Virtual Dirac Monopoles underlying the Nontrivial Phases of Rogue Waves

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We uncover the virtual monopoles underlying the nontrivial phases of the one-dimensional nonlinear excitations of rogue waves by extending the Dirac magnetic monopole theory to a complex plane. We find that the density zeros of the nonlinear waves on the extended complex plane constitute the virtual monopole fields with a quantized flux of elementary π. We then explain the exotic properties of rogue waves by means of a virtual monopole collision mechanism and find that the maximum amplitude amplification ratio and multiple phase steps of the high-order rogue waves are closely related to the number of their contained monopoles. These results open a new avenue for studying topological properties of nonlinear waves and provide an alternative way to understand their dynamics.

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1. Introduction. In the quantum world, phase serves both as a mathematical element in theoretical constructs [1] and as the physical core of experimental observations and technological implementations. [2,3] The phase singularities or non-integrable phase factors closely related to the nodes of wave functions are of particular interest in which the monopoles with quantized magnetic charges are shown to emerge at the endpoints of the nodal lines, as first claimed by P. A. M. Dirac. [4] The concept of Dirac monopoles also provides a fertile playground for theoretical ideas. [5–7] For instance, around an energy degeneracy point there is a Dirac virtual monopole vector potential field that gives rise to the celebrated Berry geometric phase for an electron in its cyclic adiabatic evolution [8–12] and is responsible for the topological properties in various physical systems. [13–16]

Recently, the nonlinear systems, i.e., their dynamics governed by nonlinear Schrödinger equations (NLSE), attract much attention for their important applications in the practical physical systems such as Bose–Einstein condensates and nonlinear optics. [17–28] Among many nonlinear excitations, rogue waves (RWs) are interesting nonlinear excitations characterized by "appearing from nowhere and disappearing without a trace", [29–34] and have been observed experimentally in these physical systems. [35–38] Even though a lot of theoretical works have been done to understand the formation mechanism and dynamics of the RWs, [39–43] their phase singularity and its connection with the exotic dynamics are not fully understood. [44,45] However, when directly applying the above Dirac's phase singularity theory to the nonlinear waves of RWs, we find that the non-integrable phase factor cannot appear because the spatial coordinate is one-dimensional (1D) so that the noncommutative property of phase gradients [4] has no meaning.

In this Letter, we report for the first time that the striking phases of RWs contain virtual magnetic monopoles.By extending Dirac's theoretical formulation to investigate the density (the modulus of wave functions) zeros of RWs in an extended complex coordinate plane, we find that a nth-order RW contains $n(n+1)$ pairs of monopoles with opposite charges, and that the collision of these monopoles and the reconnection of the corresponding vector field will lead to energy conversion from interaction energy to kinetic energy and are responsible for the exotic properties of RWs. These results on RWs motivate us to develop a general method for uncovering the topological vector field underlying nontrivial phases of 1D nonlinear waves.

2. Virtual Dirac Monopoles and Topological Vector Potential. We choose one of the simplest models, i.e., the scalar NLSE $\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + g|\psi|^2\psi$, to demonstrate our theory. The NLSE can be used to describe the dynamics of nonlinear waves in atomic gases, [46] plasma, [47] water waves, [36] and ferromagnetic materials, [48] and it plays important roles in both integrability theory [49,50] and optical communications. [51] It has a nonlinear wave solution $\psi(x,t)$, which can be a dark soliton (for $g>0$), a bright soliton, a RW, a breather (for $g<0$), or some superposition thereof. Nevertheless, the topological properties associated with these 1D nonlinear waves still need further research. [52–55]

In fact, almost a century ago, P. A. M. Dirac discussed the phase change and its underlying topological property for a general wave function $\psi(\boldsymbol{r},t) = \sqrt{n}\mathrm{e}^{\mathrm{i}\phi}$. [4] He introduced a vector potential field through the phase gradi-

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ent, i.e., $\boldsymbol{A} = (A_x, A_y, A_z) = \frac{\partial\phi}{\partial x}\boldsymbol{e}_x + \frac{\partial\phi}{\partial y}\boldsymbol{e}_y + \frac{\partial\phi}{\partial z}\boldsymbol{e}_z$, and found that the noncommutative property of $\frac{\partial A_\mu}{\partial \nu} \neq \frac{\partial A_\nu}{\partial \mu}$ ($\nu, \mu = x, y, z$ and $\nu \neq \mu$) would give rise to a non-integrable phase factor. This striking property is caused by the emergence of the nodal line, where the wave function vanishes and its phase does not have meaning. More importantly, Dirac claimed that in calculating the change in phase round a closed curve, one has to take into account the influence of the endpoint of the nodal line, which acts as a magnetic monopole of charge $\pm 1/2$ with a vector potential taking the form of $\boldsymbol{A}_{3D} = \frac{y\boldsymbol{e}_x - x\boldsymbol{e}_y}{2r(r \pm z)}$ with $r = \sqrt{x^2 + y^2 + z^2}$.[4]

However, when directly applying the above Dirac's theory to the nonlinear waves such as RWs, we find that the non-integrable phase factor cannot appear because the spatial coordinate is 1D, so that the noncommutative property has no meaning. It is noted that some new phenomena would remain hidden if one were to focus on real physical parameters.[56] Two famous examples are the Lee–Yang zeros[57] and Fisher zeros,[58] reported for imaginary magnetic fields and imaginary temperatures, respectively. The dynamical behavior of a quantum-mechanical system is determined in part by what happens in the complex plane.[59] Here, we attempt to extend the real coordinate variable x to a complex variable $Z = x + \mathrm{i}y$ to investigate the complex density function $n(Z)$ and phase function $\phi(Z)$. Then we define a complex vector potential also through the phase gradient, i.e.,

$$\begin{aligned}\mathscr{A}_\mathrm{c} &= \frac{\partial\phi(Z)}{\partial x}\boldsymbol{e}_x + \frac{\partial\phi(Z)}{\partial y}\boldsymbol{e}_y \\ &= \frac{\partial\phi(Z)}{\partial Z}(\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y). \end{aligned} \tag{1}$$

The vector potential is now defined on a two-dimensional (2D) plane of the variables of x and y, and we focus on the real part of the above vector potential, i.e., $\boldsymbol{A} = \mathrm{Re}[\mathscr{A}_\mathrm{c}]$. Following Dirac's spirit, we then pay attention to the singularity properties of the vector potential field of $\boldsymbol{A} = (A_x, A_y) = (\mathrm{Re}[\partial\phi/\partial \mathrm{Z}], -\mathrm{Im}[\partial\phi/\partial \mathrm{Z}])$ and their noncommutative property of $\frac{\partial A_x}{\partial y} \neq \frac{\partial A_y}{\partial x}$. In the 2D situation, the nodal line will reduce to some scattered points corresponding to density zeros (i.e., $n(Z_n) = 0$ with $Z_n = x_n + \mathrm{i}y_n$) where the wave function vanishes and its phase has no meaning (phase singularity).

More interestingly, we claim that for calculating the change in phase along the x axis, the density zeros (i.e., denoted by $\boldsymbol{r}_N = x_N\boldsymbol{e}_x + y_N\boldsymbol{e}_y$) of these nonlinear waves on an extended complex plane can constitute virtual magnetic monopole fields with a quantized flux of elementary $\Omega = \pi$, i.e., the charge of $\mu = 1/2$. The corresponding vector potential takes the following explicit form:

$$\boldsymbol{A} = \mathrm{Re}[\mathscr{A}_\mathrm{c}] = \sum_N \frac{\pm\mu[(x - x_N)\boldsymbol{e}_y - (y - y_N)\boldsymbol{e}_x]}{(x - x_N)^2 + (y - y_N)^2}, \tag{2}$$

where $\pm$ is the sign of monopole. Near each singular density zero, the above 2D vector potential field is exactly Dirac's monopole field of $\boldsymbol{A}_{3\mathrm{D}}$ with $z = 0$ set therein. The corresponding magnetic field will be zero everywhere except at those singular points; that is, $\boldsymbol{B} = \nabla \times \boldsymbol{A} = \boldsymbol{e}_z \sum_N \pm\Omega\delta(x - x_N, y - y_N)$.

The phase variations of such a nonlinear wave between x_1 and x_2 can then be expressed by the line integral of the vector potential, namely,

$$\begin{aligned}\Delta\phi = \phi(x_2) - \phi(x_1) &= \int_{x_1}^{x_2} \boldsymbol{A}(x, y = 0) \cdot \boldsymbol{e}_x \mathrm{d}x \\ &= \int_{(x_1,0)}^{(x_2,0)} \boldsymbol{A}(x, y) \cdot \mathrm{d}\boldsymbol{l} - \sum n\pi, \end{aligned} \tag{3}$$

where $\boldsymbol{l}$ is an arbitrary curve that connects $(x_1, 0)$ and $(x_2, 0)$, and $\sum n\pi$ denotes the total magnetic flux enclosed by the curve $\boldsymbol{l}$ and the line connecting x_1 and x_2 in the (x, y) plane.

Note that the above complex extension is done separately for the density and phase, in contrast to the previous works that directly replaces the x variable with a complex variable z in the wavefunction (e.g., see Ref. [60]) to investigate the complex extension of the dynamical equation. On the other hand, we find that the density zeros correspond to the exceptional points of the Bogoliubov–de-Gennes (BdG) excitation spectrum.[61] By introducing an arbitrary Fourier mode perturbation of the wave solution, $\psi = \psi_0(x, t) + f_1(t)\exp[\mathrm{i}kx] + f_2^*(t)\exp[-\mathrm{i}kx]$, we can readily obtain the instantaneous BdG eigenvalues of $E_\pm = \pm\sqrt{\frac{k^4}{4} + 3g^2n^2 + 2k^2gn} = \pm\sqrt{3g^2n^2}$ at $k = 0$. The density zeros correspond to the degenerate points or so-called exceptional points[62] since the BdG Hamiltonian is non-Hermitian. The topology of exceptional points can be depicted by the eigenvalue winding number that has been calculated to be 2 when ψ_0 takes on following RW form, in contrast to the square root singularity case.[62]

It is interesting to compare the above results with the Aharonov–Bohm effect,[63] which predicts a topological phase when an electron moves along a closed path around a solenoid. A 1D nonlinear wave moving on the real axis cannot see the magnetic fields scattered on the complex plane; however, it will acquire a phase due to the presence of the vector potential. The evolution of such a nonlinear wave can be understood from the transformed equation based on the topological vector potential, $\mathrm{i}\partial_t\psi_1 = \{\frac{[\hat{p}_x + \boldsymbol{A}(x,y=0)\cdot\boldsymbol{e}_x]^2}{2} + (g|\psi_1|^2 + \frac{\partial\int\boldsymbol{A}(x,y=0)\cdot\boldsymbol{e}_x\mathrm{d}x}{\partial t})\}\psi_1$ with a transformation $\psi_1 = \psi\exp[-\mathrm{i}\int\boldsymbol{A}(x, y = 0)\cdot\boldsymbol{e}_x\mathrm{d}x]$. The effective magnetic field and electric field can be derived as $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and $\boldsymbol{E} = -\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\frac{\partial\int\boldsymbol{A}(x,y=0)\cdot\boldsymbol{e}_x\mathrm{d}x}{\partial t}$, respectively.[64] In this sense, the phase variations of these nonlinear waves can be viewed as a 1D counterpart to the Aharonov–Bohm phase. Usually, the topological vector potential exhibits time dependence, which provides an alternative way to understand the dynamics of such nonlinear waves. We would like to choose RWs to demonstrate our theoretical results specifically.

3. Topological Vector Potentials for Rogue Waves. The above NLSE with $g = -1$ has a Peregrine breather which is also called the first-order RW (FRW) solution on a uniform background,[29] $\psi = [1 - \frac{4(1+2\mathrm{i}t)}{4t^2+4x^2+1}]\mathrm{e}^{\mathrm{i}t}$. The temporal evolution of the RW amplitude, depicted in Fig. 1(a),

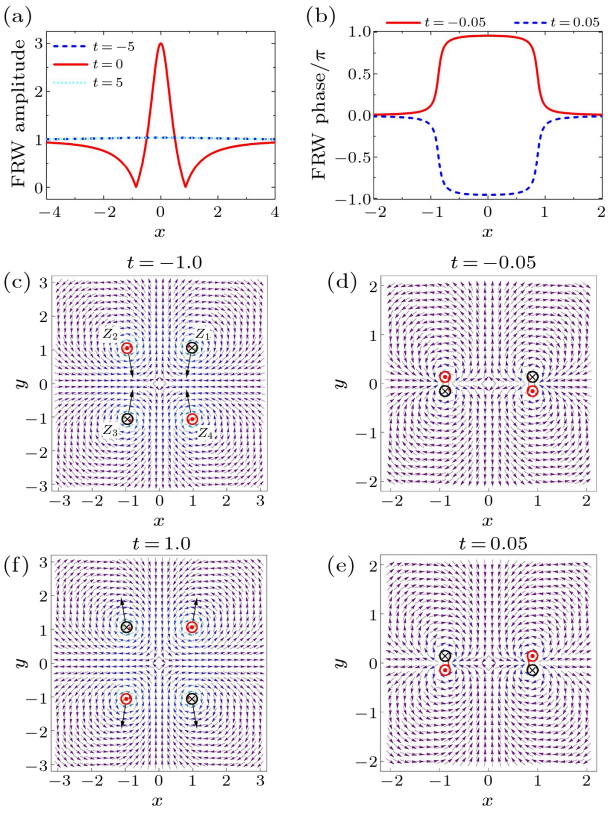

Fig. 1. (a) Amplitude distribution. (b) Phase distribution. (c)–(f) Evolution of the topological vector potential for a Peregrine breather (the first-order RW). The Dirac monopoles with positive and negative charges (i.e., $\pm\frac{1}{2}$) are indicated by $\odot$ and $\otimes$, respectively.

shows that the wave density remains almost constant until $t=-5$, after which sudden growth occurs. At $t=0$, the amplitude amplification ratio (defined as the peak amplitude divided by the background amplitude) reaches its maximum value of 3. Subsequently, the RW quickly decays, and the wave density recovers to be nearly constant. The amplitude peak is located at $x=0$, and on either side of the peak, there are two valleys at $x=\pm a_0$ ($a_0=\frac{\sqrt{3}}{2}$). Interestingly, phase jumps accompany the rise in amplitude. In Fig. 1(b), we see that there is a π phase jump corresponding to each amplitude valley,[44,45] and the direction of the phase jump at $x=-a_0$ suddenly inverts to $-\pi$ slightly after the moment when the maximum amplitude peak emerges.

The above π phase jumps and abrupt inversion are not fully understood.[44,45] In particular, analytical studies have indicated that the amplitude amplification ratios for RWs of different orders are subject to certain limits,[65] but no physical mechanism for these ceiling values has been discovered. By the way, linear modulational instability (MI) analysis describes well the growth of weak perturbations.[24,33,34,39,40] Nonlinear MI theory was further developed to address nonlinear behavior of MI beyond its growth stage.[66,67] A truncated three-wave model on frequency was suggested to describe MI dynamics effectively.[68,69] Those developments could be used to understand the RW dynamics qualitatively, but they cannot explain the striking phases and these ceiling values. Here, we attempt to elucidate these issues with the help of our developed topological vector potential theory.

It has four density zeros, i.e., $Z_{1,3}=\pm(a+\mathrm{i}b)$ and $Z_{2,4}=\pm(-a+\mathrm{i}b)$, where $a=\frac{2\sqrt{2}|t|}{\sqrt{4t^2+\sqrt{16t^4+40t^2+9}-3}}$ (the absolute operation in $|t|$ is introduced for marking monopole conveniently) and $b=\frac{\sqrt{4t^2+\sqrt{16t^4+40t^2+9}-3}}{2\sqrt{2}}$. According to Eq. (2), the vector potential $\boldsymbol{A}$ underlying the FRW takes an explicit form of $\boldsymbol{A}(x,y)=\sum_{N=1,\ldots,4}\frac{(-1)^N\mu[(x-x_N)\boldsymbol{e}_y-(y-y_N)\boldsymbol{e}_x]}{(x-x_N)^2+(y-y_N)^2}$. We can see that the vector potential $\boldsymbol{A}$ is composed of two pairs of Dirac monopoles, and in each pair, the two monopoles have charges of 1/2 with opposite signs. The temporal evolution of the potential is shown in Figs. 1(c)–1(f). The monopoles with opposite charges in each pair approach each other, collide elastically in the vertical direction at time $t=0$ with speeds of $\mathrm{d}b/\mathrm{d}t=2/\sqrt{3}$ and $\mathrm{d}a/\mathrm{d}t=0$, and then bounce back after exchanging their charges. As $t\to\pm0$, $a\to a_0$ and $b\to0$. The vector potential on the real axis then takes the following form:

$$\begin{aligned}\lim_{t\to\pm0}\boldsymbol{A}=&\lim_{b\to0}\Big\{\frac{-\mu[(x\pm a_0)\boldsymbol{e}_y+b\boldsymbol{e}_x]}{(x\pm a_0)^2+b^2}+\frac{\mu[(x\pm a_0)\boldsymbol{e}_y-b\boldsymbol{e}_x]}{(x\pm a_0)^2+b^2}\\&+\frac{\mu[(x\mp a_0)\boldsymbol{e}_y+b\boldsymbol{e}_x]}{(x\mp a_0)^2+b^2}+\frac{-\mu[(x\mp a_0)\boldsymbol{e}_y-b\boldsymbol{e}_x]}{(x\mp a_0)^2+b^2}\Big\}\\=&[-\pi\delta(x\pm a_0)+\pi\delta(x\mp a_0)]\boldsymbol{e}_x.\end{aligned}\tag{4}$$

The line integral of the above vector potential can explain the π phase jumps shown in Fig. 1(b). The phase gradient determines the density flow, and the change in the phase distribution can provide an understanding of the growth and decay of RWs.[44] The collision of the monopoles leads to a sudden rise in wave amplitude. The exchange of monopole charges after collision can well explain the striking phase reversal that induces the RW's rapid decay. This provides another possible way to understand RW's dynamical properties, serving as a good supplement for previously known mechanisms.[39–42]

Higher-order RWs admit higher amplitude peaks, more density valleys (or humps) and multiple phase steps, as shown in Fig. 2.[20,44] For a second-order RW, the maximum amplitude amplification ratio is 5. There are five phase steps distributed symmetrically with respect to the $x=0$ axis, each of which is associated with a phase jump of $\pm\pi$ [see Fig. 2(a)]. The vector field is composed of six pairs of Dirac monopoles, as shown in Fig. 2(b). They can be divided into two classes: the four pairs that are closer to the x-axis collide with each other on the x-axis, leading to four π phase jumps, whereas the other two pairs (upper and lower pairs) collide on the imaginary axis, mainly contributing to a sudden rise in the wave amplitude. For a third-order RW, the maximum amplitude amplification ratio is 7, and there are seven phase steps distributed symmetrically with respect to the $x=0$ axis [see Fig. 2(c)]. The vector field is composed of twelve pairs of monopoles, as shown in Fig. 2(d). The paired monopoles collide and merge when the RW reaches its highest peak. Among them, six pairs collide and merge on the real axis,

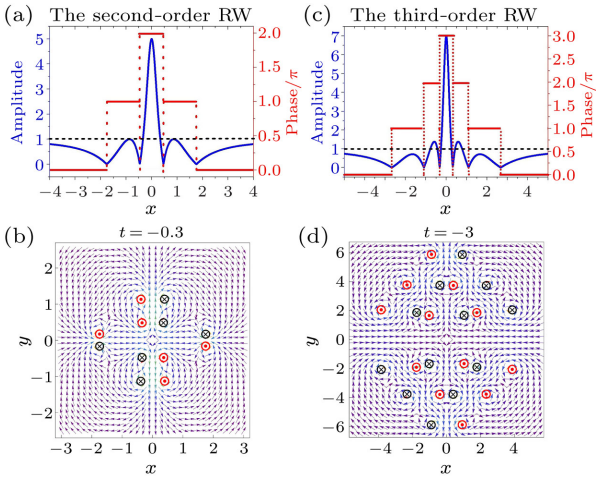

Fig. 2. (a) Amplitude and phase distributions for a second-order RW at the moment when the wave amplitude reaches its highest peak ($t = 0$). (b) Monopole vector potential at the moment slightly before the monopole collision ($t = -0.3$). (c) Amplitude and phase distributions for a third-order RW at the moment when the wave amplitude reaches its highest peak ($t = 0$). (d) Monopole vector potential at the moment before the Dirac monopole collision ($t = -3$).

and the other six pairs collide on the imaginary axis or in other locations on the complex plane. The monopole pairs that do not collide on the real axis do not contribute to multiple phase jumps, but they do influence the peak amplitude and the energy transfer involving RWs.

4. Energy Transfer and Topological Vector Potential Reconnection. The energy of a RW can be written as $\int_{-\infty}^{+\infty}[\frac{1}{2}|\partial_x\psi|^2 - \frac{1}{2}(|\psi|^2-1)^2]\mathrm{d}x$, where the first term is the kinetic energy E_k and the second term is the interaction energy E_int. For an FRW, the time-dependent kinetic energy is $E_\mathrm{k} = \frac{4\pi}{(4t^2+1)^{3/2}}$. Because total energy is conserved, the amplitude amplification of RW corresponds to energy transfer process from interaction energy to kinetic energy, as indicated in Fig. 3(a). Quantitatively, total energy transfer can be evaluated as the time integral of the kinetic energy ($\int_{-\infty}^{+\infty} E_\mathrm{k}\mathrm{d}t$). We have numerically calculated the integrals for RWs whose orders are up to 10 and have found that they are equal to the sum of absolute magnetic flux of the monopoles [see Fig. 3(b)]. From the vector field perspective, we know that monopole collisions induce the vector potential field reconnection, which brings the conversion of interaction energy into kinetic energy. This process is analogous to the magnetic field reconnection process identified in astrophysics, in which magnetic field energy is transformed into the kinetic energy of a plasma.[70]

We have also investigated the relation between the peak amplitude of a RW and the number of monopoles and found that they are closely related. According to our discussion above, an FRW admits 4 Dirac monopoles. Because the nth-order RW solution is a nonlinear superposition of $\frac{n(n+1)}{2}$ FRWs, it will contain $N = 2n(n+1)$ monopoles on the complex plane. Among them, there are $4n$ monopoles colliding on the real axis that are responsible for the multiple phase steps (the nth-order RW with the highest peak), whereas the other $2n(n-1)$ monopoles

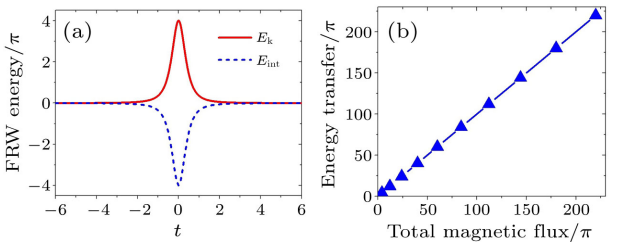

Fig. 3. (a) Temporal evolution of the kinetic energy and interaction energy of an FRW. (b) Energy transfer vs. total magnetic flux. Here, the energy transfer refers to the time integral of the kinetic energy, and the total magnetic flux represents the sum of the absolute magnetic fluxes of all monopoles for a high-order RW. The triangle is given by calculations on RWs whose orders are up to 10, and the solid line denotes the equal relation between energy transfer and total magnetic flux.

will collide in other locations on the complex plane. On the other hand, the square of the maximum amplitude amplification ratio P for an nth-order RW can be calculated to be $(2n+1)^2$.[65] We thus obtain the explicit relation: $P = 2N + 1$. Based on this observation, we can predict that the amplitude amplification ratio of a high-order RW will be subject to a certain limit because a RW contains only a limited number of Dirac monopoles due to the finite number of valleys in its geometric configuration.[20]

5. Conclusion and Discussion. With the extension to the complex coordinate plane and following Dirac's theoretical formulation of magnetic monopole, we reveal a monopole topological potential widely hidden in the phase variations of 1D nonlinear waves. The monopole vector potentials are very useful for analyzing and understanding striking phases of nonlinear waves and can be used to make topological classifications (see details in the Supplementary Materials). We also compare our results with the Aharonov–Bohm effect and suggest that the phase variations of these nonlinear waves can be viewed as a 1D counterpart to the Aharonov–Bohm phase. These results present a clear message on interpretations of these different concepts (i.e., nonlinear waves, topological field, monopole, and Aharonov–Bohm phase) in theoretical physics.

The monopole collision process can perfectly explain the striking phase jump of RWs. Moreover, we uncover that the ceiling values of the amplitude amplification ratios for RWs of different orders are determined by the total magnetic flux. The unintelligible π phase jump is found to be a manifestation of the quantized magnetic flux of a magnetic monopole. Our results provide an alternative way to understand the dynamics and topological characters of nonlinear waves. For example, we uncovered the topological vector potentials underlying the phases of dark solitons and classified these dark solitons according to virtual monopole fields.[55] The monopoles are static for dark solitons,[55] but they can generally move for time-dependent nonlinear waves, such as RWs shown here. The motion of monopoles will induce much more abundant phases dynamics. It can be further extended to study the semiclassical dynamics of BdG excitations in

Bose–Einstein condensation under the action of intrinsic topological potentials,[71] interactions between nonlinear waves,[19–23] and even in non-integrable systems.[43] In this latter case, density zeros in the complex plane that constitute magnetic monopole fields could be traced with the help of numerical algorithms.

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