Solitons in the Fifth-Order KdV Equation with a Perturbation

Wentao Li[1], Yipu Chen[1], Zhao Zhang[2], Xiangyu Yang[3], and Biao Li[1*]

[1]*School of Mathematics and Statistics, Ningbo University, Ningbo 315211, China*
[2]*School of Mathematics and Statistics, Lingnan Normal University, Zhanjiang 524048, China*
[3]*School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China*

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The fifth-order KdV equation with a perturbation term is naturally consistent with the shallow-water wave model. In this study, the leading-order asymptotic solution of the perturbed fifth-order KdV equation was derived using the multi-scale perturbation expansion method. Furthermore, the asymptotic solution was reconsidered from the perspective of conservation laws, and consistent results were obtained. In addition, the numerical solution of the slowly varying solitary wave asymptotic solution was obtained using the Fourier spectral method under appropriate initial conditions to better understand its dynamic behavior. These results enrich research on fluid dynamics.

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1. Introduction. Solitary waves are stable wave solutions that are ubiquitous in many nonlinear systems. Since Russell discovered this phenomenon in his study of shallow-water waves in the mid-19th century,[1] the concept of solitary waves has been rapidly introduced in various fields of science and engineering, including fluid mechanics,[2] nonlinear optics,[3] and plasma physics.[4] With the development of nonlinear science, numerous scientists have focused on the theoretical construction and experimental research of solitary waves. In theoretical construction, many models that conform to the solitary wave phenomenon have been constructed,[5,6] and then various characteristics of solitary waves have been investigated.[7,8] The formation and stable transmission of optical solitons have been discovered.[9]

The KdV equation is a general model for studying weak nonlinear long waves, which include leading-order nonlinearity and dispersion. In addition, it describes surface waves with long wavelengths and small amplitudes in shallow water. The delicate balance between the nonlinearity and the dispersion effect in the system ensures that the waves are not distorted during transmission; however, it explains the form of solitons composed of single hump waves. The physical significance of higher-order nonlinear and higher-order dispersion terms lies in their ability to simulate complex fluctuating properties that interact at different scales. The fifth-order KdV equation has received widespread attention[10] and is expressed as follows:

$$u_t + 30u^2u_x + 10(uu_{xxx} + 2u_xu_{xx}) + u_{xxxxx} = 0. \quad (1)$$

The N-soliton solutions to Eq. (1) was explored in detail,[11] and the Cauchy problem for initial Sobolev data with low regularity was analyzed.[12] In addition, soliton molecules and asymmetric solitons have been derived.[13]

To describe the complexity of the soliton phenomena in real fluids accurately, the effects of damping, dissipation, and medium changes on solitons have received extensive research attention.[14–18] The long-term asymptotic behavior of the solutions of integrable equations has been investigated.[19,20] The approximate solution of the integrable equation with slowly varying coefficients under the initial conditions of the soliton shows that the soliton undergoes an irreversible loss when the energy propagates in a slowly varying medium.[14] Introducing perturbation terms into integrable equations can specifically explore the effects of damping and dissipation on solitons.[15–17] In these studies, integrable equations that could simulate complex fluctuation characteristics with high-order dispersion and nonlinear terms were not investigated. Therefore, this study focuses on the perturbed fifth-order KdV equation, which can be written as

$$u_t + 30u^2u_x + 10(uu_{xxx} + 2u_xu_{xx}) + u_{xxxxx} = \varepsilon R(u), \quad (2)$$

where $u = u(x, t)$, ε is a small number, less than 1, $R(u)$ can be a function of u, a derivative of u with respect to x or t, or a function with a slow-time variable.

In this study, we investigated the asymptotic and numerical solutions of the perturbed fifth-order KdV equation. The structure of this paper is organized as follows: In Section 2, the leading-order asymptotic solution is obtained using the multi-scale perturbation expansion method. In Section 3, the leading-order asymptotic solution of Eq. (2) is investigated from the perspective of conservation laws, and the same results are obtained as in Section 2. In Section 4, a numerical solution is obtained according to the initial conditions generated by the leading-order asymptotic solution, and the influence of the perturbation term on the solitary wave is explored. Section 5 presents concluding remarks.

2. Asymptotic Analysis. This section introduces the multi-scale perturbation expansion method.[21] Using this method, the asymptotic solution of Eq. (2) can be derived.

*Corresponding author. Email: libiao@nbu.edu.cn
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The rapidly varying position variable θ and slowly varying time variable T can be defined as:

$$\theta = x - \frac{1}{\varepsilon}\int_0^T c(T')dT', \quad T = \varepsilon t, \tag{3}$$

where

$$c(T) = c_0(T) + \varepsilon c_1(T) + \varepsilon^2 c_2(T) + \varepsilon^3 c_3(T) + \cdots. \tag{4}$$

Subsequently, the asymptotic expansion form of the solution $u(x,\, t)$ of the perturbed fifth-order KdV is defined as

$$\begin{aligned} u(x,\, t) = {} & u_0(\theta,\, T) + \varepsilon u_1(\theta,\, T) + \varepsilon^2 u_2(\theta,\, T) \\ & + \varepsilon^3 u_3(\theta,\, T) + \cdots. \end{aligned} \tag{5}$$

Substituting Eq. (5) into Eq. (2) and collecting the coefficients of each order of ε, the following series of equations can be obtained:

$$\begin{aligned} & \varepsilon^0 : u_{0\theta\theta\theta\theta\theta} - c_0 u_{0\theta} + 30u_{0\theta}u_0^2 + 10u_{0\theta\theta\theta}u_0 \\ & + 20u_{0\theta\theta}u_{0\theta} = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} & \varepsilon^1 : (30u_0^2 u_1)_\theta + 10u_0 u_{1\theta\theta\theta} + 10u_1 u_{0\theta\theta\theta} + 20(u_{0\theta}u_{1\theta})_\theta \\ & - c_0 u_{1\theta} + u_{1\theta\theta\theta\theta\theta} = F_1, \end{aligned} \tag{7}$$

where

$$F_1 = c_1 u_{0\theta} + R(u_0) - u_{0T}. \tag{8}$$

According to Eq. (6) using the traveling wave transform, the leading-order asymptotic solution to Eq. (2) can be obtained as follows, and it is shown that as $\varepsilon \to 0$, $u \sim u_0$,

$$u_0 = A(T)\mathrm{sech}^2\{\lambda(\mathrm{T})\theta\}, \tag{9}$$

where

$$c_0 = 4A(T)^2 = 16\lambda(T)^4. \tag{10}$$

When $\varepsilon \to 0$, the velocity $c(T)$ of the slowly varying solitary wave approaches c_0; $A(T)$ is the amplitude of the slowly varying solitary wave, and $\{1/\lambda(T)\}$ is proportional to the wavelength.

For the leading-order asymptotic solutions of Eq. (2), to demonstrate that the asymptotic solution is valid, we must ensure that the solution to u_1 in Eq. (7) is bounded. Therefore, the terms in Eq. (7), which causes u_1 to produce long-term terms, must be eliminated. Because Eq. (7) is a nonhomogeneous differential equation with respect to the variable u_1, its homogeneous part has the corresponding adjoint equation as follows:

$$\begin{aligned} & -10u_{0\theta\theta}v_\theta - 10u_{0\theta}v_{\theta\theta} - v_{\theta\theta\theta\theta\theta} - 10u_0 v_{\theta\theta\theta} \\ & + v_\theta c_0 - 30u_0^2 v_\theta = 0. \end{aligned} \tag{11}$$

From the adjoint Eq. (11), it is clear that u_0 and 1 are the two fundamental solutions, and when $v = u_0$, $v \to 0$ is $\theta \to \pm\infty$. Therefore, to obtain the bounded solution u_1, we assume that the inhomogeneous part F_1 in Eq. (7) is orthogonal to u_0; that is,

$$\int_{-\infty}^{+\infty} F_1 u_0 d\theta = 0. \tag{12}$$

F_1 is provided by Eq. (8), which yields:

$$\frac{\partial}{\partial T}\int_{-\infty}^{+\infty} \frac{1}{2}{u_0}^2 d\theta = \int_{-\infty}^{+\infty} u_0 R(u_0) d\theta. \tag{13}$$

The above equation can be understood as the rate of change in the energy of the slow solitary wave (9), which is the same as that of the KdV equation. In addition, by substituting Eq. (9) into Eq. (13), we obtain

$$\frac{\partial}{\partial T}\lambda = \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{sech}(\lambda\theta)^2 R[2\lambda^2 \mathrm{sech}(\lambda\theta)^2] d\theta. \tag{14}$$

Thus, Eq. (9), and Eq. (14) represents the leading-order asymptotic solution of the fifth-order KdV.

3. Conservation Laws. It is important to explore the effects of perturbations from the perspective of conservative relationships. In this section, we explore the perturbed fifth-order KdV equation and the results presented in the previous section are verified.

For Eq. (2), the conservation relation can be written as

$$E_t + F_x = \varepsilon G, \tag{15}$$

where

$$E = \frac{1}{2}u^2, \tag{16}$$

$$F = \frac{15}{2}u^4 + 10u^2 u_{xx} + u u_{xxxx} - u_x u_{xxx} + \frac{1}{2}u_{xx}^2, \tag{17}$$

$$G = uR(u). \tag{18}$$

Because $u = u(\theta,\, T)$ and the variables θ and T are provided by Eq. (3), Eq. (15) can be rewritten as follows:

$$\varepsilon E_T + (F - cE)_\theta = \varepsilon G. \tag{19}$$

We perform an asymptotic series expansion on E, F, G, $c(T)$, which are

$$\begin{aligned} E &= E_0 + \varepsilon E_1 + \varepsilon^2 E_2 + \cdots, \\ F &= F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots, \\ G &= G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \cdots, \end{aligned} \tag{20}$$

where $c(T)$ is denoted by the previous Eq. (4). Substituting Eq. (20), and Eq. (4) into Eq. (19) and then comparing the coefficients of each order of ε, the results are as follows:

$$\epsilon^0 : (F_0 - c_0 E_0)_\theta = 0, \tag{21}$$

$$\epsilon^1 : (F_1 - c_0 E_1)_\theta - c_1 E_{0\theta} + E_{0T} = G_0. \tag{22}$$

Because the specific relationship with u must be obtained by substituting Eqs. (4) and (5) into Eqs. (21) and (22). Subsequently, using the asymptotic series expansion method, the equations for the coefficients of each order of ε are obtained as follows:

$$\begin{aligned} & \epsilon^0 : E_0 = \frac{1}{2}u_0^2, \\ & F_0 = \frac{15}{2}u_0^4 + 10u_0^2 u_{0\theta\theta} + \frac{1}{2}u_{0\theta\theta}^2 + u_0 u_{0\theta\theta\theta\theta} - u_{0\theta\theta\theta}u_{0\theta}, \\ & G_0 = u_0 R(u_0), \\ & \epsilon^1 : E_1 = u_0 u_1, \\ & F_1 = 30u_0^3 u_1 + 20u_0 u_1 u_{0\theta\theta} + 10u_0^2 u_{1\theta\theta} + u_1 u_{0\theta\theta\theta\theta} \\ & \qquad + u_{1\theta\theta}u_{0\theta\theta} - u_{1\theta}u_{0\theta\theta\theta} - u_{0\theta}u_{1\theta\theta\theta} + u_{1\theta\theta\theta\theta}, \\ & G_1 = u_0 u_1 R'(u_0) + u_1 R(u_0). \end{aligned} \tag{23, 24}$$

Combining Eqs. (21) and (23), which are the same as Eq. (6), we can deduce that u_0 in Eqs. (9) and (10), proposed in the previous section, is a fundamental solution to the adjoint equation of the homogeneous part of Eq. (22). To determine the expression for $\lambda(T)$, we integrate Eq. (22) with respect to θ, and then obtain

$$\frac{\partial}{\partial T}\int_{-\infty}^{\infty} E_0 d\theta + [F_1 - c_0 E_1 - c_1 E_0]_{-\infty}^{\infty} = \int_{-\infty}^{\infty} G_0 d\theta. \quad (25)$$

According to Eqs. (23) and (24), Eq. (25) can be rewritten as follows:

$$\frac{\partial}{\partial T}\int_{-\infty}^{+\infty} \frac{1}{2}u_0^2 d\theta = \int_{-\infty}^{+\infty} u_0 R(u_0) d\theta. \quad (26)$$

It can be observed that Eq. (26) is consistent with Eq. (13). Substituting Eq. (9) into Eq. (26) yields an expression for variable λ, which is consistent with Eq. (14). Thus, the leading-order asymptotic solution of the perturbed fifth-order KdV equation was deduced from the perspective of the conservation law.

4. Numerical Results and Analysis. In this section, based on the asymptotic solution in the previous two sections, we generate a numerical solution of the leading-order asymptotic solution of the perturbed fifth-order KdV equation by choosing appropriate initial value conditions and using the Fourier spectral method.

We performed a Fourier transform of Eq. (2). The following definition of Fourier transform is defined as

$$F(u) = \hat{u}(p) = \int_{-\infty}^{\infty} u(x) \mathrm{e}^{-\mathrm{i}px} dx, \quad (27)$$

Eq. (2) can be rewritten as

$$\begin{aligned}\hat{u}_t = & -30F\Big\{[F^{-1}(\hat{u})]^2 \cdot F^{-1}(\mathrm{i}p\hat{u})\Big\} \\ & -10F\Big\{F^{-1}(\hat{u}) \cdot F^{-1}[(\mathrm{i}p)^3\hat{u}]\Big\} \\ & -20F\Big\{F^{-1}(\mathrm{i}p\hat{u}) \cdot F^{-1}[(\mathrm{i}p)^2\hat{u}]\Big\} \\ & -(\mathrm{i}p)^5\hat{u} + \varepsilon F\{R(u)\}. \end{aligned} \quad (28)$$

Subsequently, the ode45 algorithm based on the Runge–Kutta method is used to solve $\hat{u}$, and the initial time is selected as the periodic boundary condition. Finally, the numerical solution of the asymptotic solution of the perturbed fifth-order KdV equation can be derived using the inverse Fourier transform of the function $\hat{u}$.

To derive the numerical solution for the asymptotic solution of the perturbed fifth-order KdV equation, it is necessary to concretize the perturbation term. In nearly integrable equations, the perturbation terms are defined as $R(u) = \delta(T)u$. Therefore, we regard the function $R(u)$ as $-u/(1+T)$ to ensure that Eq. (2) has the following expression:

$$\begin{aligned} & u_t + 30u^2u_x + 10(uu_{xxx} + 2u_xu_{xx}) + u_{xxxxx} \\ & = -\varepsilon\frac{u}{1+\varepsilon t}. \end{aligned} \quad (29)$$

Based on Eqs. (9), (10), and (14), the expression for the leading-order asymptotic solution of the perturbed fifth-order KdV equation is

$$\begin{aligned} u \approx & \, 2\lambda_0^2(1+\varepsilon t)^{-\frac{4}{3}}\mathrm{sech}^2\Big\{\lambda_0(1+\varepsilon t)^{-\frac{2}{3}} \\ & \cdot\Big[x - 48\lambda_0^4\frac{(1+\varepsilon t)^{\frac{5}{3}}-1}{5\varepsilon(1+\varepsilon t)^{\frac{5}{3}}}\Big]\Big\}, \end{aligned} \quad (30)$$

where λ_0 is the specific value of $\lambda(T)$ at time $T = 0$. According to Eq. (30), the expressions for the amplitude $A(t)$, wave width $W(t)$ and position $x(t)$ in the brief time range can be obtained:

$$\begin{aligned} & A(t) = 2\lambda_0^2(1+\varepsilon t)^{\frac{4}{3}}, \quad W(t) = \frac{1}{\lambda_0}(1+\varepsilon t)^{\frac{2}{3}}, \\ & x(t) = 48\lambda_0^4\frac{(1+\varepsilon t)^{\frac{5}{3}-1}}{5\varepsilon(1+\varepsilon t)^{\frac{5}{3}}}. \end{aligned} \quad (31)$$

For the numerical calculation using the Fourier spectrum method, we choose the interval of x with an interval length of L as $[x_0, x_N]$, and then divide the interval into N equal parts to ensure that the frequency domain p is chosen to be $[0, (N-2)\pi/L, -N\pi/L, -2\pi/L]$, $\Delta x = L/N$, and $\Delta p = 2\pi/L$. Simultaneously, let time 0 be a periodic boundary condition

$$u(x, 0) \approx 2\lambda_0^2\mathrm{sech}^2(\lambda_0 x). \quad (32)$$

Thus, by combining Eq. (28) by employing the Fourier spectrum method, a numerical asymptotic solution of the perturbed fifth-order KdV equation is obtained. The patterns of the numerical solution of the asymptotic solution of the perturbed fifth-order KdV equation and the error between the numerical and asymptotic solutions were acquired.

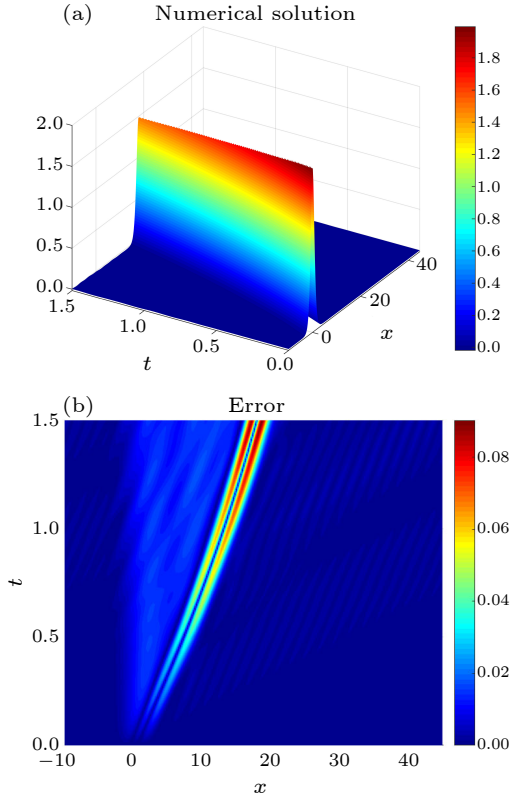

Fig. 1. Critical parameters in the numerical calculations include $x_0 = -10$, $x_N = 45$, $t_0 = 0$, $t_N = 1.5$, $N = 256$, $\Delta t = 0.01$, $\varepsilon = \frac{1}{6}$, $\lambda_0 = 1$. (a) Numerical solution of Eq. (30) with periodic boundary conditions (32), (b) Absolute value error between the numerical solution and the asymptotic solution of Eq. (30).

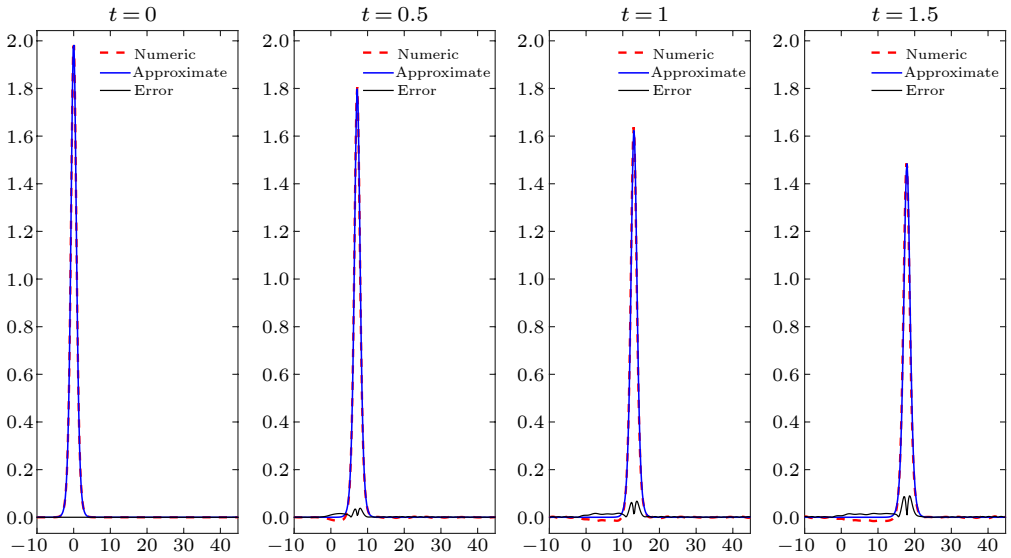

Fig. 2. Numerical solution of Eq. (30) with periodic boundary conditions (32) is described. Some important parameters in the numerical calculations are the same as in Fig. 1.

Figure 1(a) illustrates the numerical solution of the perturbed fifth-order KdV equation. As the three-dimensional plots of the asymptotic and numerical solutions are similar, an error diagram between them is provided, as shown in Fig. 1(b). Figure 1(a) illustrates that the soliton moves along the positive direction of the x-axis as time increases, and the amplitude of the soliton gradually decreases. Simultaneously, the soliton gradually produces a tail, which appears as time increases. Figure 1(b) illustrates that the error between the asymptotic solution and the numerical solution is close to 0 at $t = 0$. As time progresses, the errors between them gradually increase. To make the graphics more intuitive, a two-dimensional comparison of the numerical and asymptotic solutions was provided, as shown in Fig. 2.

Figure 2 illustrates the two-dimensional pattern of the numerical and asymptotic solutions of the perturbed fifth-order KdV equation. As time increases, the error between the asymptotic and numerical solutions in the main pulse region gradually increases. In the non-main-pulse region, the error between them approaches 0. To reduce this error, there are two different approaches: (1) The important parameters in Fig. 1 remain unchanged but the value of ε and λ_0 are reduced. (2) The asymptotic solution requires addition of a first-order correction term u_1. This aspect will be explored in future studies.

5. Conclusion. In this study, the (1+1)-dimensional perturbed fifth-order KdV equation was investigated. Multiscale perturbation expansion methods and the Fourier spectral method were applied to derive the leading-order asymptotic solution and numerical solutions of the asymptotic solution for a slowly varying solitary wave. Through the multi-scale perturbation expansion method, the leading-order asymptotic solution of slowly varying solitary waves for the perturbed fifth-order KdV equation was obtained using Eqs. (9), (10) and (14). In addition, by analyzing from the perspective of conservation laws, the same leading-order asymptotic solution was obtained as Eqs. (9), (14) and (26). Using the Fourier spectral method with $t = 0$ as the periodic boundary condition, a numerical solution to the asymptotic solution of the perturbed fifth-order KdV equation could be obtained (see Fig. 1), and the numerical and asymptotic solution were similar to each other with small errors between them (see Fig. 2).

The exploration of this nearly integrable equation is important, and the study of nearly integrable equations with perturbations can help us better understand interesting physical phenomena and the mechanisms behind them. The multi-scale perturbation expansion method can be applied to explore the asymptotic solutions to other nearly integrable equations. In the future, we will explore the first-order correction of the asymptotic solutions.

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