Predicted Critical State Based on Invariance of the Lyapunov Exponent in Dual Spaces

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Critical states in disordered systems, fascinating and subtle eigenstates, have attracted a lot of research interests. However, the nature of critical states is difficult to describe quantitatively, and in general, it cannot predict a system that hosts the critical state. We propose an explicit criterion whereby the Lyapunov exponent of the critical state should be 0 simultaneously in dual spaces, namely the Lyapunov exponent remains invariant under the Fourier transform. With this criterion, we can exactly predict a one-dimensional quasiperiodic model which is not of self-duality, but hosts a large number of critical states. Then, we perform numerical verification of the theoretical prediction and display the self-similarity of the critical state. Due to computational complexity, calculations are not performed for higher dimensional models. However, since the description of extended and localized states by the Lyapunov exponent is universal and dimensionless, utilizing the Lyapunov exponent of dual spaces to describe critical states should also be universal. Finally, we conjecture that some kind of connection exists between the invariance of the Lyapunov exponent and conformal invariance, which can promote the research of critical phenomena.

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Since the phenomenon of Anderson localization[1] was proposed, quantum disordered systems have attracted extensive research enthusiasm. When impurities are randomly added to an ideal conductor, the Bloch state in the conductor will change into a localized state with increasing average concentration of impurities. Heretofore, many landmark achievements have been obtained.[2,3] For a three-dimensional tight-binding model with short-range hopping and random on-site potential, there exists a critical energy separating the extended state and localized state, which is dubbed as the mobility edge,[4] and many rigorous mathematical methods have been developed to quantitatively analyze the Anderson localization phenomenon.[5] An example is to use the transfer matrix method to solve the Lyapunov exponent of the one-dimensional Anderson model,[6] as a one-dimensional problem is more likely to be solved analytically.

The transfer matrix method focuses on the recurrence relation of the amplitude of wave functions, and calculates the average exponential divergence rate.[7] Mathematically, let the Lyapunov exponent γ be rewritten in a form of $\exp(-\gamma)$, where the latter indicates the divergence rate between the adjacent lattice points. In position space, for an extended state, the amplitude between neighboring lattice points should be equal under the thermodynamic limit, so γ should be 0; whereas for a localized state, the amplitude between neighboring lattice points should be exponentially decaying, so γ should be greater than 0. Consequently, we can utilize the Lyapunov exponent to explicitly distinguish the state; namely, the extended state corresponds to $\gamma = 0$ and the localized state corresponds to $\gamma > 0$.

However, in a quantum disordered system, there is a class of rare and important states, namely the critical state; the rarity means that it generally only appears at the phase transition point,[8] such as the eigenstate at the mobility edge. The critical state is neither an extended state nor a localized state, i.e., its prominent feature is possessing a self-similar structure.[5] In the language of multifractal theory,[9] the minimum scaling index of the extended state should tend to 1, that of the localized state should tend to 0, and that of the critical state should be greater than 0 but less than 1. However, multifractal theory requires that the system should be numerically diagonalized first, then use numerically obtained eigenstates to calculate the minimum scaling index. This method cannot predict the nature of the eigenstate before numerically diagonalizing the Hamiltonian of a system.

In contrast, calculating the Lyapunov exponent does not need detailed information about eigenstates, and it can directly be solved by giving the Hamiltonian.[10] As long as the Lyapunov exponent of the system is obtained, we can predict the properties of the eigenstates without numerical diagonalization. For the famous Aubry–André model,[11] the Lyapunov exponent is $\gamma = \max\{0, \ln(V/2)\}$ in position space, and $\gamma_m = \max\{0, \ln(2/V)\}$ in momentum space, where V is the strength of the quasiperiodic potential. We can accurately predict when $0 < V < 2$, $\gamma = 0$, the eigenstate in position space is extended; while $V > 2$, $\gamma > 0$, the eigenstate is localized. However, confusion emerges, when $V = 2$ (the phase transition point), γ is also equal to 0; at this point, the eigenstates are all crit-

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ical states. Consequently, the Lyapunov exponent cannot distinguish the extended state from the critical state, and the $\gamma = 0$ method is invalid.

Therefore, numerous research efforts are being devoted to accurately describing the properties of critical states. In Ref. [12] the authors developed a renormalization-group theory to describe the localization properties of quasiperiodic systems. That theory can be used to obtain exact or approximate analytical expressions for the phase boundaries of extended, localized and critical phases for specific models. In Ref. [13] the authors proposed that the coupling between localized and extended states in their overlapped spectrum can provide a general recipe to construct critical states for two-chain models. In Ref. [14] the authors pointed out that there are two approaches involving the existence of critical states, one involves unbounded potential [15] and the other involves zeros of hopping terms in the Hamiltonian. However, an explicit criterion for characterizing critical states is still lacking.

To fill this theoretical gap, we provide a quantitative criterion to predict the critical state in a general sense. Let us re-examine the Lyapunov exponent in both position space and momentum space. For an extended state, we have $\gamma = 0$ and $\gamma_m > 0$; for a localized state, we have $\gamma > 0$ and $\gamma_m = 0$. In order to distinguish the above two states while maintaining the approaching 0 property of the Lyapunov exponent, a reasonable conjecture is that the critical state corresponds to the invariance of the Lyapunov exponent in dual spaces, namely its distinguishing feature satisfies the condition

$$\gamma = \gamma_m = 0. \tag{1}$$

To verify the validity of this conjecture, we introduce an exactly solvable model hosting critical states in a wide range of parameters, and exactly predict the interval of the existence of critical states through $\gamma = \gamma_m = 0$. Hence, we demonstrate that this conjecture is applicable to various critical states.

Model and Prediction. The difference equation of the model we consider reads

$$\psi_{n+1} + \psi_{n-1} + Vi\tan(2\pi\alpha n + \theta)\psi_n = E\psi_n, \tag{2}$$

where V is the complex potential strength, E is the eigenvalue of the systems, and ψ_n is the amplitude of the wave function at the nth lattice. We choose to unitize the nearest-neighbor hopping amplitude. A typical choice for an irrational parameter is $\alpha = (\sqrt{5}-1)/2$, and θ is the phase factor.

With the Hamiltonian of the system [Eq. (2)], we can determine the Lyapunov exponent γ in position space, [16–19] which is calculated by taking the product of the transfer matrix $T(\theta)$, namely $\gamma = \lim_{n\to\infty} \ln ||T_n(\theta)||/n$. Utilizing Avila's global theory, [20] we can obtain the explicit expression of γ (see the detailed calculation in the Supplemental Material), [21]

$$\gamma(E) = \max\Big\{ \operatorname{arcosh}\frac{|E+V+2|+|E+V-2|}{4}, \operatorname{arcosh}\frac{|E-V+2|+|E-V-2|}{4}\Big\}. \tag{3}$$

From Eq. (3), we can extract the allowed energies of the system. Firstly, we make $\gamma(E) = 0$, and obtain that E is within the region $[V-2, 2-V]$. Secondly, we make $\gamma(E) > 0$, and obtain that E is within the region $\{iy \mid y \in \mathbb{R}^*\}$ $(V \le 2)$ or $\{iy \mid y \in \mathbb{R}\}$ $(V > 2)$, which means that when the eigenstate is a localized state in position space, the eigenvalue is a pure imaginary number. However, when $\gamma = 0$, with regard to the eigenvalue within $[V-2, 2-V]$, it cannot be concluded that the corresponding eigenstate is an extended state or a critical state.

The next step is to calculate the Lyapunov exponent γ_m in momentum space. Firstly, we introduce the Fourier transformation

$$f_k = \frac{1}{\sqrt{L}}\sum_{n=1}^{L} e^{i2\pi\alpha kn}\psi_n,$$

thus the dual equation of Eq. (2) in momentum space is written as

$$f_{k+1} = \frac{-2\cos[2\pi(k-1)\alpha] + V + E}{2\cos[2\pi(k+1)\alpha] + V - E} f_{k-1}. \tag{4}$$

Here we should emphasize that this model is not a self-duality model, i.e., the position space difference equation [Eq. (2)] and momentum space difference equation [Eq. (4)] are significantly different, hence the feature of the model is different from the self-duality of the Aubry–André model.

From Eq. (4), an initial wave function solution can be written as

$$f_k \propto \begin{cases} 0, & k = 0, \pm2, \pm4\cdots, \\ 0, & k = 2j+1 < k_0, \\ 1, & k = 2j+1 = k_0, \\ \frac{g^{(1)}_{k-2}}{g^{(2)}_{k}} f_{k-2}, & k = 2j+1 > k_0, \end{cases}$$

then the Lyapunov exponent γ_m can be obtained by Sarnak's method, [21,22]

$$\begin{aligned}\gamma_m(E) &= \lim_{k\to\infty}\frac{1}{k-k_0}\ln\Big|\frac{f_k}{f_{k_0}}\Big| \\ &= \frac{1}{2\pi}\int_0^{2\pi}[\ln g^{(1)} - \ln g^{(2)}]d\theta,\end{aligned} \tag{5}$$

where $g^{(1)} = |-2\cos(2\pi\theta) + V + E|$, $g^{(2)} = |2\cos(2\pi\theta) + V - E|$.

Recalling the calculation process of the Lyapunov exponent γ in position space, when $\gamma > 0$, namely eigenvalues of the system are pure imaginary numbers, we have the identity

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\ln|-2\cos(2\pi\theta) + V + iy|d\theta \\ =&\frac{1}{2\pi}\int_0^{2\pi}\ln|2\cos(2\pi\theta) + V - iy|d\theta.\end{aligned} \tag{6}$$

Hence, the Lyapunov exponent in momentum space $\gamma_m = 0$. Here $\gamma_m = 0$ and $\gamma > 0$ indicate that the associated state is the extended state in momentum space, which exactly corresponds to the localized state in position space.

When $\gamma = 0$, namely eigenvalues are real numbers and within the interval $[V-2, 2-V]$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|g^{(1)}|d\theta = \begin{cases} 0, & g^{(1)} = 0, \\ \ln\left|\frac{|E+V|+\sqrt{(E+V)^2-4}}{2}\right|, & g^{(1)} \neq 0, \end{cases} \tag{7}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|g^{(2)}|d\theta=\begin{cases}0, & g^{(2)}=0,\\ \ln\left|\frac{|E-V|+\sqrt{(E-V)^2-4}}{2}\right|, & g^{(2)}\neq 0.\end{cases}\tag{8}$$

According to Eqs. (7) and (8), we obtain the identity

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|g^{(1)}|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\ln|g^{(2)}|d\theta,$$

see the Supplemental Material for the detailed calculation.[21] Hence, when $V-2\leq E\leq 2-V$, γ_m is also equal to 0, which is very similar to the result where the eigenvalues are pure imaginary numbers. The difference is that the latter have $\gamma>0$ in position space and $\gamma_m=0$ in momentum space; whereas the eigenstates of the former have $\gamma=\gamma_m=0$.

Based on the above achievements and the conjecture of Eq. (1), we can predict that, different from the common models where the critical state only exists at the phase transition point, for the model of Eq. (2), critical states exist in a wide range of parameters $0<V\leq 2$, due to $V-2\leq E\leq 2-V$ indicating $0<V\leq 2$. Consequently, we realize the prediction of a system hosting a large number of critical states through giving its Hamiltonian.

Numerical Verification and Self-Similarity. To support the analytical results given above, we perform numerical calculations and analyze the critical nature of the eigenstate. We numerically diagonalize the Hamiltonian (2) in a large size, and obtain the eigenvalue and the associated eigenstates. To verify the critical state, we should illustrate the distinctive feature of the corresponding state, namely, self-similarity.

Mathematically, self-similarity is a typical property of the fractal,[23,24] a self-similar construction is exactly or approximately similar to a part of itself. Many physical objects in nature, such as capillary distribution and leaf veins, are statistically self-similar: parts of them show the same statistical properties as the whole. An equivalent description of self-similarity is scale invariant,[25,26] where there is a smaller part that is similar to the proximate larger part at a certain amplitude. Thus, we can perform the contraction–expansion variations on the part of physical quantities of the system. If the variation amplitude meets a certain value, it means that the physical quantity is scale invariant, that is, the system has a self-similarity structure.

Therefore, we directly numerically diagonalize Eq. (2), plot the obtained eigenstate, and examine the detailed structure of the wave function to determine whether self-similarity exists. As shown in Fig. 1, the black curve is the original eigenstate of the system, and it has four wave-function peaks of different amplitudes. This state is not a localized state, but it does not look like an extended state because the peaks of the extended state should be equal. When we magnify three smaller peaks by 7, 4, and 2.2857 times successively, namely the red curves, they become very similar to the largest black peak. More importantly, we can find that $7/4\approx 4/2.2857\approx 1.75$, which indicates that three smaller peaks can be expanded to the largest one by a certain multiple 1.75. This clearly shows that the wave function of the system has scale invariance and self similarity, hence this state is definitely a critical state.

We also calculate the eigenstates corresponding to different energy levels and different sizes, and the corresponding eigenstate in momentum space is also illustrated; all numerical results are critical states as expected (see the detailed calculation in the Supplemental Material).[21] For the critical state of the known models,[27] the validity of the criterion can also be conveniently verified.

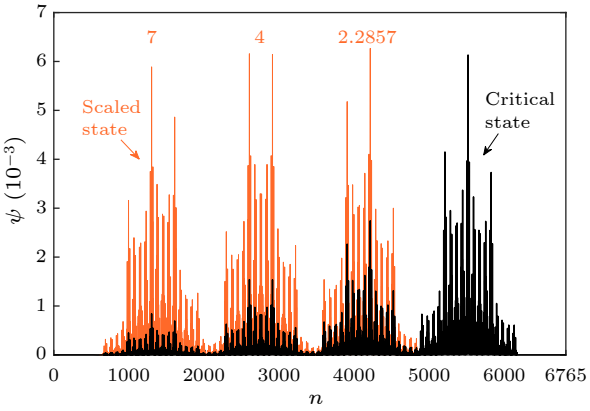

Fig. 1. The black curve represents the wave function of $E=-1$ obtained from Eq. (2) with the parameter $V=1$. The red curves represent three wave function peaks after magnifying. Starting with the smallest peak, the scaled multiples are 7, 4, and 2.2857 in turn. It is obvious that the scaled three smaller peaks are very similar to the largest peak. The total number of sites is set to be $L=6765$.

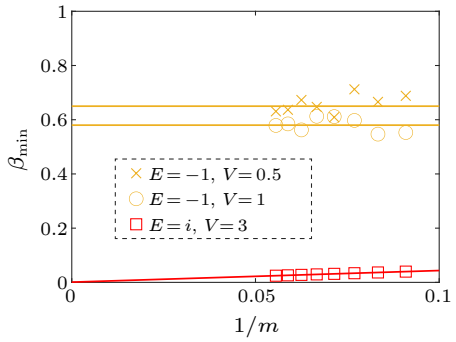

Fig. 2. Characteristics of $\beta_{\min}$ against the inverse Fibonacci index $1/m$ for various eigenvalues. In the large size limit, the brown crosses tend to 0.67 and correspond to the critical state with $E=-1$ ($V=0.5$), the brown circles tend to 0.58 and correspond to the critical state with $E=-1$ ($V=1$), the red square markers tend to 0 and correspond to the localized state with $E=i$ ($V=3$).

In addition to visually displaying the wave function, we also calculate the minimum scaling index of the critical state according to multifractal theory.[9] For the given wave function ψ_n, a scaling index β_n can be extracted from the nth on-site probability $P_n=|\psi_n|^2\sim(1/F_m)^{\beta_n}$, where F_m is the mth Fibonacci number.

Multifractal theorem states that when the wave functions are extended, the maximum of P_n scales as $\max(P_n)\sim(1/F_m)^1$, i.e., $\beta_{\min}=\min(\beta_n)=1$. On the other hand, when the wave functions are localized, P_n concentrates at the individual site and tends to zero at the other sites, yielding $\max(P_n)\sim(1/F_m)^0$ and $\beta_{\min}=\min(\beta_n)=0$. With regard to the critical state, the cor-

responding $\beta_{\min}$ is located within the interval (0, 1), and can be utilized to distinguish extended and critical states. In order to reduce finite-size effects, we examine the trend of $\beta_{\min}$ under the limit of large size.

As shown in Fig. 2, $\beta_{\min}$ is plotted as a function of the inverse Fibonacci index $1/m$, when $1/m \to 0$, the system size $L \to \infty$. It clearly shows that $\beta_{\min}$ is between 0 and 1 in the large L limit for the eigenvalues $E = -1$ ($V = 0.5$) and $E = -1$ ($V = 1$), hence the corresponding state is critical, while for the eigenenergy $E = i$ ($V = 3$), $\beta_{\min}$ asymptotically tends to 0 in the large L limit, indicating that the corresponding state is localized. We have also checked other combinations of parameters and get the same results as expected. Thus, the above numerical results are in excellent agreement with the analytical results.

What we need to emphasize here is that the critical states of the Aubry–André model are considered to originate from the self-duality of the Aubry–André model,[28] while this model also hosts critical states, but does not have the self-duality. Hence, critical states[29,30] cannot be completely classified as duality or self-duality, in other words, the duality cannot directly provide explicit information about critical states. A more accurate statement about critical states is that the Lyapunov exponents of dual spaces should be 0 simultaneously; the self-duality leads to the Lyapunov exponents of dual spaces being 0 simultaneously, but the reverse statement is incorrect, therefore, satisfying self-duality is only a special case of critical states.

In summary, we propose a conjecture to predict and identify the existence of the critical state in a system, that is, the Lyapunov exponent of the eigenstate should be 0 in both position space and momentum space. To illustrate this criterion, we introduce an exactly solvable model which does not exhibit self-duality, and predict and verify the existence of a large number of critical states in a wide range of potential strengths. This demonstrates that $\gamma = \gamma_m = 0$ is not limited to the critical state at the phase transition point or the self-duality point, but is applicable to various types of critical states.

In addition to quasiperiodic systems, the Lyapunov exponent can define all kinds of extended, localized, or critical states, even for random systems. However, the method of calculating the Lyapunov exponent for quasiperiodic systems obviously cannot be applicable to random systems, such as Avila's global theory. Another example is that the energy-level statistics of the single-particle quasiperiodic system do not conform to the Wigner–Dyson or Poisson statistics predicted by random matrix theory. Hence, from a methodological perspective, the Lyapunov exponent method is applicable for eigenstates of random matrices, but specific methods of random matrices are difficult to apply to non-random systems. Our findings provide an explicit quantitative description of the characteristics of the critical state; however, analytically calculating the critical state of high-dimensional or random systems remains a long-term and challenging task.

Furthermore, it is noteworthy that the prominent feature of critical states is the scale invariance of wave functions, which indicates that the invariance of the Lyapunov exponent under Fourier transform has a closed relation with conformal invariance. This is likely to be a new application scenario of conformal invariance theory, which can describe properties of quantum disordered systems near the critical point or within the critical interval.

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References

[1] Anderson P W 1958 *Phys. Rev.* **109** 1492
[2] Abrahams E, Anderson P W, Licciardello D C, and Ramakrishnan T V 1979 *Phys. Rev. Lett.* **42** 673
[3] Fleishman L and Licciardello D C 1977 *J. Phys. C* **10** L125
[4] Mott N 1987 *J. Phys. C* **20** 3075
[5] Lagendijk A, van Tiggelen B, and Wiersma D S 2009 *Phys. Today* **62** 24
[6] Brandes T and Kettemann S 2003 *The Anderson Transition and Its Ramifications—Localisation, Quantum Interference, and Interactions* (Berlin: Springer) pp 3–19
[7] Soukoulis C M and Economou E N 1982 *Phys. Rev. Lett.* **48** 1043
[8] Sokoloff J B and structure U B 1985 *Phys. Rep.* **126** 189
[9] Hiramoto H and Kohmoto M 1989 *Phys. Rev. B* **40** 8225
[10] Avila A, You J, Zhou Q 2017 *Duke. Math. J.* **14** 166
[11] Aubry S and André G 1980 *Ann. Israel Phys. Soc.* **3** 133
[12] Gonçalves M, Amorim B, Castro E V, Ribeiro P arXiv:2206.13549v2 [cond-mat.dis-nn]
[13] Lin X, Chen X, Guo G, Gong M 2022 arXiv:2209.03060v1 [quant-ph]
[14] Zhou X, Wang Y, Poon T J, Zhou Q, Liu X arXiv:2212.14285v2 [cond-mat.dis-nn]
[15] Liu T, Xia X, Longhi S, and Sanchez-Palencia L 2022 *SciPost Phys.* **12** 027
[16] Liu Y X, Wang Y C, Liu X J, Zhou Q, Chen S, edges E M, and breaking P S 2021 *Phys. Rev. B* **103** 014203
[17] Liu Y X, Wang Y C, Zheng Z H, and Chen S 2021 *Phys. Rev. B* **103** 134208
[18] Cai X M 2021 *Phys. Rev. B* **103** 214202
[19] Cai X M 2022 *Phys. Rev. B* **106** 214207
[20] Avila A 2015 *Acta Math.* **215** 1
[21] See the Supplemental Material for details of (i) derivation of Lyapunov exponent in position space, (iii) more numerical verification
[22] Sarnak P 1982 *Commun. Math. Phys.* **84** 377
[23] Amin K, Nagarajan R, Pandit R, and Bid A 2022 *Phys. Rev. Lett.* **129** 186802
[24] Deng X, Ray S, Sinha S, Shlyapnikov G V, and Santos L 2019 *Phys. Rev. Lett.* **123** 025301
[25] Yao H P, Khoudli A, Bresque L, and Sanchez-Palencia L 2019 *Phys. Rev. Lett.* **123** 070405
[26] Wardak A and Gong P 2022 *Phys. Rev. Lett.* **129** 048103
[27] Liu F L, Ghosh S, and Chong Y D 2015 *Phys. Rev. B* **91** 014108
[28] Liu T, Guo H, Pu Y, and Longhi S 2020 *Phys. Rev. B* **102** 024205
[29] Bai X and Xue J 2015 *Chin. Phys. Lett.* **32** 010302
[30] Yin Y, Niu Y, Ding M, Liu H, and Liang Z 2016 *Chin. Phys. Lett.* **33** 057202