Some Modified Equations of the Sine-Hilbert Type

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Three modified sine-Hilbert (sH)-type equations, i.e., the modified sH equation, the modified damped sH equation, and the modified nonlinear dissipative system, are proposed, and their bilinear forms are provided. Based on these bilinear equations, some exact solutions to the three modified equations are derived.

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Some nonlinear integro-differential equations have gained much attention since the last century. Among them, the Benjamin–Ono (BO) equation and the intermediate long wave (ILW) equation are two typical equations, which are completely integrable. The BO equation that describes the internal waves in a stratified fluid of great depth can be written as[1–3]

$$u_t + 2uu_x + Hu_{xx} = 0,$$

where H denotes the Hilbert operator defined by

$$Hu(x,\,t) := \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{u(y,\,t)}{y-x}\mathrm{d}y$$

with P being the Cauchy principal value. The ILW equation that describes the long internal gravity waves in a stratified fluid with finite depth is written in the form[4,5]

$$u_t + \delta^{-1}u_x + 2uu_x + Tu_{xx} = 0,$$

where

$$Tu(x,\,t) := \frac{1}{2\delta}\mathrm{P}\int_{-\infty}^{\infty}\coth\Big(\frac{\pi}{2\delta}(y-x)\Big)u(y,\,t)\mathrm{d}y$$

with δ being a positive parameter characterizing the depth of the fluid. It reduces to the Korteweg–de Vries equation as $\delta \to 0$ (shallow water limit), and to the BO equation as $\delta \to \infty$ (deep water limit).[6,7] Numerous research achievements regarding the ILW equation and the BO equation have been made, such as the Bäcklund transformation and an infinite number of conservation laws for the ILW equation[7] and a two-parameter family of Miura transformations and an infinite number of conserved laws for the BO equation,[8] the inverse scattering transform scheme for the ILW equation[7,9,10] and the BO equation,[11] the multi-soliton solutions for the ILW equation[6,12] and the BO equation,[13] the periodic solutions for the ILW equation[14,15] and the BO equation,[16] the interaction of the solitary waves of the ILW equation[17] and of the solitons for the BO equation.[18] More research achievements can be found in Refs. [19–21] and references therein.

In addition, the sine-Hilbert (sH) equation is another remarkable integro-differential equation associated with a matrix spectral problem,[22] named so due to its analogy with the sine-Gordon equation.[23] The sH equation reads

$$H\theta_t = -\sin\theta, \quad \theta = \theta(x,\,t). \tag{1}$$

As stated in Ref. [24], although the sH equation is a mathematical model equation, it is related to a model of dislocations in crystals[25,26] first proposed by Peierls in 1940. It exhibits many completely integrable properties, such as formulation as a Riemann–Hilbert scattering problem,[23,27] an infinite number of conservation laws,[22] kink,[28,29] and periodic solutions.[30] Later, Matsuno presented two variations of the sH equation called the damped sH equation and the dissipative nonlinear system and investigated their properties.[31–33]

As we know, different forms of modified BO and ILW equations have been studied in the literature.[21] However, to our knowledge, the modified sH equation has never been proposed formally. In this Letter, we present the following three new equations, which are the modified sH equation

$$\omega_t^2 + (H\omega_t)^2 + 2\cos\omega - 2 = 0, \tag{2}$$

the modified damped sH equation

$$(\omega_t - \epsilon H\omega_x)^2 + (H\omega_t + \epsilon\omega_x)^2 + 2\cos\omega - 2 = 0 \tag{3}$$

with ϵ being a positive constant representing the magnitude of the damping, and the modified dissipative nonlinear system

$$(\omega_t - \epsilon H\omega_x)^2 + (H\omega_t + \epsilon\omega_x)^2 + 2\mu^2\cos\omega - 2\mu^2 = 0 \tag{4}$$

with ϵ and μ being the real parameters. In the following, we will derive their bilinear forms and demonstrate how exact solutions can be obtained accordingly. The relationships between these modified equations and their original counterparts will also be provided. We provide a detailed explanation of how to derive solutions for the modified sH

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equation. For the other two equations, we provide their solutions without further elaboration.

Modified Sine-Hilbert Equation. Firstly, let us briefly review the linearization process of the sH equation, which is crucial for our subsequent derivations. We introduce the dependent variable transformation [28]

$$\theta = i \ln \frac{f^*}{f}, \tag{5}$$

with i the imaginary unit, f^* the complex conjugate of f, and

$$f = f(x,\, t) = \prod_{j=1}^{N} [x - x_j(t)],$$
$$\operatorname{Im} x_j(t) > 0 \quad (j = 1,\, 2,\, \ldots,\, N), \quad x_n \neq x_m \quad \text{for} \quad n \neq m, \tag{6}$$

and note the relation $H\theta_t = -(\ln f^* f)_t$ which can be obtained by the formula $H(\frac{1}{x-x_j}) = \frac{-i}{x-x_j}$ in the case of $\operatorname{Im} x_j(t) > 0$. Based on Eqs. (5) and (6), the sH Eq. (1) can be transformed into the bilinear equation

$$(f^* f)_t = \frac{1}{2i}(f^2 - f^{*2}). \tag{7}$$

With a specific simplification, it further leads to the linear equation

$$f_t = \frac{1}{2i}(f - f^*). \tag{8}$$

It should be noted that Eq. (8) can only be used to obtain non-periodic solutions, whereas for periodic solutions, another form of the function f needs to be considered [see Eq. (24)].

Next, we bilinearize the modified sH Eq. (2) and show how it relates to the linear Eq. (8). With f satisfying the assumption (6) and assuming that g satisfies

$$g = g(x,\, t) = \prod_{j=1}^{M} [x - \hat{x}_j(t)],$$
$$\operatorname{Im} \hat{x}_j(t) > 0 \quad (j = 1,\, 2,\, \ldots,\, M), \quad \hat{x}_n \neq \hat{x}_m \quad \text{for} \quad n \neq m, \tag{9}$$

we introduce the variable transformation for ω, which is

$$\omega = i \ln \frac{f g^*}{g f^*}. \tag{10}$$

Substituting it into Eq. (2) and using the bilinear operator formula [34] $D_x a d \cdot bc = (D_x a \cdot b)cd - ab(D_x c \cdot d) = (D_x a \cdot c)bd - ac(D_x b \cdot d)$, we obtain

$$2gg^* ff^* - (gf^*)^2 - (g^* f)^2 = 4(D_t g \cdot f)(D_t g^* \cdot f^*), \tag{11}$$

which can be simplified to

$$D_t g \cdot f = \alpha(gf^* - g^* f), \quad \alpha\alpha^* = \frac{1}{4}. \tag{12}$$

Thus we obtain the bilinear form of the modified sH Eq. (2). Clearly, in the case of $\alpha = \frac{1}{2i}$, Eq. (12) is satisfied if f and g satisfy Eq. (8). This is why we call Eq. (2) the modified sH equation.

In Ref. [28], a series of solutions of linear Eq. (8) were derived. With f and g the specific N-soliton solution and M-soliton solution among them, the solution of the modified equation can be constructed by the expression (10). This is one way to derive solutions. In addition, some new solutions can be found. To this end, we propose two methods. The first method still chooses $\alpha = \frac{1}{2i}$ but needs to introduce an additional parameter k, while the second method chooses $\alpha = \frac{e^{i\eta}}{2i}$ with $\eta \in (0,\, 2\pi)$. Equation (12) with an extra complex parameter k can be expressed as

$$D_t g \cdot f = \alpha(gf^* - kgf + kgf - g^* f). \tag{13}$$

Choosing $\alpha = \frac{1}{2i}$ and decoupling f and g, we have

$$g_t = \frac{1}{2i}(kg - g^*) = \operatorname{Im}(g) + \hat{k} g,$$
$$f_t = \frac{1}{2i}(kf - f^*) = \operatorname{Im}(f) + \hat{k} f, \tag{14}$$

where $\hat{k} = \frac{k-1}{2i}$. As the equations of f and g are the same, we only consider the equation of f. With the assumption (6), we rewrite f as [28]

$$f(x,\, t) = \sum_{j=0}^{N} (-1)^j s_j x^{N-j}, \tag{15}$$

where $s_0 = 1$. Writing $\hat{k}$ as $\hat{k} = k_1 + ik_2$, substituting Eq. (15) into Eq. (14) and comparing the real part and imaginary part on both sides of the equation of f, we can obtain the following equations

$$\operatorname{Re} \dot{s_j} = (1 - k_2)\operatorname{Im} s_j + k_1 \operatorname{Re} s_j,$$
$$\operatorname{Im} \dot{s_j} = k_1 \operatorname{Im} s_j + k_2 \operatorname{Re} s_j. \tag{16}$$

The eigenvalues of the coefficient matrix $\begin{pmatrix} 1-k_2 & k_1 \\ k_1 & k_2 \end{pmatrix}$ determine the evolution of the solution. For simplicity, we assume $k_2 = 0$, and only consider the solutions in the case of $k_1 > 0$ and $k_1 < 0$. It follows

$$\operatorname{Im} s_j = \operatorname{Im} s_j(0) e^{k_1 t},$$
$$\operatorname{Re} s_j = \operatorname{Re} s_j(0) e^{k_1 t} + \operatorname{Im} s_j(0) e^{k_1 t} t, \tag{17}$$

which gives

$$\theta_1 = i \ln \frac{f^*}{f} = 2 \arctan \frac{\operatorname{Im} f}{\operatorname{Re} f}$$
$$= 2 \arctan \frac{\sum_{j=1}^{N} (-1)^j \operatorname{Im} s_j\, x^{N-j}}{x^N + \sum_{j=1}^{N} (-1)^j \operatorname{Re} s_j\, x^{N-j}}.$$

The $\theta_2 = i \ln \frac{g^*}{g}$ can be solved similarly. Then θ_1 and θ_2 compose the solution to Eq. (2) as $\omega = \theta_2 - \theta_1$. For the purpose of visualizing the solutions, we introduce $u \equiv \omega_x$. [32] To see the structure of u, we decompose u based on $x_j(t)$ $(j = 1,\, \ldots,\, N)$ and $\hat{x}_j(t)$ $(j = 1,\, \ldots,\, M)$, which are zeros of the polynomials $f(x,\, t)$ and $g(x,\, t)$ with respect to the variable x, as in Eq. (6). For general N and

M, they need to be solved from expression (15), with coefficients expressed by s_j and $\hat{s}_j$. Then u can be expressed as

$$\begin{aligned} u &= (i\ln\frac{fg^*}{gf^*})_x \\ &= [i\ln\frac{\prod_{j=0}^{M}(x-\hat{x}_j^*)\prod_{j=0}^{N}(x-x_j)}{\prod_{j=0}^{M}(x-\hat{x}_j)\prod_{j=0}^{N}(x-x_j^*)}]_x \\ &= \sum_{j=1}^{M}\Big(\frac{i}{x-\hat{x}_j^*}-\frac{i}{x-\hat{x}_j}\Big)-\sum_{j=1}^{N}\Big(\frac{i}{x-x_j^*}-\frac{i}{x-x_j}\Big) \\ &= \sum_{j=1}^{M}\hat{u}_j-\sum_{j=1}^{N}u_j, \end{aligned} \tag{18}$$

where

$$\hat{u}_j=\frac{2\mathrm{Im}\,\hat{x}_j}{(x-\mathrm{Re}\,\hat{x}_j)^2+(\mathrm{Im}\,\hat{x}_j)^2},$$
$$u_j=\frac{2\mathrm{Im}\,x_j}{(x-\mathrm{Re}\,x_j)^2+(\mathrm{Im}\,x_j)^2}.$$

Choosing $N=M=1$, it follows that $x_1=s_1$ and $\hat{x}_1=\hat{s}_1$, and hence

$$u=\frac{2\mathrm{Im}\,\hat{s}_1}{(x-\mathrm{Re}\,\hat{s}_1)^2+(\mathrm{Im}\,\hat{s}_1)^2}-\frac{2\mathrm{Im}\,s_1}{(x-\mathrm{Re}\,s_1)^2+(\mathrm{Im}\,s_1)^2}. \tag{19}$$

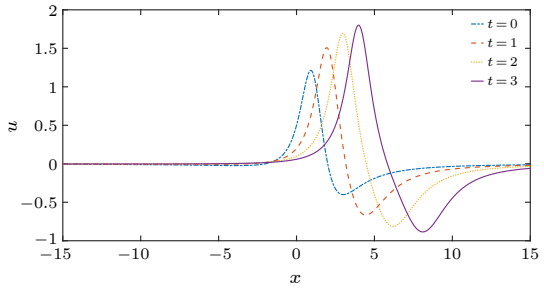

Fig. 1. Time evolution of u in Eq. (21).

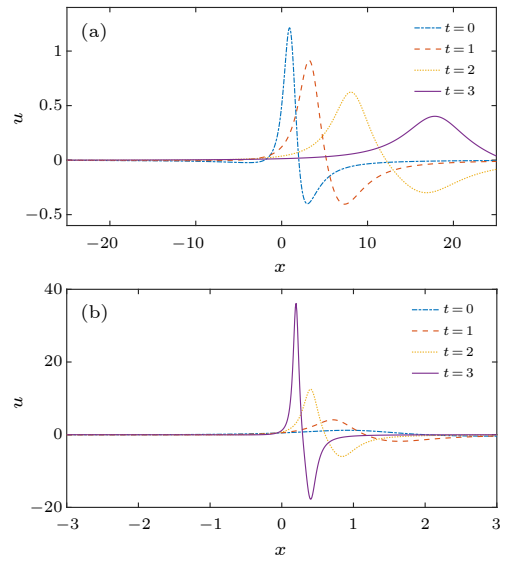

Fig. 2. The time evolution of u in Eq. (20) with the parameter (a) $k_1=0.5$ and (b) $k_1=-1$.

To obtain the specific solution, we take the initial values $\hat{s}_1(0)=1+i$ and $s_1(0)=2+2i$. Then the solution u becomes

$$\begin{aligned} u=&\frac{2e^{k_1t}}{(x-e^{k_1t}-te^{k_1t})^2+e^{2k_1t}} \\ &-\frac{4e^{k_1t}}{(x-2e^{k_1t}-2te^{k_1t})^2+4e^{2k_1t}}. \end{aligned} \tag{20}$$

In particular, taking $k_1=0$, we have

$$u=\frac{2}{(x-1-t)^2+1}-\frac{4}{(x-2-2t)^2+4}, \tag{21}$$

which is the difference between two one-soliton solutions of the sH equation given in Ref. [28]. It moves at a constant speed in the positive direction of the x-axis, with the maximum approaching $u=2$ and the minimum approaching $u=-1$ over time. Figure 1 depicts the profile of u at $t=0$, 1, 2 and 3. In the case of $k_1>0$, u moves in the positive direction of the x-axis, with a speed approximately of $O(\exp(k_1t))$ and the amplitude decaying at a rate of $O(\exp(-k_1t))$; while in the case of $k_1<0$, u moves in the negative direction of the x-axis, with a speed approximately of $O(\exp(k_1t))$ and the amplitude growing at a rate of $O(\exp(-k_1t))$. Note that the peak and trough are always located to the right of $x=0$. Figures 2(a) and 2(b) depict the evolution of u over time for $k_1=0.5$ and $k_1=-1$, respectively. As the peak and trough move in the same direction, we take it as the direction of u.

Remark 1. The bilinear Eq. (7) for the sH equation can also be transformed into the linear equation with an extra parameter k, as shown in Eq. (14). However, k must be a real number, which implies that $\hat{k}$ must be a pure imaginary number, i.e., $k_1=0$.

Next, we set $\alpha=\frac{e^{i\eta}}{2i}$ in Eq. (12). When $\eta\in(0,\pi)\cup(\pi,2\pi)$, s_j can be solved as

$$\begin{aligned} &\mathrm{Im}\,s_j=\mathrm{Im}\,s_j(0)e^{t\sin\eta}, \\ &\mathrm{Re}\,s_j=\mathrm{Re}\,s_j(0)+\mathrm{Im}\,s_j(0)\frac{\cos\eta}{\sin\eta}(e^{t\sin\eta}-1). \end{aligned} \tag{22}$$

The solution u can be obtained similarly to Eq. (18) for general N and M, and it will not be elaborated here. Choosing $N=M=1$, and taking the initial values $\hat{s}_1(0)=1+i$ and $s_1(0)=2+2i$, we obtain

$$\begin{aligned} u=&\frac{2e^{t\sin\eta}}{[x-1-\frac{\cos\eta}{\sin\eta}(e^{t\sin\eta}-1)]^2+e^{2t\sin\eta}} \\ &-\frac{4e^{t\sin\eta}}{[x-2-2\frac{\cos\eta}{\sin\eta}(e^{t\sin\eta}-1)]^2+4e^{2t\sin\eta}}. \end{aligned} \tag{23}$$

When $\eta=\pi$, with the same initial values we have

$$u=\frac{2}{(x+t-1)^2+1}-\frac{4}{(x+2t-2)^2+4},$$

which is equivalent to Eq. (21) under the transformation $t\to-t$. For $0<\eta<\pi$, $\sin\eta>0$ results in amplitude decaying at a rate of $O(\exp(-t\sin\eta))$. For $\pi<\eta<2\pi$, $\sin\eta<0$ leads to the amplitude growing at a rate of $O(\exp(-t\sin\eta))$. Regarding the motion of u, it moves

in the positive direction of the x-axis for $0 < \eta < \frac{\pi}{2}$, and in the negative direction of the x-axis for $\frac{\pi}{2} < \eta < \pi$. The peak of u approaches $x = 1 - \frac{\cos\eta}{\sin\eta}$ from the right for $\pi < \eta < \frac{3\pi}{2}$, and from the left for $\frac{3\pi}{2} < \eta < 2\pi$, but never reaches it. When $\eta = \frac{\pi}{2}$, both the peak and the trough decay at a rate of $O(\exp(-t))$. When $\eta = \frac{3\pi}{2}$, as time evolves, the peak eventually stabilizes at $x = 1$ and the trough stabilizes at $x = 2$, but the absolute values of both rapidly increase. Figures 3(a), 3(b), 3(c), and 3(d) show the profiles of u at $t = 0, 1, 2, 3$, with $\eta = \frac{\pi}{4}$, $\eta = \frac{3\pi}{4}$, $\eta = \frac{5\pi}{4}$, and $\eta = \frac{7\pi}{4}$, respectively.

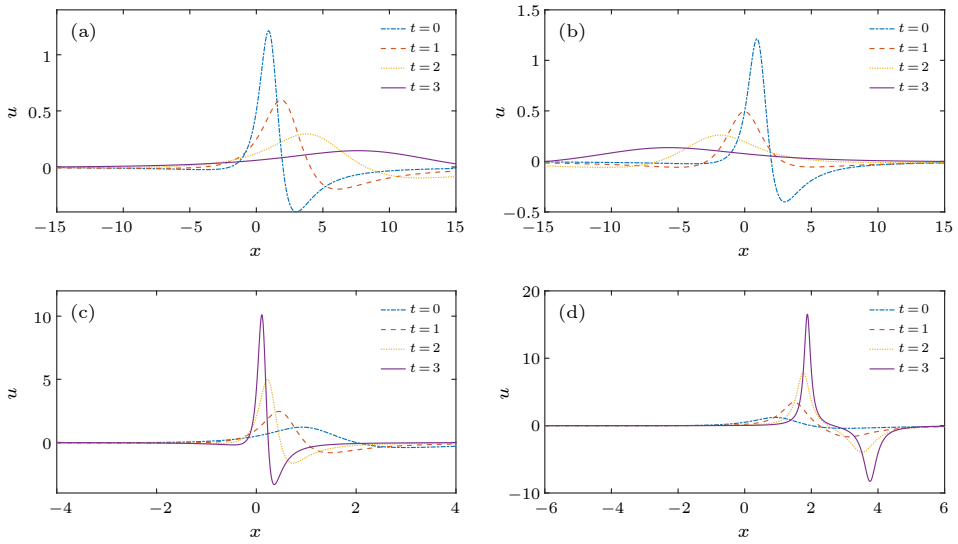

Fig. 3. Time evolution of u in Eq. (23) with the parameter (a) $\eta = \frac{\pi}{4}$, (b) $\eta = \frac{3\pi}{4}$, (c) $\eta = \frac{5\pi}{4}$, and (d) $\eta = \frac{7\pi}{4}$.

Remark 2. We get the solutions based on the bilinear Eq. (12), which is the simplification of Eq. (11). To derive Eq. (11) from Eq. (2), we make the assumption $\mathrm{Im}\, x_j(t) > 0$ $(j = 1, \ldots, N)$ and $\mathrm{Im}\, \hat{x}_j(t) > 0$ $(j = 1, \ldots, M)$, where x_j and $\hat{x}_j$ are the algebraic zeros in the expressions of f and g, as in Eq. (6). It remains to be validated that this holds in solving the simplified bilinear Eq. (12). With $\alpha = \frac{e^{i\eta}}{2i}$, taking f and g with the forms in Eq. (6) into Eq. (12), x_j $(j = 1, \ldots, N)$ satisfy $\mathrm{Im}\, \dot{x_j} = G_j\, \mathrm{Im}\, x_j$ $(j = 1, \ldots, N)$, where

$$G_j = \frac{1}{2i}\Big[e^{i\eta}\prod_{\substack{n=1\\ n\neq j}}^{N}\Big(\frac{x_j - x_n^*}{x_j - x_n}\Big) - e^{-i\eta}\prod_{\substack{n=1\\ n\neq j}}^{N}\Big(\frac{x_j^* - x_n}{x_j^* - x_n^*}\Big)\Big],$$

and $\hat{x}_j$ $(j = 1, \ldots, M)$ satisfy the same evolution equations. Thus the positivity of $\mathrm{Im}\, x_j$ and $\mathrm{Im}\, \hat{x}_j$ is ensured by $\mathrm{Im}\, x_j(0) > 0$ and $\mathrm{Im}\, \hat{x}_j(0) > 0$. In addition, x_j can be determined by solving Eq. (15) directly or analyzing the asymptotic form of the solution (one can refer to Refs. [31,32]).

Now we derive the periodic solutions of the modified sH equation. Introduce the dependent variable transformation (10), where f and g are respectively given by [30]

$$f = f(x, t) = \prod_{j=1}^{N}\frac{1}{\beta}\sin[\beta(x - x_j(t))], \quad \beta > 0,$$
$$\mathrm{Im}\, x_j(t) > 0\ (j = 1, 2, \ldots, N), \quad x_n \neq x_m \text{ for } n \neq m, \tag{24}$$

$$g = g(x, t) = \prod_{j=1}^{M}\frac{1}{\beta}\sin[\beta(x - \hat{x}_j(t))], \quad \beta > 0,$$
$$\mathrm{Im}\, \hat{x}_j(t) > 0\ (j = 1, 2, \ldots, M), \quad \hat{x}_n \neq \hat{x}_m \text{ for } n \neq m. \tag{25}$$

As f and g (f^* and g^*) have no zeros in the lower (upper) half complex x plane and are periodic on the unit cycle, it leads to $H\omega_t = -(\ln g^* g / f^* f)_t$ in Ref. [35]. Thus we can obtain the bilinear equation, which is still Eq. (12). It is noted that f and g in Eqs. (24) and (25) respectively reduce to Eqs. (6) and (9) when $\beta \to 0$.

Similar to the non-periodic case, if both f and g satisfy the following linear sH equations [30]

$$f_t = \frac{1}{2i}(f - f^*) - i\lambda f, \quad g_t = \frac{1}{2i}(g - g^*) - i\lambda g, \tag{26}$$

then they also satisfy Eq. (12). In other words, the periodic solutions to the modified sH equation can also be generated by the periodic solutions to the sH equation. Here λ is a positive constant with the constraint $\lambda \to 0$ when $\beta \to 0$. In the following, we present one solution constructed from the solutions given in Ref. [30] in the case of $N = M = 1$. We remark that more solutions can be obtained by using the methods proposed in the non-periodic case. Taking x_1 and $\hat{x}_1$ as

$$x_1(t) = at + b_1 + i\frac{\sinh^{-1}(2\beta a)}{2\beta},$$
$$\hat{x}_1(t) = at + b_2 + i\frac{\sinh^{-1}(2\beta a)}{2\beta},$$

where $a = \frac{\sqrt{\lambda^2+\lambda}}{\beta}$, and b_1 and b_2 are real-valued constants, we have the periodic solution

$$\omega = i \ln \frac{\sin[\beta(x-\hat{x}_1^*(t))]\sin[\beta(x-x_1(t))]}{\sin[\beta(x-\hat{x}_1(t))]\sin[\beta(x-x_1^*(t))]},$$

which yields

$$\begin{aligned} u = \omega_x = i\beta\{ &\cot[\beta(x-\hat{x}_1^*(t))] - \cot[\beta(x-\hat{x}_1(t))] \\ &+ \cot[\beta(x-x_1(t))] - \cot[\beta(x-x_1^*(t))]\}. \end{aligned} \quad (27)$$

The temporal period of u is $\frac{\pi}{a}$ and the spatial period is π. Figure 4 shows the solution u at different times $t = 1$, $1+\frac{\pi}{4a}$, $1+\frac{\pi}{2a}$, and $1+\frac{3\pi}{4a}$.

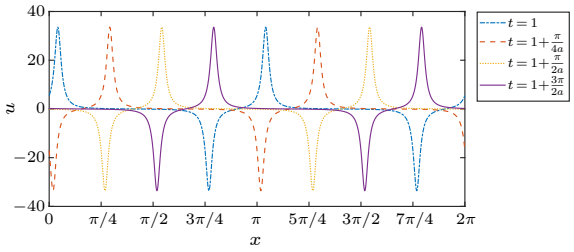

Fig. 4. Time evolution of u in Eq. (27) with the parameters $\beta = 1$, $\lambda = 1$, $b_1 = 1$, and $b_2 = 5$.

Modified Damped Sine-Hilbert Equation. We derive the bilinear form of the modified damped sH equation and show how it relates to the damped sH equation. An exact solution is presented. Here we only consider the non-periodic case. Firstly, let us briefly review the linearization process of the damped sH equation. The damped sH equation is

$$H\theta_t = -\sin\theta - \epsilon\theta_x, \quad \epsilon > 0, \quad (28)$$

where a damping term $-\epsilon\theta_x$ is added, suppressing the blowup of solutions of the sH equation.[32] It can be linearized to

$$f_t = \frac{1}{2i}(f - f^*) - i\epsilon f_x \quad (29)$$

by the dependent variable transformation (5) and (6). To obtain the bilinear form of the modified damped sH equation, substituting Eq. (10) into Eq. (3), we have

$$\begin{aligned} &2gg^*ff^* - (gf^*)^2 - (g^*f)^2 \\ =&4(D_tf\cdot g)(D_tf^*\cdot g^*) + 4\epsilon^2(D_xf\cdot g)(D_xf^*\cdot g^*) \\ &+ 4i\epsilon[-(D_tf\cdot g)(D_xf^*\cdot g^*) + (D_tf^*\cdot g^*)(D_xf\cdot g)], \end{aligned}$$

where f and g satisfy Eqs. (6) and (9). Thus we have the bilinear equation

$$D_tg\cdot f = \alpha(gf^* - g^*f) - i\epsilon D_xg\cdot f, \quad \alpha\alpha^* = \frac{1}{4}. \quad (30)$$

Obviously, Eq. (30) holds if both f and g satisfy the linear Eq. (29). With x_j $(j = 1, \ldots, N)$ and $\hat{x}_j$ $(j = 1, \ldots, M)$ the zeros of f and g, in the process of solving Eq. (30), $\mathrm{Im}\, x_j(t) > 0$ holds true when $\mathrm{Im}\, x_j(0) > 0$ and $\epsilon > 0$. The same applies to $\mathrm{Im}\, \hat{x}_j(t)$. The simplest non-periodic solution may be expressed as

$$\omega = i\ln\frac{[x-((\epsilon/2)t^2+a_2t+b_2-i(\epsilon t+a_2))][x-((\epsilon/2)t^2+a_1t+b_1+i(\epsilon t+a_1))]}{[x-((\epsilon/2)t^2+a_2t+b_2+i(\epsilon t+a_2))][x-((\epsilon/2)t^2+a_1t+b_1-i(\epsilon t+a_1))]}.$$

More solutions can be derived following the procedure given above by solving the bilinear Eq. (30), setting $\alpha \neq \frac{1}{2i}$ or adding the extra term for f_t and g_t.

Modified Dissipative Nonlinear System. We derive the bilinear form of the modified dissipative nonlinear system and show how it relates to the dissipative nonlinear system. An exact solution is presented. Here we only consider the non-periodic case. Firstly, let us briefly review the linearization process of the dissipative nonlinear system. The dissipative nonlinear system is

$$\theta_t = -\mu\sin\theta + \epsilon H\theta_x, \quad (31)$$

which has stationary, blowup, and decaying solutions according to the sign of μ and ϵ.[31] It can be linearized to

$$f_t = -\frac{\mu}{2}(f - f^*) - i\epsilon f_x \quad (32)$$

by the dependent variable transformation (5) and (6). To obtain the bilinear form of the modified dissipative nonlinear system, substituting Eq. (10) into Eq. (4), we have

$$\begin{aligned} &2gg^*ff^* - (gf^*)^2 - (g^*f)^2 \\ =&\frac{1}{\mu^2}\{4(D_tf\cdot g)(D_tf^*\cdot g^*) + 4\epsilon^2(D_xf\cdot g)(D_xf^*\cdot g^*) \\ &+ 4i\epsilon[-(D_tf\cdot g)(D_xf^*\cdot g^*) + (D_tf^*\cdot g^*)(D_xf\cdot g)]\}, \end{aligned}$$

where f and g satisfy Eqs. (6) and (9). Thus we have the bilinear equation

$$D_tg\cdot f = \mu\alpha(gf^* - g^*f) - i\epsilon D_xg\cdot f, \quad \alpha\alpha^* = \frac{1}{4}. \quad (33)$$

Obviously, Eq. (33) holds if both f and g satisfy the linear Eq. (32). With x_j and $\hat{x}_j$ as the zeros of f and g, $\mathrm{Im}\, x_j(t) > 0$ holds true when $\mathrm{Im}\, x_j(0) > 0$ and $\epsilon > 0$. The same holds for $\mathrm{Im}\, \hat{x}_j(t)$. The simplest non-periodic solution may be expressed as

$$\omega = i\ln\frac{\{x-[b_2-i(a_2-(a_2-\epsilon/\mu)(1-e^{-\mu t}))]\}\{x-[b_1+i(a_1-(a_1-\epsilon/\mu)(1-e^{-\mu t}))]\}}{\{x-[b_2+i(a_2-(a_2-\epsilon/\mu)(1-e^{-\mu t}))]\}\{x-[b_1-i(a_1-(a_1-\epsilon/\mu)(1-e^{-\mu t}))]\}}.$$

The methods applied to the modified sH equation can also be used to find more solutions.

In conclusion, the principle of the linear superposition of solutions plays an essential role in the world governed by linear differential equations. Unfortunately, in the nonlinear world governed by nonlinear differential equations, this basic principle is lost. Generally speaking, it is difficult to find nonlinear superposition formulae of nonlinear differential equations (if they exist!). However, research in soliton theory suggests that the corresponding

nonlinear superposition formula can usually be obtained for a nonlinear evolution equation under consideration if the so-called Bäcklund transformation for the equation can be found. Interestingly and surprisingly, it has been demonstrated (see, e.g., Refs. [28,31,32,36–44] and references therein) that certain nonlinear integrable equations exhibit a unique linear structure, enabling the linear superposition of solutions. This remarkable characteristic proves to be beneficial in deriving significant outcomes, particularly in generating explicit solutions. Inspired by the linearization of the sH equation, we propose three modified sH type equations, i.e., the modified sH equation, the modified damped sH equation, and the modified nonlinear dissipative system. We also gave their bilinear forms and some exact solutions. In deriving non-periodic solutions of the modified sH equation, we propose two methods, which allow us to obtain a broader class of solutions. These solutions are essentially determined by linear equations, thus more solutions can be generated using the linear superposition principle.

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