Entanglement Signatures of (2+1)D Dirac Fermions Subjected to Randomness

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In this work, we investigate disordered Dirac fermions from the perspective of quantum entanglement, which provides a different angle compared to the ordinary perturbative renormalization group (RG) analysis. We consider Dirac fermions subjected to random hopping and random flux, which respectively fall into the chiral Gaussian orthogonal ensemble (cGOE) and chiral Gaussian unitary ensemble (cGUE) universality classes. Existing studies based on perturbative calculations suggest that both types of randomness are marginal. Here, through numerical simulations of the corresponding lattice models, we find that these two different types of randomness exhibit distinct entanglement features, signaling completely different properties in contrast to the perturbative RG analysis. In particular, although the entropy area-law is generally held for both types of randomness, we identify that the subleading term of the entanglement entropy is enhanced by random flux but not by random hopping. This subleading term is known as the entropic $\mathcal{F}$-function in the clean limit without disorder. Our observations indicate that disordered theories in cGOE and cGUE are essentially different, which recalls careful analysis on the RG calculations.

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1. Introduction. Understanding the coarse-graining of degrees of freedom of theories under renormalization group (RG) transformations is of fundamental importance, not only for characterizing the parameter space of theories but also for understanding phase transitions and related collective phenomena.[1–4] As the RG transformations keep removing (integral out) microscopic information from the ultraviolet to the infrared, such characterization is expected to be a monotonically decreasing function along RG flows, indicating the irreversibility of RG transformations. The first established RG irreversibility theorem is the Zamolodchikov's c-theorem[5] in (1+1) dimensions, which states that there exists a c-function that monotonically decreases along RG flows and attains a stationary value corresponding to the central charge of the underlying conformal field theory at the RG fixed points.[5,6] Extensions of Zamolodchikov's c-theorem to higher dimensions have been proposed.[7–10] For instance, in three dimensions, one can define a so-called $\mathcal{F}$-function to constrain the RG flow.

Technically, performing a non-perturbative calculation of RG irreversibility defining functions, such as c-function and $\mathcal{F}$-function, is a highly challenging task. The primary difficulty arises from the intrinsically non-local nature of the RG irreversibility functions, which precludes their extraction via local correlation functions.[7] Moreover, it is noteworthy that the RG irreversibility functions are intimately related to the entanglement entropy (EE).[11] In this context, the aforementioned RG irreversibility theorems can alternatively be derived from the perspective of quantum information theory.[12,13] With the help of EE, in two dimensions, the c-function and related central charge have been widely studied in integrable and non-integrable systems.[14,15] In three dimensions, the $\mathcal{F}$-function for non-interacting free Dirac fermion field[16–18] has been validated and confirmed numerically very recently.[19] However, to the best of our knowledge, the behaviour of the $\mathcal{F}$-function for three-dimensional Dirac fermion field deviating away from the free RG fixed point has yet to be explored.

In this work, we investigate the properties of (2+1)-dimensional Dirac fermions subjected to randomness[20–25]

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from the perspective of quantum entanglement. Specifically, we consider two types of randomness that preserve chiral symmetry:[20–25] (i) random hopping, which belongs to the chiral Gaussian orthogonal ensemble (cGOE) universality class, and (ii) random flux, which belongs to the chiral Gaussian unitary ensemble (cGUE) universality class.[26] Within the framework of perturbative RG theory (up to the two-loop level), both types of randomness are marginal without flows. This suggests the formation of an RG fixed line along which chiral disorder does not induce any instability. However, since the two types of randomness belong to different symmetry classes,[24,27,28] they are expected to exhibit distinct behaviors. Indeed, we find that the two types of randomness host very different entanglement properties. Although both models share an area-law leading term in EE, their subleading terms behave differently: In the random hopping model, the subleading term is independent of disorder strength; whereas in the random flux model, it increases monotonically with disorder strength. We deduce that this difference is rooted in the topological nature of disordered theories, which is closely tied to their underlying symmetry class.[21,23,24,26–32]

2. Model and Perspective. We investigate two lattice models that exhibit Dirac fermion behavior at low energy: the π-flux square lattice model and the honeycomb lattice model. The Hamiltonian for both models reads

$$H_{\mathtt{latt=sq/hc}} = -\sum_{\langle ij\rangle} t_{ij} c_i^\dagger c_j + \text{h.c.}, \tag{1}$$

where $c_i^\dagger$ (c_i) represents the spinless fermion creation (annihilation) operator at site i, and t_{ij} denotes the hopping amplitude between nearest-neighbor sites. For the π-flux square lattice model, the hopping amplitude t_{ij} is modulated according to $t_{ij} = (-1)^{(i_x - j_x)(\frac{i_x + j_x + 1}{2} + i_y)} t$, where i_x (j_x) and i_y (j_y) denote the site coordinates along the x- and y-axes, respectively.[33] For simplicity, we set $t = 1$ as the unit of energy. The π-flux square lattice model features two distinct energy bands, which touch linearly at the Dirac points $K, K' = (\pm\frac{\pi}{2a}, 0)$ in the first Brillouin zone, where a is the atomic spacing. For the honeycomb lattice, the hopping amplitude is uniform, given by $t_{ij} = t$. Similarly, this model also features two energy bands that meet linearly at the Dirac points $K, K' = (\pm\frac{4\pi}{3\sqrt{3}a}, 0)$, where a is the nearest-neighbor distance between sublattices A and B. The schematic diagrams of the lattice structures and their corresponding Brillouin zones are presented in Fig. 1.

In this paper, we focus on the stability of the Dirac fermion under the influence of random chiral disorders. To be specific, we examine two types of disordered systems: (i) the random hopping model, characterized by the modification $t_{ij} \to t_{ij} + \delta t_{ij}$, and (ii) the random flux model, characterized by the modification $t_{ij} \to t_{ij} e^{i\delta\phi_{ij}}$.[23,24] The mapping of lattice models with disorder onto continuous models is presented in Section I of the Supplementary Materials (SM).

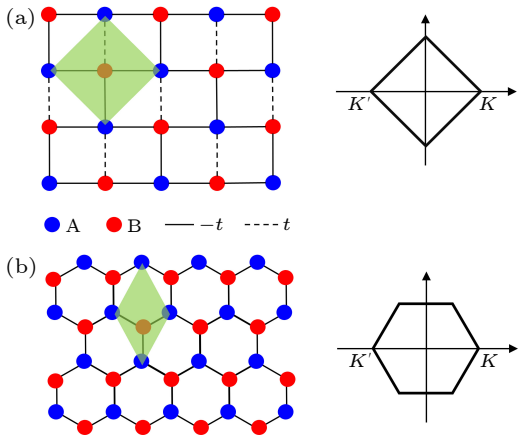

Fig. 1. Lattice structure (left) and Brillouin zone (right) of the (a) π-flux square lattice model and (b) honeycomb lattice model. The shaded area represents the primitive cell.

In the random hopping model [case (i)], δt_{ij} is uniformly distributed in the range $-W < \delta t_{ij} < W$ (with $W < 1$ to avoid flux insertion). Since time-reversal symmetry is preserved, this model falls into the cGOE universality class, corresponding to the BDI class in the classification of Ref. [35]. In the random flux model [case (ii)], $\delta\phi_{ij}$ is treated as white noise with a disorder strength (variance) g. This flux randomness breaks the time-reversal symmetry (symmetry analysis of randomness in Section II of the SM), placing the model in the cGUE class, which corresponds to the AIII in Ref. [35]. Both types of randomness preserve chiral symmetry near the Dirac point, maintaining the block-off-diagonal structure of the effective Hamiltonian.[23,36] The chiral disorder can lead to specific quantum interference effects that modify the system's single-particle and transport properties. As a result, the distinct behaviors of 2D Dirac fermions with chiral disorder, especially regarding their density of states and conductivity, have been extensively explored.[21–24,35] The density of states exhibits a power-law dependence on energy and features a peak at zero energy under certain disorder strengths. Meanwhile, the conductivity approaches a finite, universal value at zero energy. These findings confirm that chiral symmetry in these models yields a delocalized state at zero energy, thereby significantly affecting the localization properties of the systems.

3. Perturbative RG Analysis. First, we demonstrate the criticality of the zero-energy state through a perturbative RG analysis in the effective continuous limit (continuous-limit expressions are shown in Section I of the SM). Following Wilson's approach, we start with the disorder-averaged partition function, $\langle Z\rangle = \int D[\bar\Psi, \Psi]_\Lambda e^{\langle S\rangle}$, where the action in the presence of random hopping is given by

$$S = \int_{E,\boldsymbol{k}} \sum_\alpha \bar\Psi^\alpha_{\boldsymbol{k},E}(E - v\tau_3 \boldsymbol{k}\cdot\boldsymbol{\sigma})\Psi^\alpha_{\boldsymbol{k},E} + \frac{\Delta}{2}\int_{t,t',\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{q}} \sum_{\alpha\beta}\sum_{i=1,2} (\bar\Psi^\alpha_{\boldsymbol{k}}\sigma_i\tau_0\Psi^\alpha_{\boldsymbol{k}+\boldsymbol{q}})_t (\bar\Psi^\beta_{\boldsymbol{k}_1}\sigma_i\tau_0\Psi^\beta_{\boldsymbol{k}_1-\boldsymbol{q}})_{t'}. \tag{2}$$

Here, σ_i and τ_i denote the Pauli matrix in the sublattice

and valley spaces, respectively. v_0 denotes the velocity at the Dirac point in the continuous limit: for the π-flux square lattice model, $v_0 = 2ta$, while for the honeycomb lattice model, $v_0 = \frac{\sqrt{3}}{2}ta$. For random flux disorder, the action takes an analogous form with the only difference being the replacement of τ_0 by τ_3 in the disorder potential term. The disorder coupling Δ varies depending on the lattice structure and the specific type of randomness considered,

$$\Delta = \begin{cases} 2W^2/3, & \text{random hopping in } \pi\text{-flux square lattice;} \\ W^2/2, & \text{random hopping in honeycomb lattice;} \\ 2g, & \text{random flux in } \pi\text{-flux square lattice;} \\ 3g/2, & \text{random flux in honeycomb lattice.} \end{cases} \tag{3}$$

Through the one-loop calculation, we find that both types of randomness exhibit identical RG flow equations for energy and disorder strength in both randomness cases (details in Section III of the SM),

$$\frac{\mathrm{d}E}{\mathrm{d}\ell} = \Big(1 + \frac{\Delta}{\pi v_0^2}\Big)E, \quad \frac{\mathrm{d}\Delta}{\mathrm{d}\ell} = 0, \tag{4}$$

where the length scale ℓ serves as the running parameter. The results suggest that the system of chiral fermions near $E = 0$ represents a critical regime with respect to the disorder strength. Similar conclusions have been drawn from various analyses involving the broad distribution of the local density of states[24] and the application of non-Abelian bosonization.[35] Moreover, we extend the calculation to the two-loop level for the disorder strength Δ, and the result remains consistent, confirming that the RG flow of Δ is absent for both randomness. The perturbative RG calculations indicate that both types of randomness are marginal and indistinguishable, raising doubts about the physical significance of categorizing the random hopping (cGOE) and the random flux (cGUE) into two distinct symmetry classes.

4. The Entanglement Entropy. While conventional perturbative RG analysis fails to reveal the distinction between random hopping (cGOE) and random flux (cGUE), we turn to the calculation of the EE, which is closely tied to RG irreversibility. The EE qualifies the amount of entanglement between subsystem A and its complement part. For fermionic Gaussian theories, the EE can be determined through the two-point correlation matrix $G_{ij} = \langle c_i^\dagger c_j \rangle$ of subsystem A,[37–39]

$$\begin{aligned} S_{\mathrm{EE,A}} &= -\mathrm{Tr}(\rho_\mathrm{A} \ln \rho_\mathrm{A}) \\ &= -\sum_k g_k \ln g_k + (1-g_k)\ln(1-g_k), \end{aligned} \tag{5}$$

where $\{g_k\}$ are eigenvalues of correlation matrix G_{ij}. When dividing a lattice system into A and B, we set subsystem A as a disk with a radius R and the complementary portion of the system as subsystem B (shown in the inset of Fig. 2 as proposed in the $\mathcal{F}$-theorem.[40] To avoid the non-smooth boundary due to the lattice bipartition,

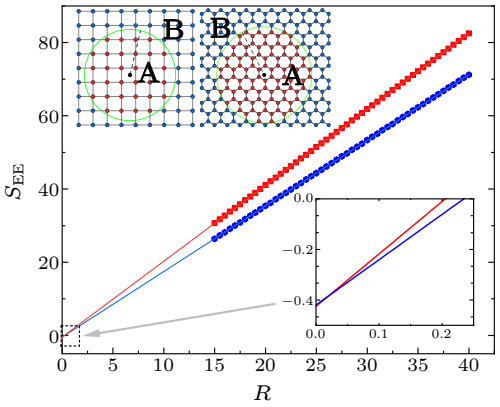

Fig. 2. The S_{EE} as a function of the subsystem size R in the absence of disorder for the π-flux model (square dots) and the honeycomb model (hexagon dots). The lines represent the best linear fits: $S_{\mathrm{EE}} \approx 2.074R - 0.425$ for the π-flux model and $S_{\mathrm{EE}} \approx 1.791R - 0.419$ for the honeycomb model. Inset: Behaviour of the fitted curves around $R = 0$. The system size is 120×120 sites.

for each given size R we choose different centers of the disk and calculate the averaged EE.

We start by discussing the behavior of the EE in the absence of randomness. For a (2+1)-dimensional Dirac fermion (the Fermi level is set to $E = 0$), the EE for a disk subsystem is expected to follow the form[41]

$$S_{\mathrm{EE}} = \alpha R - \mathcal{F}, \tag{6}$$

where α is a non-universal prefactor that depends on the microscopic details of the model, and $\mathcal{F}$ is a the subleading constant term. The first term, proportional to the boundary length $2\pi R$ of the subsystem, is a common feature in filed theories and is known as the area-law term.[42–44] This term demonstrates that the entanglement is mainly established by local correlations across the boundary. The second term, dubbed as the $\mathcal{F}$-function, is expected to be a positive constant that takes the value of sphere free energy at the RG fixed point.[11,40,45–47] For the (2+1)-dimensional Dirac fermion,[17,18,48–50] the sphere free energy is analytically accessible, yielding $\mathcal{F}_{\mathtt{theo}} = \frac{1}{8}(2\ln 2 + \frac{3\zeta(3)}{\pi^2}) = 0.21895948\cdots$ (for a single Dirac valley, we have $F = -\log Z_{S^3} = \frac{1}{8}(2\ln 2 + \frac{3\zeta}{\pi^2}) \approx 0.219$). A previous numerical study[19] estimates $\mathcal{F} = 0.21029$ via a lattice discretization of the Dirac equation.[51] To make a comparison, we first numerically calculate the EE for the free Dirac fermion without randomness on two different lattice models. The results are depicted in Fig. 2. To ensure the subsystem approximates a disk, we set the radius to $R \geq 15$. The main features of the numerical results are consistent with Eq. (6): (1) The leading term is linearly proportional to the subsystem size R, in a direct support of the area law; (2) The subleading term is nonzero, with the two models yielding nearly identical values: $\mathcal{F}_{\pi-\mathtt{sq}} \approx 0.425$ for the $\pi-$flux square lattice model and $\mathcal{F}_{\mathtt{hc}} \approx 0.419$ for the honeycomb model. Since both lattice models we considered involve two valleys, the expected theoretical value for the subleading term is $2\mathcal{F}_{\mathtt{theo}} = 0.4379\cdots$, in good agreement with our numerical results.

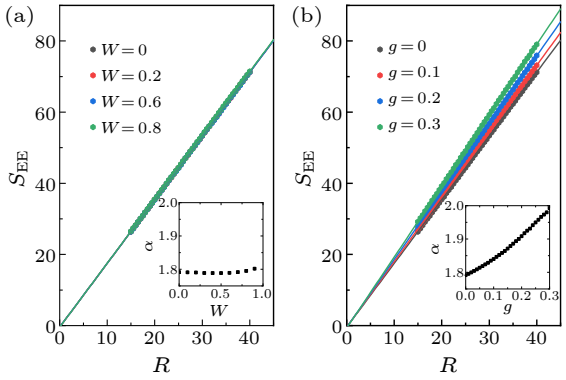

Fig. 3. The $S_{\rm EE}$ for the honeycomb lattice in the presence of (a) random hopping (disorder strength $\Delta = \frac{W^2}{2}$) and (b) random flux (disorder strength $\Delta = \frac{3g}{2}$). Inset: Dependence of the prefactor α on disorder strength. Disorder averaging is performed over 100 realizations, with 20 random choices of the circle center for each realization, leading to a total of $100 \times 20 = 2000$ samples for each data point.

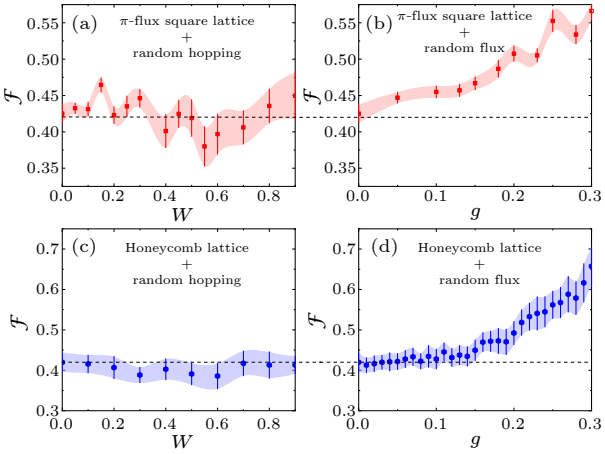

Fig. 4. The subleading $\mathcal{F}$-term in the EE for the Dirac fermion subjected to the chiral disorder. In the random hopping model, the subleading term of the EE for (a) π-flux square lattice and (c) honeycomb lattice. In the random flux model, the subleading term of the EE for (b) π-flux square lattice and (d) honeycomb lattice. The results are averaged over 1000 disorder configurations and the error bars measures the standard deviation. The dashed line represents the expectation value of $\mathcal{F}$-term for the Dirac fermion without randomness.

Then, we turn to investigate the effect of disorder on the EE. Figures 3(a) and 3(b) present the results obtained for the honeycomb lattice model in the presence of the random hopping and random flux, respectively. As expected from the locality of the models, the EE follows the typical area-law scaling, as described by Eq. (6), for both types of randomness across a range of disorder strengths. Upon applying the scaling form in Eq. (6), we identify notable differences between the two types of randomness. In the case of random hopping, the EE is nearly independent of the disorder strength W, resulting in overlapping curves for different W values. In contrast, for random flux, we find that the area-law coefficient (the prefactor of leading area-law term that proportional to the subsystem size R) exhibits a monotonic increase with disorder strength g[52] [see the inset of Fig. 3(b)]. A similar trend is also observed in the π-plux square lattice model.

More importantly, for the two types of randomness, the subleading $\mathcal{F}$ term in the EE displays different dependencies on the disorder strength. In Fig. 4 we present the extracted subleading $\mathcal{F}$ term of the EE for two different models and two different types of randomness. It is found that the value of $\mathcal{F}$ gradually increases with the disorder strength g of flux randomness, on both the π-flux square and the honeycomb lattices [see Figs. 4(b) and 4(d)]. In contrast, for hopping randomness, $\mathcal{F}$ fluctuates around a constant value that is quite close to the free-fermion value $\sim 2\mathcal{F}_{\texttt{theo}}$, without a clear increase or decrease trend [Figs. 4(a) and 4(c)]. Furthermore, the $\mathcal{F}$-term is determined by the intercept of a linear fit of the EE as a function of radius R. Since the resulting intercept is much smaller in magnitude than the EE itself, we have also performed a Z-core standardization to further verify the reliability of our conclusions (details in Section IV of the SM).

We have clearly demonstrated that the hopping randomness and flux randomness respectively yields distinct behaviors of subleading term of the EE. We deduce that this difference is rooted in topology of the cGOE and cGUE. To be specific, the hopping randomness and the flux randomness fall into the cGOE (BDI in the ten-fold classification) and the cGUE (AIII in the ten-fold classification) classes, respectively.[35] From the symmetry point of view, these disordered theories can be described by the fermionic replicated nonlinear σ-models (NLσMs), which are charaterized by the target space of field configurations.[26,32,53] On one hand, target space of cGUE is $U(N)\times U(N)/U(N)$ (N is the replica index).[28] As a result, for cGUE class it must be augmented by a $\mathbb{Z}$-topological term, namely the Wess–Zumino–Novikov–Witten (WZNW) term. The WZNW term pins the theory to a nonperturbative conformal fixed point.[22,25] This conformal structure produces a delocalized critical phase characterized by algebraic correlations and multifractal wavefunctions, in contrast to the conventional NLσM without such a term. Due to this topological term, the description of Eq. (2) is insufficient. One may expect some non-trivial corrections to Eq. (4), however a full RG analysis of Eq. (2) in addition to WZNW term is still out of reach. On the other hand, target space of cGOE is $U(2N)/Sp(2N)$,[28] which does not host a topological term. In this regarding, equation (4) is sufficient for hopping randomness. Our calculations based on the EE are non-perturbative, which may reveal the subtle difference from the presence/absence of topological WZNW term. This difference of cGOE and cGUE is beyond the conventional perturbative RG theory, to our best knowledge, which has not been uncovered before.

5. Discussion and Summary. We have investigated the EE of Dirac fermions in the presence of randomness belonging to cGOE and cGUE universality classes. Our results confirm that the leading contribution to the EE still satisfies the area law, even in the presence of chiral disorder, which breaks Lorentz symmetry—specifically, the random flux disorder considered here. Crucially, the subleading term of the EE, corresponding to the $\mathcal{F}$-function

in the clean limit, exhibits distinct behaviors for the two types of randomness: in the cGOE case, the subleading term is almost a constant independent of hopping randomness; whereas in the cGUE case, it gradually increases with disorder strength. This conclusion is cross-checked by testing on both π-flux square and honeycomb lattice models, which serve as realizations of Dirac fermions. A plausible understanding is rooted in topological properties of cGUE where its topological WZNW term stabilizes Dirac fermion field rendering it immune to perturbations. These findings not only uncover a refined structure of RG diagram regarding the Dirac fermion field subjected to randomness, but also demonstrate that EE can identify information beyond perturbative RG theory.

To the best of our knowledge, the present study might be the first attempt to access the subleading EE term in disordered theories, where randomness typically leads to a dynamical exponent $z \neq 1$.[54–56] This opens up a general question about the application of the $\mathcal{F}$-theorem to disordered theories without Lorentz symmetry, which is also potentially helpful for understanding RG flows for non-Lorentz theories.[57–60] Addressing this question requires detailed studies of the entanglement and RG properties for fields subject to randomness. For example, it would be interesting to extend the current study by performing an RG analysis that includes the topological term arising from the cGOE-class randomness. Such an analysis would offer deeper insights when compared with the present observation of the subleading term in EE. Additionally, it is also interesting to explore the behavior of this subleading term in other disordered theories. For instance, besides cGOE and cGUE, EE may also peek into the RG diagram of the chiral sympletic class (CII).[35] However, a challenge lies in that this class cannot be directly realized on a two-dimensional lattice; one has instead to consider its two-dimensional boundary of a three-dimensional topological band insulator.[28] Moreover, how various types and strengths of interactions affect stability of Dirac fermions is another fascinating problem for future exploration.[61] The possible "dangerously irrelevant" RG flow induced by long-ranged electron–electron interaction,[62] as well as suppression of Fermi velocity due to local Hubbard interaction,[63] are well worth investigating through EE.

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