Complete Universal Scaling in First-Order Phase Transitions

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Phase transitions and critical phenomena are among the most intriguing phenomena in nature and society. They are classified into first-order phase transitions (FOPTs) and continuous ones. While the latter shows marvelous phenomena of scaling and universality, whether the former behaves similarly is a long-standing controversial issue. Here we definitely demonstrate complete universal scaling in field driven FOPTs for Langevin equations in both zero and two spatial dimensions by rescaling all parameters and subtracting nonuniversal contributions with singular dimensions from an effective temperature and a special field according to an effective theory. This offers a perspective different from the usual nucleation and growth but conforming to continuous phase transitions to study FOPTs.

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Phase transitions and critical phenomena are among the most intriguing phenomena in nature and society. According to a modern classification,[1] they are classified into first-order phase transitions (FOPTs) and continuous phase transitions. FOPTs exist more widely, obviously have a longer history, and are usually described by nucleation and growth or spinodal decomposition.[2–5] Continuous phase transitions exhibit critical phenomena near their critical points[6] and are characterized by universal scaling behavior. While scaling and universality in critical phenomena have been well established and accurately described by the renormalization-group theory,[7–10] the theories for FOPTs are far less accurate.[11–16] A way out is to see whether or not FOPTs exhibit universal scaling behavior. However, this is a long-standing controversial issue.[16–64]

Consider a free energy[7–10]

$$f_4(\phi) = \frac{1}{2}a_2\phi^2 + \frac{1}{4}a_4\phi^4 - H\phi \tag{1}$$

for an order parameter ϕ and its ordering field H, where a_2 is a reduced temperature and a_4 a coupling constant. It exhibits a continuous transition at $a_2 = 0$ and $H = 0$ but field-driven FOPTs for all $a_2 < 0$ between two ordered phases with $\phi_{\rm eq} = \pm\sqrt{-a_2/a_4} = \pm M_{\rm eq}$ at $H = 0$. Dynamics can be studied by the usual purely dissipated Langevin equation

$$\frac{\partial\phi}{\partial t} = -\lambda\frac{\partial f_4}{\partial\phi} + \zeta = -\lambda(a_2\phi + a_4\phi^3 - H) + \zeta, \tag{2}$$

where λ is a kinetic constant and ζ a Gaussian white noise whose averaged moments satisfying $\langle\zeta\rangle = 0$ and $\langle\zeta(t)\zeta(t')\rangle = 2\lambda\sigma\delta(t-t')$ with a noise amplitude or temperature σ. For a field linearly changing with time t with a constant rate R, the transition from $-M_{\rm eq}$ to $M_{\rm eq}$ does not occur at the equilibrium $H = 0$. Rather, it is delayed to a larger field $H > 0$. This is hysteresis that increases with the rate R, as seen in Fig. 1(a). Regarding the averaged order parameter, $M = \langle\phi\rangle$, as magnetization, this hysteresis is just the usual magnetic hysteresis. It has been continuously shown in a large number of contexts that such a hysteresis may exhibit a power-law dependence on R with a constant exponent, showing universal scaling.[16,26,28–34,36–41,43–62] However, even more than thirty years ago, it was found from direct numerical solutions of Eq. (2) that the exponent depends instead on σ and thus no single power law exists.[35] Recently, repeated calculations of the same model leads to a claim that the hysteresis shows power-law scaling over three orders of magnitude variation of the rate R but nonuniversal exponents, which was employed to explain the smaller-than-expected exponent observed in experiments and previously reported scattered exponents.[63] In the following, we will demonstrate that the measured exponents are only effective. Universal exponent and complete scaling emerge once an effective theory is fully taken into account.[53–55] Moreover, we are able to extend this complete universal scaling to the more realistic two-spatial-dimensional situation in which new features appear. This offers a perspective different from the usual nucleation and growth[2–5] but conforming to continuous phase transitions to study FOPTs.

To start, without the noise, we note that the transition can only take place beyond the so-called spinodal point ($H_{\rm s}$, $M_{\rm s}$), the stars in Fig. 1(a), at which the free energy barrier between the two phases vanishes. As such, let $\phi = M_{\rm s} + \varphi$,[53,54] one finds $f_4(\phi) = f_4[M_{\rm s}] + f_3(\varphi) + \frac{1}{4}a_4\varphi^4$ with

$$f_3(\varphi) = \frac{1}{2}\tau\varphi^2 + \frac{1}{3}a_3\varphi^3 - h\varphi, \tag{3}$$

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where $\tau = a_2 + 3a_4M_s^2$, $h = H - H_s$, and $a_3 = 3a_4M_s$, the first two are an effective reduced temperature and an effective field, respectively, that become zero exactly at $M_s = \pm\sqrt{-a_2/3a_4}$, $H_s = a_2M_s + a_4M_s^3 = 2a_2M_s/3$. Therefore, $\tau = 0$ and $h = 0$ are located exactly at the spinodal point, similar to $a_2 = 0$ and $H = 0$ at the critical point. Accordingly, we will see that the transition is controlled by the cubic f_3 theory near the spinodal where the quartic term is negligible instead of the original quartic f_4 theory.

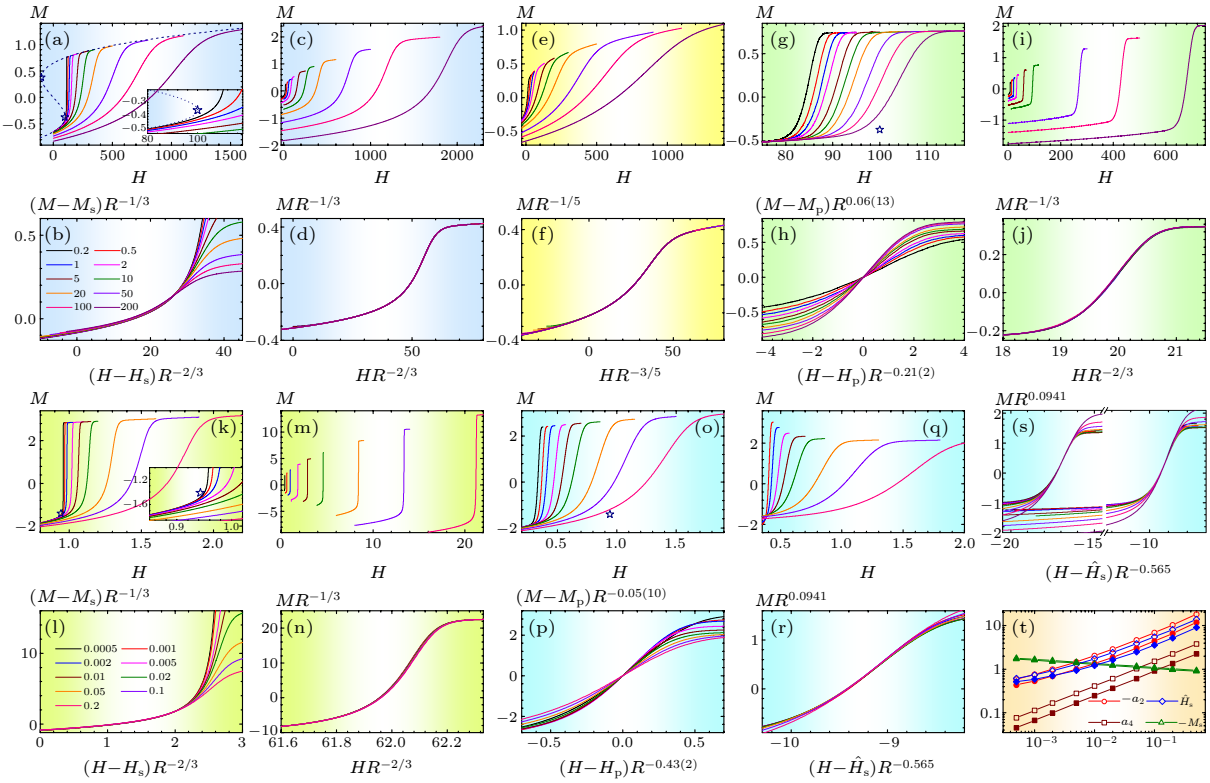

Fig. 1. (a) M versus H for a series of rate R given in (b) from numerical solutions of Eq. (2). This R range is adopted in (a)–(j). Here $a_2 = a_{20} = -400$, $a_4 = a_{40} = 948$, $\lambda = 0.0005$,[63] $\sigma = 0$. The dashed curve is the mean-field equation $H = a_2M + a_4M^3$ and the stars mark the spinodal points. The inset enlarges the curves near one star. (b) Rescaling of all curves in (a) according to Eq. (5). Curves of small rates are partly cut to show the poor collapsing. (c) Similar to (a) but for $a_2 = a_{20}(R_0/R)^{-1/3}$, $a_4 = a_{40}(R_0/R)^{1/3}$, and $R_0 = 10$. (d) The same rescaling of all curves in (c) now exhibits complete collapsing. (e) Similar to (a) but with $a_2 = 0$, exactly the critical value. (f) Rescaling of all curves in (e) according to Eq. (4) perfectly collapses. (g) Similar to (a) but with $\sigma = \sigma_0 = 0.0004$. The transition occurs before the spinodal star. (h) Rescaling of all curves in (g) using H_p and M_p associated with the fluctuation peaks together with the exponents that fit H_p and M_p with R, respectively. The exponents and their fitting standard errors are given in the axis titles. (i) Similar to (c) but with $\sigma = \sigma_0(R_0/R)^{-1}$. (j) Rescaling of all curves in (i) according to Eq. (5) exhibits complete collapsing. (k) Similar to (a) but for the two-dimensional Eq. (7) and another series of R given in (l), $a_2 = a_{20} = -1$, $a_4 = a_{40} = 1/6$ and a small $\sigma = \sigma_0 = 0.0001$ to show the universality. This R range is applied to (k)–(s) and displayed in (t). (l) Rescaling of all the curves in (k) according to Eq. (5). In contrast to (b), the collapsing in the lower part is quite good because $a_4R^{1/3}$ is now small. [(m), (n)] Similar to (c) and (d). Good collapses emerge again. (o) Similar to (k) but with $\sigma = \sigma_0 = 4$. (p) Similar to (h), collapsing occurs only at one point. [(q), (r)] Similar to (o) and (p) but with $a_4 = a_{40}(R_0/R)^{1/r\nu}$, $\sigma = \sigma_0(R_0/R)^{-(2\beta/\nu+z)/r}$, $R_0 = 0.005$, and a_2 given by Eq. (10) and displayed in (t). $M_{s0}^2 = 1.9$ and $z = 1.85$ and so $\beta/r\nu = -0.0941$ and $\beta\delta/r\nu = 0.565$. Complete collapsing emerges, but different from (j) and (n) due to crossovers (see the text). (s) Another view of the collapsed curve in (r) (right) with $R = 0.0005$ and $R = 0.5$ (purple) added and another collapsing for $R_0 = 0.002$ and $M_{s0}^2 = 2.5$ (left). (t) R dependence of the parameters used in (q), (r), and (s). Open and filled symbols represent results for $R_0 = 0.005$ and $R_0 = 0.002$, respectively. Lines connecting symbols are only a guide to the eyes. Two panels lettered consecutively on the same column form a pair except for the last two panels and panels of an identical background color compose a block that illustrates the usual and complete scaling.

Next, we make a scale transformation to the dynamic equation, Eq. (2), with either f_4 or f_3.[9,10,53] This is a rather standard technique in the field-theoretic framework of renormalization[9,10] and is to change every quantity O to O' through $O = O'b^{-[O]}$ for a scaling factor b, where we have utilized the square brackets to denote the scale dimension. To keep the transformed dynamic equation identical with the original one, one must have $[\lambda t] = -[a_2]$, $[H] = [\phi] + [a_2]$, $[\zeta] = [\phi] - [\lambda t]/2 = [\sigma]/2$, and $[a_n] = [a_2] - (n-2)[\phi]$ for $n = 3$ or 4, represent-

ing either a_3 or a_4 (and implicitly, of course, τ and h replacing a_2 and H, respectively, in the case of $n = 3$). Note that λ serves only as the unit of time. According to the renormalization-group theory,[7–10] in mean-field theory, one sets $[a_n] = 0$. This yields $[\phi] = [a_2]/(n-2)$, $[H] = (n-1)[a_2]/(n-2)$, $[\sigma] = n[a_2]/(n-2)$. The original and transformed equations must share the same solution since they are identical. Accordingly, $\phi(a_2, a_n, H, \lambda t, \sigma) = b^{-[\phi]}\phi(a_2 b^{[a_2]}, a_n b^{[a_n]}, Hb^{[h]}, \lambda t b^{[\lambda t]}, \sigma b^{[\sigma]})$. We then set the zero of the time at $H = 0$ such that $H = R\lambda t$ (similar for h), replace λt with R, and let $b = R^{-1/r}$ for $r \equiv [R]$, which obeys $r = [H] - [\lambda t] = (2n-3)[a_2]/(n-2)$.[65] These lead to the finite-time scaling (FTS) forms,[65–68]

$$\phi = R^{1/5} g_4(a_2 R^{-2/5}, HR^{-3/5}, \sigma R^{-4/5}, a_4), \tag{4}$$

$$\varphi = R^{1/3} g_3(\tau R^{-1/3}, hR^{-2/3}, \sigma R^{-1}, a_4 R^{1/3}, a_3), \tag{5}$$

for the quartic, Eq. (1), and cubic, Eq. (3), theories, respectively, where g_3 and g_4 and other forms of g below are universal scaling functions. These scaling forms differ from all previous ones in the presence of the noise amplitude. In addition, note that for the cubic theory, it is $[a_3] = [a_4] + [M_\mathrm{s}] = 0$ instead of $[a_4] = 0$. This means that a_4 is dimensional in the cubic theory and we have thus included it in the scaling function in Eq. (5). The consideration of all the parameters is different from all previous studies of scaling in FOPTs. We will see that this is important for complete universal scaling.

The cubic FTS form can be rewritten as $h = R^{2/3} g(\varphi R^{-1/3}, \tau R^{-1/3}, \sigma R^{-1}, a_4 R^{1/3}, a_3)$. It becomes obvious that an exact mean-field exponent 2/3 for the dependence of H on R can be obtained only when all the arguments of g are constant. Otherwise, only effective exponents can be obtained since g depends on R. In Fig. 1(a), we plot the transition curves M ($= \phi$ here) versus H for a large range of R for the same a_2, a_4 and λ for $\sigma = 0$.[69] The usual method to show scaling is to obtain some exponents by fitting H at some special loci with R. Here, M_s is preferred because $\varphi = M - M_\mathrm{s}$ and the first argument of g equals zero. The result is about 0.64 but varies with the R range used, slightly smaller than 2/3. It is only effective as mentioned in the sense that all the remaining arguments of g depend on R. The curve collapse shown in Fig. 1(b) is not so good indeed. However, when a_2 and a_4 change with R properly so that $\tau R^{-1/3}$ and $a_4 R^{1/3}$ are fixed according to Eq. (5), the M vs H curves in Fig. 1(c) perfectly collapse onto a single curve in Fig. 1(d). This is complete universal scaling with the perfect curve collapse and the universal exponents given in Eq. (5). It does emerge in FOPTs once the theory for them is fully respected, even though the apparent exponents appear to be different from the universal ones.

It is remarkable that we do not need to subtract M_s and H_s in Fig. 1(d), because they both scale with φ and h owing to the rescaling of a_2 and a_4 and hence just contribute rate-independent constants to an overall displacement of the rescaled curve. This seems redundant here as they are exactly known. However, it becomes crucial when they are not easy to be directly determined in the following. It is essential to note that the curves are obtained from direct numerical solutions of Eq. (2) with the quartic f_4, not a corresponding equation with the cubic f_3. However, the transition is controlled by the effective cubic theory with its exponents and thus follows Eq. (5) rather than the quartic theory. What does the latter govern instead? It governs the critical behavior, as is clearly seen in Figs. 1(e) and 1(f).

It is now straightforward to consider the noise. Due to the finite noise, the transition is activated and can proceed before H_s, the star in Fig. 1(g). A good choice of the special loci is then the peak of the fluctuation, $\langle\phi^2\rangle - \langle\phi\rangle^2$, at which $hR^{-2/3}$ is a constant.[66] However, for the same a_2, a_4, λ, and σ, the exponents obtained from the peak positions H_p and M_p at H_p change appreciably with σ and the R range. The original curves in Fig. 1(g) cannot collapse at all, see Fig. 1(h). Nevertheless, upon scaling the noise as well as a_2 and a_4 with R according to Eq. (5), the curves in Fig. 1(i) then completely overlap in Fig. 1(j), exhibiting again complete universal scaling. Therefore, the previous-found exponents that vary with σ are only effective. The thirty-year old problem of the varying exponent is again correctly described by the cubic theory with completely universal exponents.

The important point here is that the spinodal point may be considered nonexistent since the transition now is stochastic and can occur before reaching H_s as mentioned above. This is similar to usual FOPTs that can proceed through nucleation and thus it is generally believed that there exist no sharp spinodal curves for systems with short-ranged interactions.[2–5] However, with the present method, the complete universal scaling firmly indicates that the FOPTs do exhibit universal scaling even with the noise and are indeed uniquely described by the cubic theory. We emphasize that the cubic theory in the noisy systems is obviously not trivial even though one might think so in the noiseless systems, since, as pointed out, existence of some spinodal points is not transparent in the noise-activated systems.

These results stimulate us to apply the theory to more realistic FOPTs with spatial extension, though the noise-activated escape problem described by Eq. (2) has widespread applications.[70] We need to consider spatial fluctuations and hence the Landau–Ginzburg free-energy functional[53–55]

$$F(\phi) = \int d^d x \Big\{ f_4(\phi) + \frac{1}{2}(\nabla\phi)^2 \Big\} \tag{6}$$

in d-dimensional space. Dynamics is again governed by the Langevin equation, i.e., model A dynamics,[71]

$$\frac{\partial\phi}{\partial t} = -\frac{\delta F}{\delta\phi} + \zeta = -(a_2\phi - \nabla^2\phi + a_4\phi^3 - H) + \zeta, \tag{7}$$

with the correlation of the noise changes to (for $\lambda = 1$) $\langle\zeta(\mathbf{x}, t)\zeta(\mathbf{x}', t')\rangle = 2\sigma\delta(t - t')\delta(\mathbf{x} - \mathbf{x}')$. To describe the FOPTs below the critical point, the lesson learned above justifies our expanding again ϕ around a fluctuations-

displaced spinodal point denoted again by $M_{\rm s}$ and $H_{\rm s}$, resulting in the cubic theory, viz., replacing ϕ with φ and f_4 with f_3 in F.

The renormalization-group theory of the cubic theory has been set up.[53,54] Here we briefly recapitulate the key points. We first make the scale transformation with an additional $x = bx'$ for coarse graining. The standard analysis is to demand $F = F'$ instead of directly $[a_3] = 0$ above. This completely fixes the resultant scaling dimensions to be $[t] = -[a_2] = -2$, $[\varphi] = (d-2)/2$, $[h] = (d+2)/2$, $[a_3] = (6-d)/2$, $[\sigma] = 2[\varphi] - [t]$. Now $[a_2] = 2$ fixed by the gradient and we thus refer Eq. (2) as a zero-dimensional case.[72] Also, in $d = d_{\rm c} = 6$, the upper critical dimension, $[a_3] = 0$ and the zero-dimensional results above are restored with $[a_2] = 2$. These dimensions are related to the exponents β, δ, and ν, and z by $z = [t]$, $1/\nu = [a_2]$, $\beta/\nu = [\varphi]$, $\beta\delta/\nu = [h]$, which have identical meaning and symbols with their respective critical exponents. In addition, $[\sigma] = 2\beta/\nu + z$ generally. Below $d_{\rm c}$, fluctuations become important and a renormalization-group analysis has to be employed, giving rise to universality and scaling controlled by a nontrivial fixed point. A direct consequence is that there exist anomalous dimensions η's such that $z = 2 + \eta_t$, $1/\nu = 2 + \eta_\nu$, $\beta/\nu = (d - 2 + \eta)/2$, $\beta\delta/\nu = (d + 2 - \eta)/2$. In addition, Eq. (5) is generalized to

$$M - M_{\rm s} = R^{\frac{\beta}{r\nu}} \hat{g}(\tau R^{-\frac{1}{r\nu}}, hR^{-\frac{\beta\delta}{r\nu}}, \sigma R^{-\frac{\sigma}{r}}, a_4 R^{\frac{1}{r\nu}}, a_3), \tag{8}$$

which is also identical to its critical counterpart except for the presence of $M_{\rm s}$ and $H_{\rm s}$, where $r = z + \beta\delta/\nu$.[65–68] These demonstrate the uniformity in treating both orders of phase transitions. All parameters are again rescaled for complete universal scaling in contrast to previous studies.[16,53,54] In $d_{\rm c}$, the exponents recover their mean-field values, $\beta = 1$, $\delta = 2$, $\nu = 1/2$, and $z = 2$, because all η's are zero, and Eq. (8) returns to Eq. (5). These indicate that the zero-dimensional results above are controlled by the trivial Gaussian fixed point above $d_{\rm c}$.[54,73] Therefore, the fixed points of the cubic theory can uniformly describe the universal scaling behavior of both the zero- and two-dimensional systems.

Before we study general cases, we choose a very small σ such that spatial fluctuations can be ignored and the mean-field results, Eq. (5), is expected to be applicable. Figures 1(k)–1(m) demonstrate both the universality of the above method and the validity of our solutions.[69] The curves for the two largest rates slightly deviate from the main curves in Fig. 1(n), because spatial fluctuations start to take effect. This is different from Fig. 1(i), where the thick curve is due to the large noise. However, when σ gets larger, the mean-field or the two-dimensional exponents (see below) can no longer collapse the curves at all, neither can the usual method shown in Figs. 1(o) and 1(p).

The new feature here is that there exist extra contributions with singular dimensions to the effective temperature and the spinodal field. Because $d = 2 < d_{\rm c}$, $[a_3] = 2$. In addition, in two dimensions, $\delta = -6$ and $\nu = -5/2$ are exactly known[74] along with $\beta = 1$,[53,54] since the Yang–Lee edge singularity[75] above the critical temperature and the FOPTs below it fall into the same universality class.[54,55,76] Accordingly, using $a_3 = 3a_4 M_{\rm s}$, one finds $[a_4] = [a_3] - [M_{\rm s}] = 2 - \beta/\nu = 12/5$. Also, $[a_2] = 1/\nu = -2/5$, $[a_4 M_{\rm s}^2] = 8/5$, $[a_2 M_{\rm s}] = -4/5$, and $[a_4 M_{\rm s}^3] = 6/5$. These indicate that the two terms both in $\tau = a_2 + 3a_4 M_{\rm s}^2$ and in $\hat{H}_{\rm s} = a_2 M_{\rm s} + a_4 M_{\rm s}^3$ arising from the quartic contributions now possess different dimensions. Moreover, $[a_2 M_{\rm s}]$ and $[a_4 M_{\rm s}^3]$ do not equal $[h] = \beta\delta/\nu = 12/5$. Consequently, we cannot directly multiply a_2 and H by $R^{-1/r\nu}$ and $R^{-\beta\delta/r\nu}$, respectively, to achieve complete scaling as done above.

The key is to choose $a_2 = \tau - 3a_4 M_{\rm s}^2 - \delta a_2$ such that $a_2 + 3a_4 M_{\rm s}^2 + \delta a_2 = \tau$ and to subtract the singular quartic contributions $\hat{H}_{\rm s}$ from the real $H_{\rm s}$, which includes contributions δh from the cubic fluctuations, viz., $H_{\rm s} = \hat{H}_{\rm s} + \delta h$. Here δa_2 denotes the fluctuation contributions from the quartic model to a_2. These contributions again have different dimensions for different numbers of loops in a standard loop expansion. They are different from the cubic fluctuation contributions to τ which share identical dimensions with τ and do not need to be explicitly computed in the present approach. Besides a_2, quartic fluctuations do not further change $H_{\rm s}$ itself due to symmetry. Cubic fluctuation contributions to h also have the same dimensions and need not be considered either. Therefore, the FTS form, Eq. (8), can be rewritten as

$$\begin{aligned} M = R^{\frac{\beta}{r\nu}} \hat{g}((a_2 + 3a_4 M_{\rm s}^2 + \delta a_2) R^{-\frac{1}{r\nu}}, \\ (H - \hat{H}_{\rm s}) R^{-\frac{\beta\delta}{r\nu}}, \sigma R^{-\frac{\sigma}{r}}, a_4 R^{\frac{1}{r\nu}}, a_3), \end{aligned} \tag{9}$$

where we have neglected $M_{\rm s}$ and δh in anticipation of the complete scaling method. In this sense, Eq. (9) is exact except for corrections to scaling.[54,77]

Now, to test Eq. (9), for a transition, run with R, to coincide with a reference curve, run using R_0, a_{20}, a_{40}, and σ_0, after rescaling a_4 and σ according to Eq. (9), we choose a_2 to satisfy $\tau R^{-1/r\nu} = (a_{20} + 3a_{40} M_{\rm s0}^2 + \delta a_{20}) R_0^{-1/r\nu}$, a constant, or,

$$a_2 = \{a_{20} + \delta\hat{a}_2 + 3a_{40} M_{\rm s0}^2 [1 - (R_0/R)^{-\frac{2}{r}}]\} (R_0/R)^{\frac{2}{5r}}, \tag{10}$$

with $\delta\hat{a}_2 = \delta a_{20} - \delta a_2 (R_0/R)^{-2/5r}$, by adjusting $M_{\rm s0}$ and $\delta\hat{a}_2$, which are determined by a_{20}, a_{40}, and σ_0 as well as R_0. In this way, although we do not compute τ_0 and δa_{20} and even the real $H_{\rm s}$, we can demonstrate through curve collapsing whether there exists universal scaling and whether it is, if exists, described by the cubic theory, since the above theory, Eq. (9), including a_2 and the resultant $\hat{H}_{\rm s}$ of the quartic contributions is analytically exact. $M_{\rm s0}$ and $\delta\hat{a}_2$ may be determined in a loop perturbation expansion theoretically, though it is difficult to obtain exact results. Here, we estimate them numerically.

There is a difficulty that z is only known to two loops unlike the static exponents.[54] We can estimate it by searching for the best curve collapse. This means that we have three free parameters z, $M_{\rm s0}$, and $\delta\hat{a}_2$. However,

this does not mean that they are arbitrary. In fact, they can be quite definitely estimated. Given z and M_{s0}, $\delta\hat{a}_2$ is fully determined, while the precisions of the first two parameters are only affected by the quality of the curve collapses judged by the naked eye. Generally, given R and others parameters, the slope of the curve depends on a_2. For $R_0 = 0.005$, we find that z cannot exceed 1.85 in order for $R = 0.01$ to fall on top of R_0. On the other hand, it must be larger than 1.8 and 1.835 for $R = 0.02$ and $R = 0.05$, respectively, and the larger slightly the better. We thus choose $z = 1.85$ with $M_{s0}^2 = 1.9(1)$, see Figs. 1(q)–1(s). This z is not far away from the two-loop value of 1.753.[16,54] The resultant a_2 and hence $\hat{H}_s$ go smoothly with R as seen in Fig. 1(t) although we separately adjust $\delta\hat{a}_2$ for each curve, showing the consistency of the results. The same z works well for $R_0 = 0.002$ with $M_{s0}^2 = 2.5(1)$, Fig. 1(s). Note that changing R_0 changes all a_2, a_4 and σ for every R and thus is drastic. From Figs. 1(r) and 1(s), good scaling collapses do emerge. The R range in Fig. 1(r) already covers more than two orders of magnitude even though we have not shown the curve of $R = 0.0005$ as it slightly deviates. However, it follows the rule together with the largest rate $R = 0.5$ as seen in Fig. 1(s), totally three orders of magnitude. Also note that unlike a_4 and M_s, a_2 and $\hat{H}_s$ do not exhibit simple power laws with R as expected, see Fig. 1(t), since they both consist of a series of powers with different exponents. This renders the curves in Fig. 1(q) different from those in Figs. 1(c), 1(i), and 1(m).

Due to expected crossovers, the complete universal scaling here also differs from those in Figs. 1(j) and 1(n). These are crossovers from the metastable states [most explicitly seen from the lower curves in Fig. 1(s)] to the intermediate transition states and from the latter to the stable states. Both the metastable and stable states are different from the transition states and are expected to describe by fixed points and exponents different from the cubic one. Crossovers between these fixed points then result in the upper and lower crossover regions that connect the intermediate transition region, which depends on R like the crossover regions. Yet another crossover may occur when a_2 is too close to the critical point so that critical fluctuations come into play, as may be the case of $R = 0.0005$. These crossovers are interesting questions to be studied in the future. In addition, excessive R has a large σ and may lead to a new transition[47] that gives rise to different scaling. This may also cause the no-so-good collapse of the large R. Except for these special situations and a very small a_4 that may lead to a very weak FOPT, we believe that the present theory is applicable to all the remaining ranges of the parameters that result in a generic FOPT. We note that lack of exact results for both the three parameters and the information for the crossovers leads to some uncertainty ranges in parameter selection and may deteriorate the collapse. Nonetheless, the good scaling collapses over more than two orders of magnitude, which have never been achieved before with only a change of R,[16,58] firmly confirm universal scaling in FOPTs described by the cubic theory. The subtraction of the non-universal fluctuation contributions from the quartic theory can be regarded as an elimination of the contribution from the irrelevant nucleation.

We have demonstrated complete universal scaling in field-driven FOPTs controlled by the renormalization-group fixed points of the effective cubic theory with proper consideration of the quartic contributions for the Langevin equations in both zero and two spatial dimensions. This indicates that FOPTs can indeed be studied uniformly using the framework for continuous phase transitions. The theory and method can be straightforwardly applied to field-driven FOPTs, though for higher-dimensional systems, lack of precise exponents may need extensive parameter adjustments. For temperature-driven FOPTs, complications arise[59,60] and are being studied.

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