Non-Hermitian CHSH* Game with a Single Trapped-Ion Qubit

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The Clauser–Horne–Shimony–Holt (CHSH) game provides a captivating illustration of the advantages of quantum strategies over classical ones. In a recent study, a variant of the CHSH game leveraging a single qubit system, referred to as the CHSH* game, has been identified. We demonstrate that this mapping relationship between these two games remains effective even for a non-unitary gate. Here we delve into the breach of Tsirelson's bound in a non-Hermitian system, predicting changes in the upper and lower bounds of the player's winning probability when employing quantum strategies in a single dissipative qubit system. We experimentally explore the impact of the CHSH* game on the player's winning probability in a single trapped-ion dissipative system, demonstrating a violation of Tsirelson's bound under the influence of parity-time ($\mathcal{PT}$) symmetry. These results contribute to a deeper understanding of the influence of non-Hermitian systems on quantum games and the behavior of quantum systems under $\mathcal{PT}$ symmetry, which is crucial for designing more robust and efficient quantum protocols.

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The advantages of quantum strategies over classical ones have recently attracted significant attention in game theory. As an example, the Bell inequality,[1,2] a crucial and renowned method to test local realism in quantum mechanics, can be recast into the Clauser–Horne–Shimony–Holt (CHSH) game.[2] In this game, two space-like separated players, Alice and Bob, are privately assigned a classical bit a and b by a judge, and neither player knows the other's bit. Alice and Bob discuss their strategies before receiving the classical bits, and return bits x and y based on agreed-upon strategies. The two players would win the game when $x \oplus y = a \cdot b$, otherwise the judge wins. If two players only use classical strategies, the optimal winning probability is $w(\text{CHSH}) = 0.75$. However, if certain quantum resources are utilized, the winning probability will change and be bounded to a certain range, known as Tsirelson's bound which is an upper or lower limit to quantum mechanical correlations between distant events.[3]

Recently, a variant of the CHSH game, i.e., the CHSH* game was proposed and studied theoretically[4] and experimentally.[5] The CHSH* game requires only a single player, without involving demanding experimental resources such as two space-like separated parties. Any strategy of the traditional CHSH game can be mapped into a qubit through unitary operations and projective measurements. We previously studied the CHSH* game in a single photonic qubit, the simplest non-trivial quantum system, revealing how Tsirelson's bound emerges under the condition of consecutively measurements of a single qubit.[5] This result motivates us to extend the research to non-Hermitian qubit systems.

The interest for exploring the CHSH* game in non-Hermitian systems is that, in conventional quantum mechanics, Hermiticity is regarded as a fundamental assumption to guarantee some conventional quantum principles, such as entanglement monotonicity,[6,7] the no-signaling principle,[8,9] quantum speed limit,[10] and quantum indistinguishability between quantum states.[11] These principles obey Hermitian quantum mechanics unquestionably, but the violations of some principles were verified in non-Hermitian systems, particularly in $\mathcal{PT}$-symmetric quantum systems,[8–10,12–14] which is proposed by Bender and colleagues in 1998.[15] Recently, $\mathcal{PT}$ symmetry has been constructed in quantum systems including single photons,[16] cold atoms,[17] and trapped ions,[18,19] etc. It is interesting to conduct a comprehensive study on how the quantum principles behave in $\mathcal{PT}$-symmetric settings.[20,21] Moreover, it is instructive for practical applications based on $\mathcal{PT}$ symmetry, as it may help to identify the specific advantages $\mathcal{PT}$ symmetry offers over Hermitian quantum protocols and the extent to which these benefits are realized. Inspired by these works, an intriguing question arises: Does the advantage of quantum strategies change in a CHSH* game using a $\mathcal{PT}$-symmetric qubit? This provides valuable insights into the intricate interplay between quantum protocols and non-Hermitian dynamics,

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enabling a profound comprehension of their mutual interaction. Moreover, this helps us understand how dissipation and decoherence affect Tsirelson's bound, providing valuable insights into the robustness of quantum features under environmental interactions.

In this Letter, we investigate the dynamics of the CHSH* game in both a Hermitian and a $\mathcal{PT}$-symmetric system based on a single trapped-ion qubit, and experimentally determine the player's probability of winning. Our results demonstrate the violation of Tsirelson's bound in a dissipative qubit system.

Theory. There is a mapping relationship between the CHSH game and the CHSH* game.[4] In the CHSH* game, there is one player only and the parameters a, $b \in \{0, 1\}$ are sent to the player using a classical channel. The player can sequentially perform two unitary operations U_a and U_b on a given qubit and obtain the returning outcome c from the projective measurement of the final state ($c = 0$ if $|0\rangle$ is detected; $c = 1$ if $|1\rangle$ is detected). The player will win the game if the returning outcome $c = a \cdot b$. The winning probability[5] $P_{\rm w} = (1/4)(\sin\theta + \cos\theta + 2)$ reaches the maximum value is $\cos^2(\pi/8) \approx 0.8536$ when $\theta = \pi/4$, where θ represents the angle corresponding to the rotation operation applied by the unitary operation on the qubit. Meanwhile, we can get the minimum value of $\cos^2(5\pi/8) \approx 0.1464$ when $\theta = 5\pi/4$. These two values of the winning probability are the upper and lower limits of Tsirelson's bound.

We implement the Hermitian case as follows: We first assume that the initial state is prepared in the $|0\rangle$ state and the Pauli Z operator is used as the final measurement. The optimal strategy can be obtained by optimizing the unitary operations $U_{\rm a}$ and $U_{\rm b}$. For the CHSH game, as shown in Fig. 1(a), a Hadamard gate and a controlled-NOT (CNOT) gate are used to generate a pair of entangled particles shared by two players, whereby Alice performs operation $U_{\rm a}^{\rm T}$ on her particle and Bob does not do any operation. We can get (for details see Note S1 of the Supplementary Material)

$$U_{\rm a}^{\rm T} \otimes I \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) = I \otimes U_{\rm a} \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right). \quad (1)$$

The physical meaning of Eq. (1) is that Alice performs $U_{\rm a}^{\rm T}$ operation on her particle and Bob does not operate on his particle, which is equivalent to that Alice does not operate on her own particle and Bob operates on his own particle with $U_{\rm a}$ operation. Equation (1) indicates a simplified case of the CHSH game mapping to the CHSH* game.

Here, we extend this scheme to the case of the $\mathcal{PT}$-symmetric system. Alice performs $U_{\rm a}^{\rm T}$ operation and Bob performs non-unitary $O_{\rm b}$ operation on the composite system, it can be expressed as

$$U_{\rm a}^{\rm T} \otimes O_{\rm b} \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) = (I \otimes O_{\rm b} U_{\rm a}) \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right), \quad (2)$$

where the $U_{\rm a}^{\rm T} \otimes O_{\rm b}$ operations on the composite system is equivalent to that Alice does not operate, Bob performs $O_{\rm b}U_{\rm a}$ operation on his own particle. Equation (2) can be deduced using Eq. (1) (for details see Note S1 of the Supplementary Material). Figures 1(a) and 1(b) show how to map a CHSH game to a CHSH* game in $\mathcal{PT}$-symmetric systems by replacing $U_{\rm b}$ with $O_{\rm b}$.

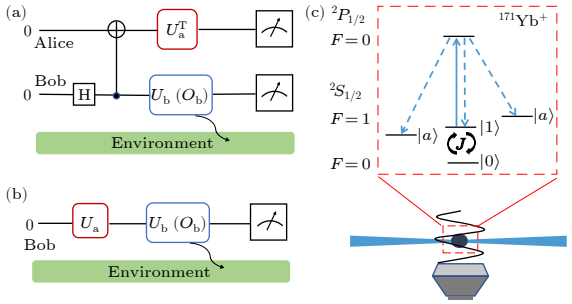

Fig. 1. (a) A quantum logic circuit of the CHSH game. Alice and Bob share a space-like separated entangled particle pairs and perform operations $U_{\rm a}^{\rm T}$, $U_{\rm b}$ ($O_{\rm b}$) to their particles to obtain results, respectively. (b) Single-system scheme of the CHSH* game. The player applies gates $U_{\rm a}$, $U_{\rm b}$ ($O_{\rm b}$) in sequence, then measures the result. When considering this game in $\mathcal{PT}$-symmetric systems, the second operator $U_{\rm b}$ is replaced by $O_{\rm b}$ which is a non-unitary operation. (c) Schematic diagram of the experimental setup.

The process of the CHSH* game in a dissipative trapped-ion qubit is described by a passive $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian H,

$$H = H_0 + i\gamma(\sigma_z - I) = H_{\mathcal{PT}} - i\gamma I, \quad (3)$$

where H_0 is the Hamiltonian of original unitary operation $U_{\rm b}$, $H_{\mathcal{PT}}$ is the $\mathcal{PT}$-symmetric Hamiltonian with balanced gain $+i\gamma$ and loss $-i\gamma$. $H_{\mathcal{PT}}$ commutes with the $\mathcal{PT}$ operator,[15] expressed as $[H_{\mathcal{PT}}, \mathcal{PT}] = 0$, where $\mathcal{P} = \sigma_x$, $\mathcal{T} = *$ denotes complex conjugation operation. Thus the non-unitary operation can be written as $O_{\rm b} = e^{-iHt}$, with t being the evolution time. The normalized winning probability $P'_{\rm w}$ of the player in the $\mathcal{PT}$-symmetric system is (see details in Note S2 of the Supplementary Material)

$$P'_{\rm w} = \frac{1}{4} \times \Big[- \frac{J\kappa \sinh(2\kappa t)}{J^2 - \gamma^2 \cosh^2(2\kappa t)} + \frac{2J^2 + (J^2 - 3\gamma^2)\cosh(2\kappa t) - 3\gamma\kappa \sinh(2\kappa t)}{J^2 - (\gamma^2 \cosh(2\kappa t) + \gamma\kappa \sinh(2\kappa t))} \Big], \quad (4)$$

where J is the coupling strength in H_0, γ is the dissipation rate, and $\kappa = \sqrt{J^2 - \gamma^2}$.

Experiments. In our experimental system, we utilize a single $^{171}Yb^+$ to study the $\mathcal{PT}$-symmetric CHSH* game. Experimental studies on $\mathcal{PT}$ symmetry can be found in the earlier works of our research group.[10,17,19] The experimental scheme is shown in Fig. 1(c). Two hyperfine states $|0\rangle = |F = 0, m_F = 0\rangle$ and $|1\rangle = |F = 1, m_F = 0\rangle$ are chosen as a qubit. The transition between the two states is coupled through microwave with the frequency of 12.643 GHz, and the coupling strength J is determined by fitting the Rabi frequency. A 369.5 nm dissipation laser beam drives the excitation from $|1\rangle$ to $^2P_{1/2}|F = 1, m_F = 0\rangle$ electronic excited state, and the laser contains only π polarization component. The electronic excited state spontaneously decays to three magnetic levels $^2S_{1/2}|F = 1, m_F = 0, \pm 1\rangle$

with same probability. Here, the two auxiliary levels $|a\rangle = |F = 1, m_F = 0, \pm 1\rangle$ are considered as environmental states.

In experiments, we prepare the initial state in the $|0\rangle$ state and then perform the $O_\mathrm{b}U_\mathrm{a}$ operation, the experiment timing sequence is shown in Fig. 2. The operation U_a (i.e., U_0 or U_1) is accomplished by microwave driving of the qubit, where U_0 does not change the initial state and can be considered as the identity matrix, having no effect on the ion. U_1 corresponds to applying a microwave $\pi/2$ pulse around the x axis. The operation matrix of O_b (i.e., O_0 or O_1) is implemented by simultaneously applying microwave and dissipative laser beams. O_0 corresponds to simultaneously applying dissipative laser and a microwave π pulse around the $-x$ axis. O_1 corresponds to simultaneously applying dissipative laser and a microwave π pulse around the x axis. Finally, we use the standard fluorescence counting rate threshold method [18,22] to detect the population of state $|0\rangle$ or $|1\rangle$ (for details see Note S3 of the Supplementary Material). Take note that, in non-unitary evolution, the detection beam, which includes $\sigma_\pm$ and π polarization, simultaneously pumps all three ground levels $^2S_{1/2}|F = 1, m_F = 0, \pm 1\rangle$, resulting in indistinguishable population information. Therefore, the probability in state $|1\rangle$ cannot be directly measured. Instead, it can be accurately ascertained by implementing an additional Rabi π flip before detection to exchange the populations of $|0\rangle$ and $|1\rangle$, given that the total probability across all four ground states ($|0\rangle$, $|1\rangle$, and $|a\rangle$) remains equal to 1. [10]

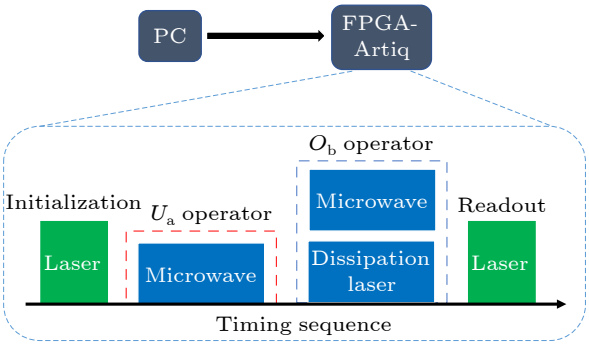

Fig. 2. Timing sequence for testing the non-Hermitian CHSH* game. A field-programmable gate array system is used to ensure precise and simultaneous control of microwave and dissipation laser beams.

Results and Discussion. The player's winning probability is theoretically determined by the coupling rate J and the dissipation rate γ as described in Eq. (4). In experiments, we perform four operations (O_0U_0, O_1U_0, O_0U_1, O_1U_1) to obtain the time-evolving populations of both $|0\rangle$ and $|1\rangle$ states, and then calculate the winning probability of each case. Then the total winning probability is the average number of four individual probabilities. The raw data of time-evolving population is shown in Note S3 of the Supplementary Material, and the winning probability of each operation is in Note S5. Figure 3 shows the player's winning probability oscillating periodically over time for different dissipation rates under fixed J, which is measured to be $J = 2\pi \times 21.8$ kHz through Rabi oscillation measurements during the data collection process. In the CHSH* game, the player needs to choose an appropriate strategy to maximize the winning probability. In Ref. [5], this is carried out by using a rotating polarizer in an optical system to rotate the polarization of a single photon for varying the operations. In our system, the time evolution of the dissipative qubit provides the strategy, where the different evolution times are equivalent to the different operations. Thus, we vary the duration of the operation O_b to maximize the player's winning probability.

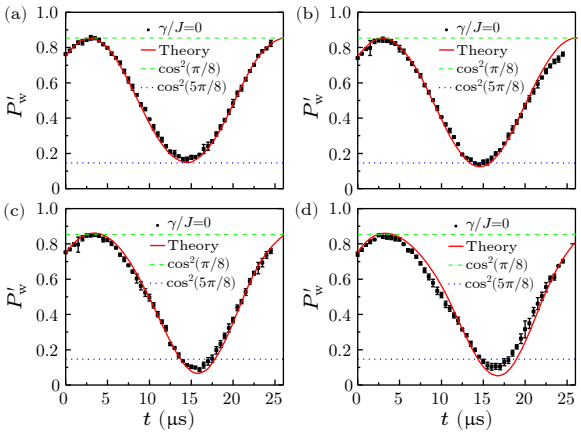

Fig. 3. The player's probability of winning under different dissipation intensities γ/J. The red curves represent the theoretical line described by Eq. (4), the black circles and error bars are the experimental results, and the green (blue) line indicates the upper (lower) bound on the player's probability of winning when $\gamma/J = 0$. (a) The probability of winning under the Hermitian system, where $\gamma/J = 0$. [(b), (c), (d)] Corresponding to an open system, where γ/J takes 0.1, 0.3, and 0.4, respectively. In experiments, our coupling strength $J = 2\pi \times 21.8$ kHz, and points are taken with steps of 0.5 μs.

In Fig. 3, the red curves represent the theoretical line described by Eq. (4), and the black circles with error bars represent the experimental results, with each point corresponding to a specific quantum strategy. To obtain the optimal quantum strategy, we collected a total of 50 data points with steps of 0.5 μs. The green line represents the upper bound of the player's winning probability at $\cos^2(\pi/8) \approx 0.8536$, and the blue line represents the lower bound at $\cos^2(5\pi/8) \approx 0.1464$. Figure 3(a) corresponds to $\gamma/J = 0$, representing the Hermitian case, at $t = 2.87$ μs, the player's winning probability reaches the maximum at 0.8525 ± 0.0072, while at $t = 14.33$ μs, it reaches minimum at 0.1645 ± 0.0151. In Fig. 3(c), representing $\gamma/J = 0.3$, at $t = 3.25$ μs, the player's winning probability exceeds the green line, reaching 0.8490 ± 0.0043, and at $t = 15.78$ μs, it falls below the blue line, reaching 0.0868 ± 0.0108. Similarly, in Fig. 3(b) for $\gamma/J = 0.1$, and Fig. 3(d) for $\gamma/J = 0.4$, the player's winning probability bounds exhibit noticeable changes with increasing dissipation strength. These results indicate that in the non-Hermitian system, Tsirelson's bound is broken, and the upper and lower bounds of the player's winning probability are altered. There exist slight deviations between

experimental results and theoretical values due to the instabilities of experimental conditions. Clearly, the experimental results for the lower bound in the non-Hermitian case are markedly lower than those for the lower bound in the Hermitian case. Moreover, as the dissipation intensity γ/J continues to increase, the variations in the upper and lower bounds of the player's winning probability exhibit a non-monotonic trend in numerical simulation, which will be investigated further in future research with more stable control of the dissipation.

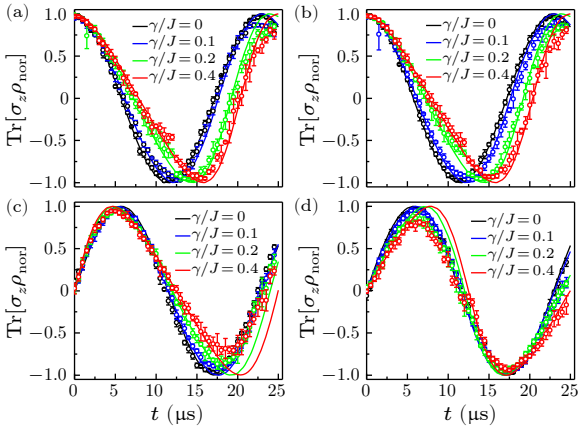

Fig. 4. State evolution dynamics in experiments. The four solid lines depict the theoretical outcomes, with various colors indicating the state evolution dynamics under different dissipation intensities γ/J. The error bars are derived from the experimental data. Here, $\gamma/J = 0$ (black line), $\gamma/J = 0.1$ (blue line), $\gamma/J = 0.3$ (green line), $\gamma/J = 0.4$ (red line). The player's winning probability is determined by four operations: (a) O_0U_0, (b) O_1U_0, (c) O_0U_1, (d) O_1U_1.

For further understanding the breached Tsirelson's bound in $\mathcal{PT}$ symmetric systems, we investigate the Lindblad master equation of the multi-level system (for details see Note S4 of the Supplementary Material). It reveals that changes in the player's winning probability are caused by the non-linear interaction. After normalization, the density operator ρ_{nor} of the qubit can be expressed as

$$\frac{d\rho_{\mathrm{nor}}}{dt} = -iJ\left[\sigma_x,\, \rho_{\mathrm{nor}}\right] + \gamma\left\{\sigma_z, \rho_{\mathrm{nor}}\right\} - 2\gamma\left[\mathrm{Tr}\left(\sigma_z\rho_{\mathrm{nor}}\right)\right]\rho_{\mathrm{nor}}. \tag{5}$$

Here, the term $\mathrm{Tr}\left(\sigma_z\rho_{\mathrm{nor}}\right)$ represents a nonlinear factor that induces non-uniform evolution dynamics.[23–25] This is illustrated by the uneven evolution speeds shown in Fig. 4, where the subtraction of the normalized population of state $|1\rangle$ from that of state $|0\rangle$ yields the nonlinear term $\mathrm{Tr}\left(\sigma_z\rho_{\mathrm{nor}}\right)$ (the data of normalized populations of state $|1\rangle$ and state $|0\rangle$ are shown in Note S3 of the Supplementary Material). Figure 4 depicts the experimental results of state evolution dynamics under different dissipation strengths for the four operations: (a) O_0U_0, (b) O_1U_0, (c) O_0U_1, (d) O_1U_1, constituting the player's winning probability. Various colors denote different dissipation strengths γ/J. The black curve corresponds to $\gamma/J = 0$, the blue to $\gamma/J = 0.1$, green to $\gamma/J = 0.3$, and red to $\gamma/J = 0.4$. It can be observed that as γ/J increases, the evolution time of state $|0\rangle$ to $|1\rangle$ gradually lengthens, while the evolution time from state $|1\rangle$ to $|0\rangle$ shortens. This indicates that the uneven flipping induced by the nonlinear term becomes more pronounced. Relevant experimental work has been observed in Ref. [10], where this uneven evolution in a dissipative qubit is experimentally confirmed. In the non-Hermitian CHSH* game, the winning probability is obtained by adding the probability of occurrence of four operations. The dissipation leads to uneven evolution, resulting in the change of the probability of the occurrence of individual operations, thereby affecting the player's winning probability.

In summary, we have successfully implemented the CHSH* game using a single $^{171}\mathrm{Yb}^+$ ion in our experimental setup and measured the winning probability. Building upon this, we have proposed and realized a non-Hermitian scheme for the CHSH* game in a two-level $\mathcal{PT}$-symmetric system, revealing violations of Tsirelson's bound. The investigation provides valuable insights into the interplay between quantum protocols and non-Hermitian dynamics, helping us understand the rich possibilities and complexities of Tsirelson's bound under the $\mathcal{PT}$-symmetric system. Furthermore, the non-Hermitian CHSH* game can be mapped to the non-Hermitian CHSH game, laying the foundation for further investigations into the CHSH game under local $\mathcal{PT}$-symmetric operations.

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