# Nonlinear-Optical Analogies in Nuclear-Like Soliton Reactions: Selection Rules, Nonlinear Tunneling and Sub-Barrier Fusion–Fission

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The main goal of our study is to reveal unexpected but intriguing analogies arising between optical solitons and nuclear physics, which still remain hidden from us. We consider the main cornerstones of the concept of nonlinear optics of nuclear reactions and the well-dressed repulsive-core solitons. On the base of this model, we reveal the most intriguing properties of the nonlinear tunneling of nucleus-like solitons and the soliton self-induced sub-barrier transparency effect. We describe novel interesting and stimulating analogies between the interaction of nucleus-like solitons on the repulsive barrier and nuclear sub-barrier reactions. The main finding of this study concerns the conservation of total number of nucleons (or the baryon number) in nuclear-like soliton reactions. We show that inelastic interactions among well-dressed repulsive-core solitons arise only when a "cloud" of "dressing" spectral side-bands appears in the frequency spectra of the solitons. This property of nucleus-like solitons is directly related to the nuclear density distribution described by the dimensionless small shape-squareness parameter. Thus the Fourier spectra of nucleus-like solitons are similar to the nuclear form factors. We show that the nuclear-like reactions between well-dressed solitons are realized by "exchange" between "particle-like" side bands in their spectra.

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We dedicate this work to the 90th anniversary of the Father of optical solitons, Akira Hasegawa, who 50 years ago, together with Fred Tappert, discovered the beauty of optical solitons to the entire Scientific World.[1]

The golden anniversary of optical solitons[2,3] made it possible not only to analyze the brilliant achievements in this and related fields of science, but also to formulate new interesting problems. If we consider nuclei from a soliton point of view, we come to the main conclusion that for their existence in 3D space and time, generally speaking, complete compensation of the effects of nonlinear self-action, self-diffraction in space and dispersion in time is necessary. What is also fundamentally important? We must take into account the existence of negative binding energy and the possibility of soliton fusion and fission reactions that satisfy the fundamental law of conservation of the total number of nucleons (or baryon number) and total electrical charge.

Historically, de Broglie's ideas and optical-quantum-mechanical analogies form the basis of the famous optical model in nuclear physics. The term "optical model" itself was introduced[4] because the atomic nucleus was considered as an "opaque glass ball" with a complex refractive index, at which a beam of particles diffracts, partially transmits and absorbs. Thus, the optical model potential is defined as a potential which produce the average reflection coefficient, and it must be complex.[5] The absorption cross section arising from the complex potential includes the inelastic processes and reactions.[6–8]

It should be especially emphasized that the use of nonlinear optical analogies in the development of nuclear models seems to be a completely natural next step, just as nonlinear optics itself appeared with an increase in laser power. This step was made recently in the work of Ref. [9], where a deep analogy was shown between the nonlinear dynamics of "soliton-like nuclei" having the Fermi density distribution and the nuclear reactions that take place among atomic nuclei.

The main purpose of this study is threefold. First we briefly analyze the most important features of the concept of nonlinear optics of nuclear reactions.[9] The main objects of this model are the so-called well-dressed repulsive-core solitons with the properties of real nuclei. Secondly, we focus our attention on the most intriguing details of the nonlinear soliton tunneling and the soliton self-induced sub-barrier transparency effects. We compare the tunneling scenarios of "soliton-like nucleus" characterized by the Fermi density distribution with the previously studied nonlinear tunneling of canonical solitons governed by the nonlinear Schrödinger equation model (NLSE). Our third purpose is to describe novel interesting and stimulating analogies with nuclear sub-barrier reactions.

*Keystones of the Model.* (1) The concept of nonlinear optics of nuclear reactions[9] is based on the following three key points.

First of all, we proceed from the fact that the experimentally obtained nuclear density distribution can be very accurately approximated by the two-parameter Fermi function

$$\rho(r) = \frac{\rho_0}{1 + \exp[(r - R(A))/a_0]}, \tag{1}$$

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where $\rho_0$ is interpreted as the saturated central nuclear density, $R(A) = r_0 A^{1/3}$ is the radius of the charge distribution with the radial parameter $r_0$ varying from 0.95 fm to 1.07 fm, and $a_0$ is the diffuseness parameter measured in the range of 0.4–0.5 fm.[7,8] $A = N$ is the integer mass number, i.e., the total number $N$ of nucleons (neutrons and protons) in a nucleus, or baryon number.

Looking closely at the Fermi distribution (1), we highlight the main parameter

$$\delta_{\rm F} = \exp(-R(A)/a_0), \tag{2}$$

which is called the shape squareness parameter,[9] meaning that it determines the steepness of the edge of the Fermi distribution of nuclear matter (1). It can therefore be expressed in the following equivalent form

$$\rho_{\rm F}(r) = \frac{\rho_0}{1 + \delta_{\rm F} \exp(r/a_0)}. \tag{3}$$

As an example, we present some estimates of this dimensionless small parameter $\delta_{\rm F} = \exp(-r_0 A^{1/3}/a_0)$ for several nuclei: $\delta_{\rm F}(A = 4) = 0.04$, $\delta_{\rm F}(A = 12) = 0.01$, $\delta_{\rm F}(A = 20) = 0.0044$, $\delta_{\rm F}(A = 100) = 9.3 \times 10^{-5}$, and $\delta_{\rm F}(A = 208) = 7.0 \times 10^{-6}$.

The presence of the small dimensionless parameter $\delta_{\rm F} \ll 1$ in the Fermi distribution of nuclear matter $\rho_{\rm F}(r)$ makes it easy to convert it into a symmetrical one-dimensional form with an accuracy defined by the parameter $\delta \ll 1$:

$$\rho(x) = \frac{\rho_0}{1 + \delta \cosh(x/a_0)}, \tag{4}$$

where $\delta = 2\delta_{\rm F}$.

A comparison between the Fermi density distributions and the symmetric soliton shape satisfying Eq. (4) shows practically complete coincidence of two functions, starting with the shape squareness parameters: $\delta_{\rm F} = 0.01$.

(2) The second keystone of our model is to find such a nonlinear evolution equation whose soliton-like solution will be in good agreement with the Fermi nuclear density distribution. Our main idea is based on solving the inverse problem of diffraction and dispersion: We have shown that to achieve stationary non-dispersive and non-diffracting self-trapped (soliton-like) modes of de Broglie matter-wave dynamics, it is necessary to introduce nonlinear compensating potentials into the wave equation for the "soliton-like nucleus" with the Fermi density distribution.[9]

If the three-dimensional stationary self-trapping problem is reduced to $(1 + 1)$- dimensional slab-like physical picture, the slab soliton-like nucleus with the Fermi-like nuclear density distribution

$$\varphi(x, t) = \frac{\sqrt{2}\exp(it/2)}{[1 + \delta \cosh(2x)]^{1/2}} \tag{5}$$

will satisfy the following evolution equation

$$i\frac{\partial \varphi}{\partial t} + \frac{1}{2}\frac{\partial \varphi}{\partial x^2} + |\varphi|^2\varphi - \frac{3}{8}(1 - \delta^2)|\varphi|^4\varphi = 0. \tag{6}$$

Introducing the parameter $\alpha = 3(1 - \delta^2)/8$ and taking into account both the Galilean invariance and the scaling symmetry of the model (6)

$$\varphi \to \eta\varphi, \quad t \to t/\eta^2, \quad x \to (x - Vt)/\eta, \quad \alpha \to \alpha/\eta^2, \tag{7}$$

let us rewrite the main equation (6) in the form well known in nonlinear optics:

$$i\frac{\partial \varphi}{\partial t} + \frac{1}{2}\frac{\partial \varphi}{\partial x^2} + |\varphi|^2\varphi - \alpha|\varphi|^4\varphi = 0 \tag{8}$$

with the Galilean invariant solution

$$\varphi_{\rm sol}(x, t) = \frac{\sqrt{2}\eta \exp[iVx - \frac{i}{2}(V^2 - \eta^2)t]}{[1 + \delta \cosh(2\eta(x - Vt))]^{1/2}}, \tag{9}$$

where $\delta = \sqrt{1 - 8\alpha\eta^2/3}$.

It is important to note that Eqs. (6) and (8) have solutions not only in the form of bright solitons (5) and (9), but also in the form of dark and gray solitons. An interested reader can find a regular method for obtaining various solutions of Eqs. (6) and (8) in the work of Ref. [10] written specifically for pedagogical purposes.

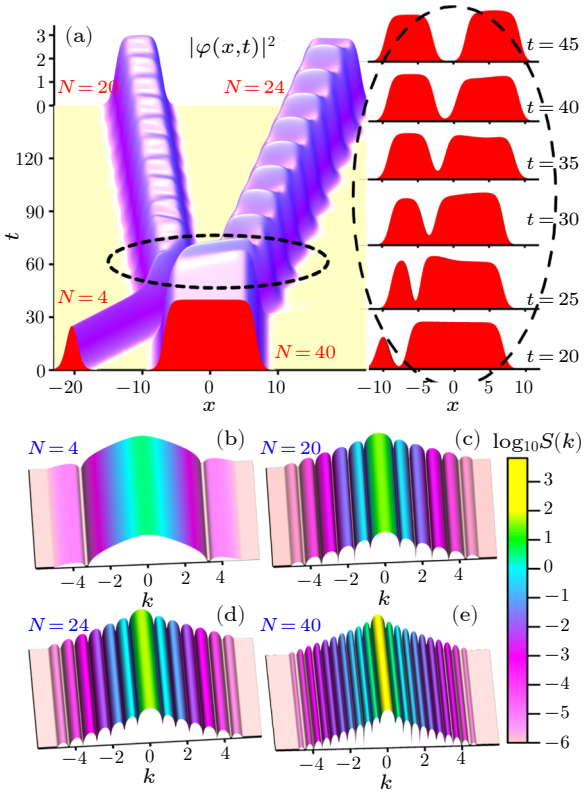


**Fig. 1.** Nuclear-like soliton reaction. (a) Analogy with the alpha-particle ($N = 4$) induced fission reaction on the nucleus with $N = 40$ after choosing the initial velocity $V = -0.4965$. Details of the dynamics of fission and redistribution of "solitonic nucleons" (marked with black dotted line) are shown in a stroboscopic style. (b)–(e) Fourier spectra $S(k)$ of nucleus-like solitons shown in decimal logarithmic scale for the numbers of constituent nucleons $N$: (b) $N = 4$, (c) $N = 20$, (d) $N = 24$, and (e) $N = 40$. Noteworthy is the good separation of spectral components in the momentum space, as well as the complete preservation of the initial and final numbers of components during the reaction.

(3) The third cornerstone of our model concerns the fulfillment of basic conservation laws: conservation of energy, momentum, and baryon number or total number of nucleons $N$ in reactions. Analogies between nuclear reactions and nonlinear solitary waves imply conservation of

total number of nucleons. This means that the nucleon, as a constituent part of the nucleus, cannot be divided into parts, just as some part of the nucleon cannot be created in the reaction. Therefore we define a number of particles in the soliton-like nucleus as an integral of the Fermi-like density distribution with strict requirements that the values of this integral can only be integers,

$$N = \int_{-\infty}^{\infty} |\varphi(x,\, t)|^2 dx = \frac{\eta}{\sqrt{1-\delta^2}} \ln\Big(\frac{1+\sqrt{1-\delta^2}}{1-\sqrt{1-\delta^2}}\Big). \tag{10}$$

The soliton-like nuclear reaction must be self-arranged in such a way that allow reaction channels which contain only integer numbers of nucleons. Thus, both the velocities and phases of the interacting solitons must be chosen to satisfy this requirement. The main features of the nuclear-like soliton reaction are shown in Fig. 1. The energy flow mechanisms studied within the framework of the model (8) are always the same: the energy transfer occurs only in a narrow channel formed between solitons. From an optical point of view, the energy exchange process can be considered as the formation of a gray soliton, as shown in Fig. 1(a). Details of the energy flow among soliton-like nuclei indicated by the black dotted line are shown in a stroboscopic style in Fig. 1(a). The shape of the spectrum allows the nucleus-like solitons shown in Fig. 1(a) to be called well-dressed. Let us also note the formal analogy with nuclear form factors for transferred moments (see the books[8] for details). It is interesting to emphasize that similar spectral structures appear after the points of optimal compression of multisoliton pulses[11] and in parton-like soliton structures that arise in nonlinear analogues of coherent states.[12]

Our model allows us to determine another remarkable analogy between Fermi-like optical solitons and nuclei: the soliton binding energy per "nucleon". This energy is a part of the total energy,

$$\langle E\rangle = \frac{\langle\varphi \mid \widehat{H} \mid \varphi\rangle}{\langle\varphi \mid \varphi\rangle} = \langle T_{\rm kin}\rangle + \langle E_{\rm bind}\rangle, \tag{11}$$

$$\begin{aligned}\langle E_{\rm bind}\rangle &= \int_{-\infty}^{\infty}\Big(-\frac{1}{2}|\varphi|^4 + \frac{\alpha}{3}|\varphi|^6\Big)dx \Big/ \int_{-\infty}^{\infty}|\varphi|^2 dx \\ &= -\frac{\eta^2}{2(1-\delta^2)}\Big[\frac{1}{2} + \frac{1}{2}(1-\delta^2) \\ &\quad - \sqrt{1-\delta^2}/\ln\Big(\frac{1+\sqrt{1-\delta^2}}{1-\sqrt{1-\delta^2}}\Big)\Big],\end{aligned} \tag{12}$$

obtained from the conserved full Hamiltonian

$$\widehat{H} = \int_{-\infty}^{\infty}\Big[\frac{1}{2}|\varphi_x|^2 - \frac{1}{2}|\varphi|^4 + \frac{\alpha}{3}|\varphi|^6\Big]dx. \tag{13}$$

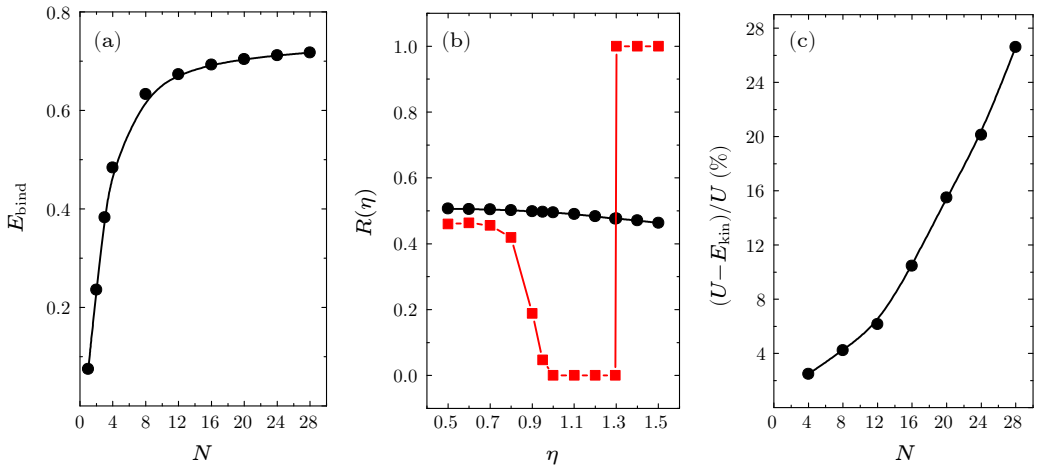


**Fig. 2.** Dependence of the absolute value of the binding energy of nucleus-like soliton on the "nucleon number" $N$ at $\alpha = 0.25$. (b) The NLSE soliton self-induced sub-barrier transparency effect (red squares) compared with the linear tunneling scenario (black circles). Normalized reflection coefficient $R(\eta)$ as a function of initial amplitude of the NLSE soliton $\eta$ (red squares) and linear wave packets tunneling through the Gaussian barrier $U(x) = U\exp(-(x/b)^2)$ after choosing the parameters: $U = 0.51$; $b = 5.0$; $v = -1.0$. The classic kinetic energy $E_{\rm kin} = v^2/2$ of the incident wave packets was chosen slightly below the height of the potential barrier. (c) Dependence of the "window" $(U - E_{\rm kin})/U$ of self-induced sub-barrier transparency on the number of constituent "nucleons" $N$ after choosing the parameters: $U = 0.32$; $b = 5.0$; $v = -0.8$.

Figure 2(a) shows the dependence of the soliton binding energy on a nucleon number $N$. Comparing it with the experimental curve for stable nuclei,[8] one can find the similar behavior: the initial rapid increase for light nuclei and saturation for medium-weight nuclei. Of course, the soliton model cannot describe slow decrease of the curve for heavier nuclei, in view of the absence of the Coulomb repulsion in this simple model. Similar to the nuclear binding energy, the soliton binding energy (12) indicates possibility of energetically favorable fusion reactions between two light nuclei, which can fuse together to form a heavier one. Thus the fusion reactions are the source of nucleosynthesis of elements up to $A = N \leq 56$.

Based on the curve in Fig. 2(a), we can calculate the threshold energy and the $Q$-value[8] for a nuclear reaction. In particular, this curve indicates that the reaction of synthesis of light nuclei is always exothermic reaction, because the $Q$-value is the difference between masses of

nuclei in the entrance channel and the exit channel, or respectively, the difference between the binding energies of products and initial nuclei. For example, the $Q$-value of the reaction presented in Fig. 1 is calculated by

$$E_{\rm bind}(20)+E_{\rm bind}(24)-E_{\rm bind}(4)-E_{\rm bind}(40)\approx 0.2>0. \tag{14}$$

*Important Features of Nuclear-Like Reactions with Solitons.* (1) Diffraction-free and dispersion-free solutions of the wave equation (6), having the Fermi-like density distribution, are formed as a result of the competition of self-attraction $\sim|\varphi|^2$ and self-repulsion $\sim-|\varphi|^4$, therefore, the general self-interaction potential takes the so-called W-shape with a specific repulsive core. Using this property, we called Fermi-like solutions of model (6)–(8) as the solitons with a "repulsive core". Along with satisfying all selection rules, it is necessary to exclude the possibility of interference effects, in which the model (6)–(8) goes beyond the limits of its applicability: $|\varphi|^2-\alpha|\varphi|^4>0$. Note that it is necessary to control the maximum allowed soliton density $|\varphi|^2<\alpha^{-1}$ and select only those results of soliton nuclear-like reactions that contain strictly integer values of integrals (10) (integer number of nucleons), and do not emit dispersive waves (unlike conventional studies with solitons in non-integrable systems). These main restrictions are well illustrated in Fig. 1.

(2) Inelastic interactions among "repulsive core" solitons appear only when the parameter $\delta$ decreases until the "cloud" of "dressing" spectral side-bands appears in the frequency or spatial (angular) spectra of temporal or spatial solitons, respectively. Thus, as the number of trapped soliton-like nucleons in the soliton-like nucleus increases, the gradient of a step-function profile of the density distribution is leading to the appearance of novel and novel spectral side-bands as is shown in Figs. 1(b)–1(e). If we calculate the nuclear form-factor for the well-known simplest model of the uniform nuclear density distribution (find this model, for example, in the book[8]) and compare it with the Fourier spectra of the "repulsive core" and "well-dressed" solitons having "quantized" parameter $\delta$ that are shown in Figs. 1(b)–1(e), we find the analogous behavior and very similar structure.

(3) As was shown in our previous work,[9] the analogy between optical solitons and nuclear-like matter-wave solitons can be written in the form that connects the main parameter of the problem $\delta$ with the parameter $\delta_{\rm F}$ introduced for the Fermi distribution of nuclear density:

$$2\delta_{\rm F}=\delta, \tag{15}$$

$$2\exp(-r_0A^{1/3}/a_0)=\sqrt{1-8\alpha\eta^2/3}. \tag{16}$$

Concluding a brief review of the main points and most important features of the concept of nonlinear optics of nuclear reactions, it should be emphasized that Eq. (8) has long been known in various fields of physics. For the first time, Eq. (8) was obtained and solved in 1976 by Kovalev and Kosevich.[13] In their paper, the authors considered a one-dimensional bosonic system with delta-like pair attraction and three-body repulsion and demonstrated that such a state can be considered as a drop of condensate. In nonlinear optics, model (8) appeared in 1979 in the article by Pushkarov, Pushkarov and Tomov.[14] A detailed review of the key literature in this area and references to Skyrme's ideas in nuclear physics can be found in our previous work.[9]

*Nonlinear Tunneling of "Soliton-Like Nuclei" and Sub-Barrier Fusion–Fission Reactions.* Nonlinear tunneling of solitons, predicted by Newell back in 1978,[15] continues to be one of the most intriguing problems in nonlinear science and constantly attracts new interest, both experimental and theoretical. Traditionally, the scattering of solitons on external potentials has been mostly studied within the framework of the canonical NLSE model (see, for example, the works,[16–19] recent papers,[20,21] and references therein). One of the most intriguing features of the NLSE soliton tunneling scenarios is the emergence of two new effects: the soliton self-induced sub-barrier transparency and "shooting out" of high-energy NLSE solitons from an arbitrary $N$-soliton nonlinear superposition.[18] Therefore, several open questions arise. What new phenomena should be expected for the nucleus-like soliton tunneling through a classically forbidden potential barrier? Is the effect of self-induced sub-barrier transparency preserved for them as well? What other effects arise instead of the "shooting out" of high-energy NLSE solitons?

Self-induced sub-barrier transparency for the NLSE solitons can be defined as the appearance of the "transparency window" in the reflection (or transmission) coefficients depending on the amplitudes of tunneling solitons as is shown in Fig. 2(b). To explain this effect, it is very useful to consider the formal analogy between the Schrödinger soliton tunneling through a classically forbidden potential barrier and Gamow's scenario[22] of quantum-mechanical tunneling and alpha particle decay [see Fig. 3(c)]. The total tunneling scenario, as can be seen from Figs. 3(a)–3(c), becomes completely analogous to quantum-mechanical tunneling, and the effective potential takes the form of the so-called "toy soliton nuclear model" represented by Fig. 3(c).

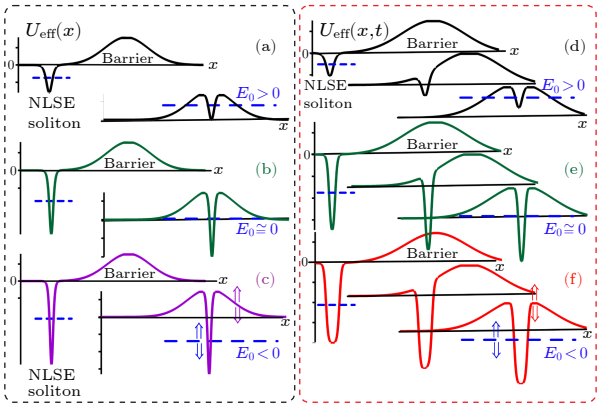


**Fig. 3.** Comparison of qualitative examples revealing the "hidden" role of the soliton binding energy in the tunneling dynamics of [(a), (b), (c)] NLSE solitons and [(d), (e), (f)] nucleus-like solitons. Effective potentials $U_{\rm eff}(x)$ are shown for different solitons placed on the edge and on the top of the barrier. Effective energy levels: [(a), (d)] $E_0>0$; [(b), (e)] $E_0\approx 0$ and [(c), (f)] $E_0<0$, demonstrate remarkable analogy with Gamow's scenario of $\alpha$-particle tunneling and decay, and explain the different tunneling scenarios for solitons with varying form factors (hence, different binding energies).

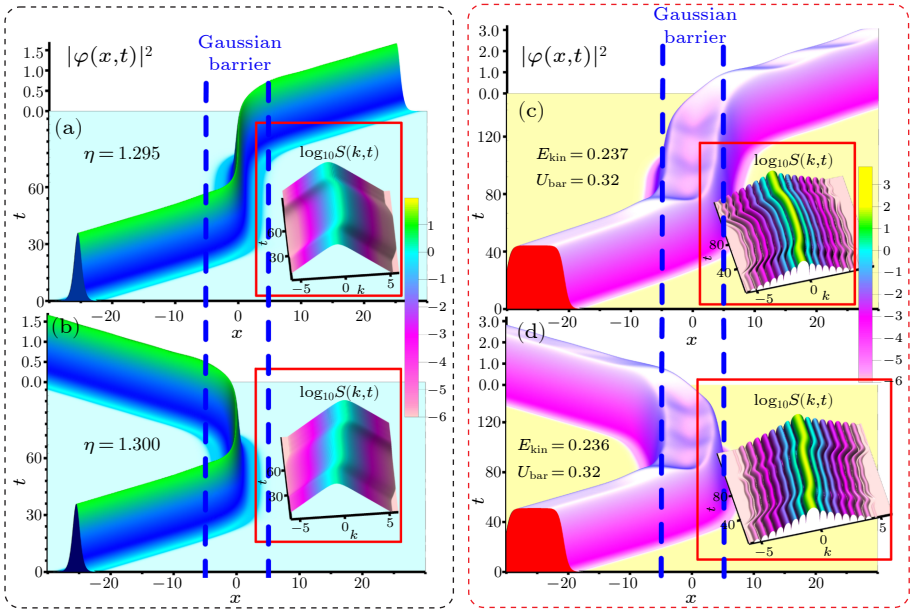


**Fig. 4.** The soliton self-induced sub-barrier transparency effect. Comparison of an abrupt transition from full transparency to full reflection for [(a), (b)] NLSE solitons with different amplitudes and [(c), (d)] nucleus-like soliton $N = 28$ with different kinetic energies.

Note that NLSE solitons can be characterized by their own energy levels $E(\eta_{\rm NLSE}) = -\eta^2_{\rm NLSE}/2 < 0$, also known in Bose–Einstein condensates as chemical potentials. These levels are related to the NLSE soliton binding energy as $E(\eta_{\rm NLSE}) = 3/2E_{\rm bind}$, where $E_{\rm bind} = -\eta^2/3$. Under the action of the external repulsive potential $U_0$, solitons with different amplitudes, and therefore different binding energies, can be "uplifted" from their bound-state energy level $E(\eta_{\rm NLSE}) < 0$ to the metastable energy level $E(\eta_{\rm NLSE}) = -\eta^2_{\rm NLSE}/2 + U_0 > 0$ in the effective potential as is shown in Figs. 3(c) and 3(d). The boundaries between these two regimes: sub-barrier complete transmission, or classical-like total reflection from forbidden barrier, as is demonstrated in Figs. 4(a) and 4(b), turn out to be very sharp. Namely, when the amplitude of tunneling NLSE soliton changes only in the third digit, the nonlinear mode of complete transparency suddenly changes to the mode of classical-like repulsion from a classically forbidden barrier [see detailed scenarios in Figs. 4(a) and 4(b)].

The transition to a more realistic model given by Eq. (8) significantly complicates the total physical scenario of nonlinear tunneling.

On the one hand, with an increase of the number of "nucleons" $N$ in the nucleus-like solitons with the Fermi density distribution (3), the effective potential takes a form similar to the real saturated nuclear-like potential as is shown in Figs. 3(d)–3(f). Thus, unlike the NLSE soliton, the bound-state level of the nucleus-like soliton cannot move down indefinitely, and the "uplifting" mechanism arising under the influence of the repulsive barrier, now depends on the saturation effect as is demonstrated in Figs. 3(d)–3(f).

On the other hand, the binding energy of the nucleus-like soliton is also saturated [see Fig. 2(a)] and consequently, the barrier itself can now induce soliton nuclear-like reactions.

First of all, we emphasize that the effect of self-induced transparency is preserved for the nucleus-like solitons, and we have observed this effect in our computer experiments up to a sufficiently large number of "constituent nucleons" in the tunneling soliton. The dependence of the width of the self-induced transparency region on the number of nucleons in the tunneling nuclear-like soliton, obtained in our computer experiments, is shown in Fig. 2(c).

Also noteworthy is the fact that the boundary between the self-induced transparency regime and classical reflection from the forbidden potential barrier remains to be very sharp for nucleus-like solitons similar to the tunneling scenario of the NLSE solitons. If we compare the results given in Figs. 4(a) & 4(b) and Figs. 4(c) & 4(d) we may conclude that a change in the third digit of the NLSE soliton amplitude or in the kinetic energy of a nucleus-like soliton leads to a radical change in two opposite regimes: complete self-induced sub-barrier transparency and classical reflection.

The most important prediction of the nonlinear optical model (8) for "soliton-like nuclei" with Fermi density distribution (4) and saturating binding energy (12) is the possibility of sub-barrier soliton-like nuclear reactions induced in the process of nonlinear tunneling. Notice that the dependence of the self-induced transparency window on the number of constituent "nucleons" $N$ [shown in Fig. 2(c)] is displayed only up to $N = 28$. At large values of $N$, nuclear-like reactions occur during sub-barrier nonlinear tunneling, similar to those shown in Figs. 5(a) and 5(b).

For example, soliton-like nuclei formed in the induced fission reaction can be ejected in both one and the opposite directions; so that the first nucleus-like soliton

leaves the barrier and only after a certain delay the second soliton appears, as shown in Fig. 5(a). In this case, the tunneling soliton penetrates inside the barrier region, and the energy exchange occurs at the top of the barrier [Fig. 5(a)]. The parameters (the kinetic energy and repulsion potential) are the same as in the study of the "window" $(U - E_{\rm kin})/U$ of self-induced sub-barrier transparency [see Fig. 2(c)], but the number of nucleons is chosen to be large enough, $N = 40$, to emphasize the transition from sub-barrier transparency to sub-barrier fission-like reaction producing new nuclei [compare Fig. 4(c) and Fig. 5(a)]. In Figs. 5(a) and 5(b), we compare two fission-like reactions: those that occur at the top of the barrier and those at its leading edge. Full spectrum of the reaction $(N = 40) \to (N = 28) + (N = 12)$ is shown in Fig. 5(c). The imperative nature of the law of conservation of the number of nucleons means that the results of fission-like reaction shown in Fig. 5(b) also consist only of integers nucleon number $N$. As a result, in the spectral (momentum) domain, we observe a redistribution between the spectral side bands, as is shown in Fig. 5(c) (a spectral "thin" structure appears because solitons are ejected in opposite directions).

As we have already noted, in the "toy solitonic model of the nucleus" the effective potential is surrounded by a repulsive shell as shown in Fig. 2(d). Let us place our soliton on the top of the barrier so that the edges of the barrier simulate the repulsive potential that must be overcome for a reaction to occur. The results of such a computer experiment are presented in Figs. 5(d)–5(f). One thing is clear: Nuclear-like fusion can only be observed when soliton-like nuclei penetrate under the surrounded barrier.

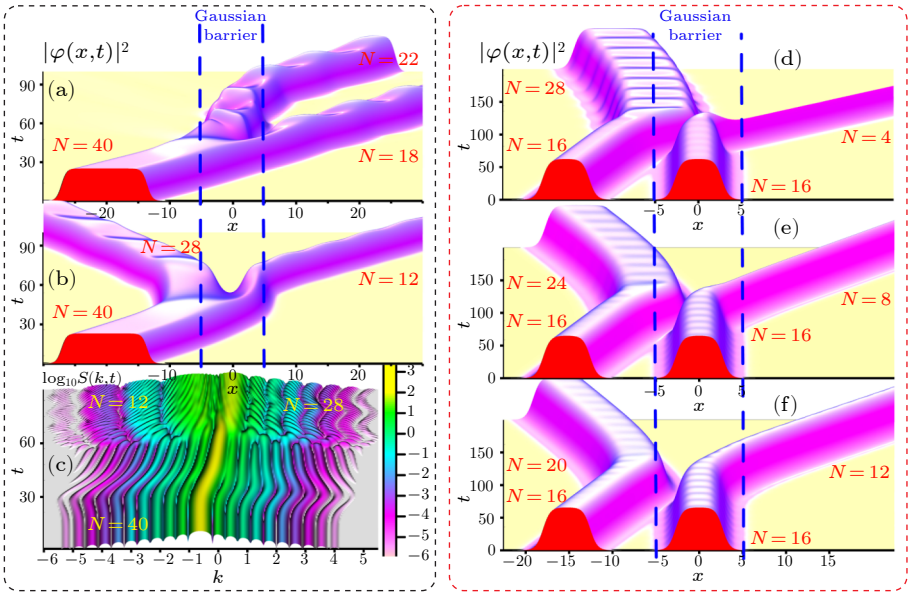


**Fig. 5.** Sub-barrier reactions of nucleus-like solitons. [(a), (b), (c)] Fission-like reactions: (a) $(N = 40) \to (N = 22) + (N = 18)$; [(b), (c)] $(N = 40) \to (N = 28) + (N = 12)$. [(d), (e), (f)] Fusion-like reactions: (d) $(N = 16) + (N = 16) \to (N = 28) + (N = 4)$; (e) $(N = 16) + (N = 16) \to (N = 24) + (N = 8)$; (f) $(N = 16) + (N = 16) \to (N = 20) + (N = 12)$. Reactions occur during nonlinear tunneling of nucleus-like solitons through the classically forbidden Gaussian barrier: $U(x) = U\exp(-(x/b)^2)$, after choosing the parameters: (a) $U = 0.32; b = 5.0; v = -0.8$; [(b), (c)] $U = 0.4$; $b = 5.0$; $v = -0.645$; (d) $U = 0.007$; $b = 5.0$; $v = -0.1$; (e) $U = 0.0135$; $b = 5.0$; $v = -0.1$; (f) $U = 0.02$; $b = 5.0$; $v = -0.1$.

Summing up the results of our work, we especially emphasize two most surprising facts. First, considering the nuclei themselves as the soliton-like objects with Fermi's density distribution (3) and using the smallness of the shape squareness parameter $\delta$ (15) for the nuclei, we solved the inverse problem and derived the basic equations (6)–(8), the diffraction-free and dispersion-free solutions of which are nucleus-like solitons.

Secondly, despite the fact that the resulting equations (6)–(8) are known in low-temperature physics [11] and nonlinear optics [12] for about 50 years, the following remarkable facts have not yet been clarified. Inelastic interactions among the nuclei-like solitons arise only upon appearance and the "exchange" between "particle-like" side-bands in their spectra. The most surprising features among the huge number of nuclear-like reactions observed within the framework of this simple model (6)–(8), there are those that satisfy the principle of conservation of the "number of nucleons" (10), similar to the law of conservation of the baryon number in nuclear physics.

Our analysis of the nonlinear soliton tunneling would be incomplete if we did not mention nonlinear tunneling of nonautonomous solitons (see, for example, Refs. [23–35] and references therein). The term nonautonomy itself suggests that the parameters (coefficients) of the nonlinear evolution equations become the functions of time or coordinates. Nonautonomous solitons arise in the framework of the inverse scattering transform (IST) method with varying spectral parameter and they significantly expand the concept of canonical solitons moving with constant ve-

locities and amplitudes. The main property of classical solitons (introduced for autonomous systems) to interact elastically is preserved, but nonautonomous solitons generally move with varying amplitudes, speeds, and spectra adapted both to the external potentials and to the dispersion and nonlinearity variations.

The effects of nonlinear tunneling of nonautonomous solitons arise, for example, in problems of optimal amplification of solitons in communication systems, or their coupling in fibers with different dispersion, in which, in order to preserve the soliton itself, we must control all linear and nonlinear parameters of the system. These problems have received much attention in the past [24–31] and continue to be an active area of research today. [32–35]

Due to their remarkable properties, solitons were proposed from the very beginning as idealized mathematical structures to describe "extended elementary particles", "soliton models of atoms", "nonlinear vacuums", and they continue to be a very active area of research. [36–45]

The analogy we have considered here only demonstrates a possible connection between the nonlinear dynamics of "soliton-like nuclei" having the Fermi density distribution and the nuclear reactions that take place among atomic nuclei. We have demonstrated novel nonlinear tunneling scenarios of nucleus-like solitons through the classically forbidden repulsive barriers, and we predict possibility to realize "nuclear-like sub-barrier reactions" among optical solitons. Our main goal was to demonstrate that such properties of the model as saturation of the binding energy, the ability to find among the whole variety of "soliton-like nuclei" interactions and tunneling scenarios those that preserve the "number of nucleons" the appearance of spectral side-bands, all these properties cannot be accidental. This gives us reason to hope that the intriguing features of sub-barrier fission and fusion reactions with "soliton-like nuclei" shown in Figs. 5(a)–5(f) can shed additional light on fundamental problems of nuclear reactions of astrophysical interest. [46]

# References


[1] Hasegawa A and Tappert F 1973 *Appl. Phys. Lett.* **23** 142
[2] Hasegawa A 2023 *Optik* **279** 170769
[3] Dudley J M, Finot C, Genty G, and Taylor R 2023 *Opt. Photon. News* **34** 26
[4] Feshbach H, Porter C E, and Weisskopf V F 1954 *Phys. Rev.* **96** 448
[5] Hodgson P E 1963 *The Optical Model of Elastic Scattering* (Oxford: Clarendon Press)
[6] Woods R D and Saxon D S 1954 *Phys. Rev.* **95** 577
[7] Hofstadter R 1957 *Ann.Rev. Nucl. Sci.* **7** 231
[8] Amsler C 2015 *Nuclear and Particle Physics* (Bristol: IOP Publishing)
[9] Serkin V N and Belyaeva T L 2023 *Opt. Commun.* **549** 129831
[10] Serkin V N, Belyaeva T L, Alexandrov I V, and Melchor G M 2001 *Proc. SPIE* **4271** 292
[11] Dianov E M, Nikonova Z S, Prokhorov A M, and Serkin V N 1986 *Sov. Tech. Phys. Lett.* **12** 311
[12] Belyaeva T L, Kovachev L M, and Serkin V N 2020 *Optik* **210** 164483
[13] Kovalev A S and Kosevich A M 1976 *Sov. J. Low Temp. Phys.* **2** 449
[14] Pushkarov K I, Pushkarov D I, and Tomov I V 1979 *Opt. Quantum Electron.* **11** 471
[15] Newell A C 1978 *J. Math. Phys.* **19** 1126
[16] Belyaeva T L and Serkin V N 2012 *Eur. Phys. J. D* **66** 153
[17] Belyaeva T L, Serkin V N, Hernandez-Tenorio C, and Garcia-Santibañez F 2010 *J. Mod. Opt.* **57** 1087
[18] Serkin V N, Hasegawa A, and Belyaeva T L 2013 *J. Mod. Opt.* **60** 116
[19] Serkin V N and Belyaeva T L 2019 *Optik* **176** 38
[20] Uthayakumar T, Al Sakkaf L, and Al Khawaja U 2021 *Phys. Rev. E* **104** 034203
[21] Damgaard Hansen S, Nygaard N, and Mølmer K 2021 *Appl. Sci.* **11** 2294
[22] Gamow G 1928 *Z. Phys.* **51** 204
[23] Serkin V N, Hasegawa A, and Belyaeva T L 2017 *Hidden Symmetry Reductions and the Ablowitz-Kaup-Newell-Segur Hierarchies for Nonautonomous Solitons.* In: *Odyssey of Light in Nonlinear Optical Fibers* (New York: CRC Press) p. 145
[24] Serkin V N and Belyaeva T L 2001 *J. Exp. Theor. Phys. Lett.* **74** 573
[25] Gao Y, Tang X Y, and Lou S Y 2009 *Chin. Phys. Lett.* **26** 030502
[26] Dai C Q, Wang Y Y, and Zhang J F 2010 *Opt. Express* **18** 17548
[27] He J D and Zhang J F 2011 *J. Phys. A* **44** 205203
[28] Mahalingam A and Mani Rajan M S 2015 *Opt. Fiber Technol.* **25** 44
[29] Jia H P, Yang R C, Dai C Q, and Guo Y Y 2019 *J. Mod. Opt.* **66** 665
[30] Hernandez-Tenorio C, Belyaeva T L, and Serkin V N 2007 *Physica B* **398** 460
[31] Serkin V N and Belyaeva T L 2018 *Optik* **158** 1289
[32] Belyaeva T L and Serkin V N 2022 *Nonlinear Dyn.* **107** 1153
[33] Zhou Q 2022 *Chin. Phys. Lett.* **39** 010501
[34] Maddouri K, Triki H, Wang N, and Zhou Q 2024 *Phys. Lett. A* **512** 129584
[35] Wang H T, Yang H J, Tian Y, and Liu W J 2024 *Chaos* **34** 053125
[36] Rebbi C and Soliani G 1984 *Solitons and Particles* (Singapore: World Scientific)
[37] Rybakov Y P and Saha B 1995 *Found. Phys.* **25** 1723
[38] Manton N S 2008 *Nonlinearity* **21** T221
[39] Kovachev L M 2004 *Physica D* **190** 78
[40] Kovachev L M, Georgieva D A, and Kovachev K L 2012 *Opt. Lett.* **37** 4047
[41] Zhao Y, Lei Y B, Xu Y X *et al.* 2022 *Chin. Phys. Lett.* **39** 034202
[42] Zhou Q, Zhong Y, Triki H *et al.* 2022 *Chin. Phys. Lett.* **39** 044202
[43] Yao Y, Yi H, Zhang X, and Ma G 2023 *Chin. Phys. Lett.* **40** 100503
[44] Wang S B, Ma G L, Zhang X, and Zhu D Y 2022 *Chin. Phys. Lett.* **39** 114202
[45] Wazwaz A M 2023 *Chin. Phys. Lett.* **40** 120501
[46] Belyaeva T L, Aguilera E F, Martinez-Quiroz E *et al.* 2009 *Phys. Rev. C* **80** 064617