Carlo results after $\tau = 2 \times 10^{-3}\,\mathrm{MeV}^{-1}$, as shown by the solid curve. The extracted ground-state energy of ^{4}He, with statistical error, is $E = -27.78(6)\,\mathrm{MeV}$. The imaginary-time evolution lowers the RVMC energy by about 2 MeV.

We compare the present ground-state energy against the previous RBHF results with the same interaction Bonn A.[25] The RBHF method is a mean-field-based method, so the results are drastically influenced by the treatment of the center-of-mass motion, i.e., the projection before or after variation approximations. In contrast, the present RDMC energy well reproduces the experimental value $E = -28.3\,\mathrm{MeV}$ within $\sim 2\%$. This highlights the importance of the present virtually exact calculations. In the future, it should be interesting to compare the present results with Faddeev–Yakubovsky or no-core shell model calculations using the same Bonn forces.

Relativistic ab initio Predictions. Figure 3 depicts the ^{3}H and ^{4}He ground-state energies obtained from the RDMC calculations with Bonn A, B, and C potentials. The nonrelativistic results from virtually exact *ab initio* methods[49–52] with various NN forces are also shown for comparison. They cover nearly all existing classes of realistic NN forces, including AV18,[1] CD-Bonn,[3] Nijmegen I and II,[53] and chiral NN forces up to the third order,[54] the fourth order,[55,56] and the fifth order.[57,58] For the chiral forces, the cutoff is either $\Lambda = 500\,\mathrm{MeV}$ in momentum space or $R_0 = 1.0\,\mathrm{fm}$ in the coordinate space.

The nonrelativistic results are systematically underbound for both ^{3}H and ^{4}He, so the $3N$ forces are often introduced and adjusted to the ^{3}H or ^{4}He ground-state energies. In such an *ad hoc* way, both the ground-state energies of ^{3}H and ^{4}He can be reproduced since they are correlated (Tjon line[59]). In contrast to the nonrelativistic case, the relativistic results are much closer to the experimental values without any adjustment, especially for the results obtained with Bonn A. In addition, the correlation between the ^{3}H and ^{4}He energies, namely the Tjon line, has also been revealed in the present relativistic calculations.

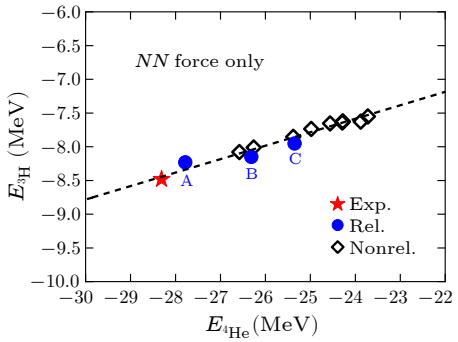

Fig. 3. The ground-state energies of ^{3}H and ^{4}He obtained from the present RDMC calculations with Bonn A, B, and C potentials, compared to the nonrelativistic results with various NN forces (taken from Refs. [49–52]). The dashed line is a linear fit to the nonrelativistic results (Tjon line).

The charge radii are calculated from the point-proton radii taking into account the nucleons' finite size, as described in Ref. [60]. The charge radii of ^{3}H for Bonn A, B, C potential are 1.78(1), 1.77(1), 1.79(1) fm, respectively. The charge radii of ^{4}He for Bonn A, B, C potential are 1.69(1), 1.71(1), 1.73(1) fm, respectively. The experimental values are 1.7591(363) fm for ^{3}H and 1.6755(28) fm for ^{4}He.[61] The calculated radii agree with the experimental values at percentage level.

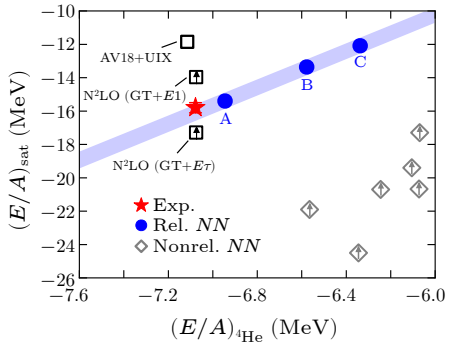

Fig. 4. Energy per nucleon for the ^{4}He ground state and the saturation point of symmetric nuclear matter obtained from various *ab initio* calculations. The experimental value of ^{4}He energy and the empirical value of the saturation energy[62] are shown for comparison. The blue band is shown to guide the eyes. The upper arrow indicates that the saturation density is overestimated by over 20% compared to the empirical value.[62] The nonrelativistic results for ^{4}He and nuclear matter are respectively taken from Refs. [10,49–52] and Refs. [15,63–65], and the relativistic results for the nuclear matter are from the RBHF calculations.[22]

Figure 4 depicts various *ab initio* results for the energy per nucleon of the ^{4}He ground state and the saturation point of symmetric nuclear matter. With only NN forces, all the nonrelativistic calculations underbind the ^{4}He ground state, but overbind the nuclear saturation energy and overestimate the corresponding saturation density. The ^{4}He energy can be reproduced by adjusting the $3N$ forces as in Refs. [4,10]. However, with the same $3N$ forces, an accurate reproduction of the saturation properties is still a challenge. As seen in Fig. 4, the results obtained with AV18+UIX[63] remarkably underbind the saturation energy, even though the saturation density is well reproduce through fitting. The results with the N^{2}LO chiral forces[15] somehow improve the saturation energy, depending on the choice of the $3N$ operator form ($E1$ or $E\tau$), while the corresponding saturation density always comes out too high.

On the contrary, the relativistic calculations with Bonn A provide the only result that can simultaneously reproduce the ^{4}He energy and the nuclear saturation energy and density. One should keep in mind that the calculation is free from any adjustment since Bonn A is a high-precision two-nucleon potential fitted only to two-body observables. For the results with Bonn B and C, the discrepancies against the experimental values are larger, mainly due to their larger tensor force components, which imply more repulsion in few- and many-body systems. However,

in contrast to the nonrelativistic results, the relativistic results with Bonn A, B, and C exhibit regular correlations, i.e., the energy discrepancies for ^{4}He and nuclear saturation are in the same direction. This provides a quantitative understanding of the connection between the light nuclei and nuclear matter saturation properties. It implies that one could reconcile light nuclei and nuclear matter in relativistic *ab initio* calculations with only NN force.

At this point, one may ask how large the $3N$-force effects should be in the relativistic *ab initio* calculations. A modern understanding of $3N$ forces is provided by chiral effective field theory, where the contributions of NN forces are leading order, and the contributions of $3N$ forces are higher order corrections[66–68] and are expected to be small. Note that the Bonn NN force adopted in the present work is based on a phenomenological meson-exchange picture, which does not allow systematic improvements as well as the construction of consistent $3N$ forces. Recently, significant progress has been made in constructing relativistic chiral forces in the relativistic frameworks.[37,69–71] The relativistic QMC methods developed in this work pave the way for accurate *ab initio* calculations of light and medium-mass nuclei with relativistic chiral NN forces, as well as the construction of relativistic chiral $3N$ forces.

Summary. We develop the relativistic variational Monte Carlo and relativistic diffusion Monte Carlo methods, and calculate the ground-state energies for $A \leq 4$ nuclei using the Bonn force with an unprecedented high accuracy. In contrast to the nonrelativistic results with NN forces that underbind $A = 3$, 4 nuclei, the relativistic results are much closer to the experimental values without any adjustment, especially for the results obtained with Bonn A. From the present results of $A \leq 4$ nuclei and the previous results of nuclear matter with the same Bonn NN forces,[22] a correlation between the energies of light $A \leq 4$ nuclei and nuclear saturation is revealed in the relativistic *ab initio* calculations. It highlights the importance of relativistic effects in a simultaneous description of light nuclei and nuclear matter. It also provides a quantitative understanding of the connection between the light nuclei and nuclear matter saturation properties, which has been an outstanding problem in nuclear *ab initio* calculations for decades.

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