Interacting Solitons, Periodic Waves and Breather for Modified Korteweg–de Vries Equation

Vladimir I. Kruglov[1*] and Houria Triki[2]

[1] *Centre for Engineering Quantum Systems, School of Mathematics and Physics, The University of Queensland, Brisbane, Queensland 4072, Australia*
[2] *Radiation Physics Laboratory, Department of Physics, Faculty of Sciences, Badji Mokhtar University, P. O. Box 12, 23000 Annaba, Algeria*

(Received 24 July 2023; accepted manuscript online 23 August 2023)

We theoretically demonstrate a rich and significant new families of exact spatially localized and periodic wave solutions for a modified Korteweg–de Vries equation. The model applies for the description of different nonlinear structures which include breathers, interacting solitons and interacting periodic wave solutions. A joint parameter which can take both positive and negative values of unity appeared in the functional forms of those closed form solutions, thus implying that every solution is determined for each value of this parameter. The results indicate that the existence of newly derived structures depend on whether the type of nonlinearity of the medium should be considered self-focusing or defocusing. The obtained nonlinear waveforms show interesting properties that may find practical applications.

DOI: 10.1088/0256-307X/40/9/090503

Breathers and solitons are spatially localized structures that naturally arise in a variety of physical systems such as optical fibers,[1,2] Bose–Einstein condensates,[3,4] waveguide arrays,[5] plasmas,[6] stratified fluids,[7] and so on. Experimental observation of breathers in water wave tanks has been also reported recently.[8] In optical contexts, solitons are based on the balance between the dispersion and nonlinearity of a medium while breathers are formed as the result of the evolution of a modulated continuous wave field disintegrating into a train of short pulses.[9,10] In addition, breathers are always located on a constant background, whereas envelope solitons can exist on a plane wave background (Ma solitons[11]) or on a zero background.[12]

Most studies of breathers and solitons are concerned with physical media for which the nonlinear Schrödinger equation is the main underlying model for governing the wave evolution. There exist, however, certain types of nonlinear materials where the theoretical descriptions of wave dynamics require the use of other kinds of nonlinear wave equations. It should be noted that there are many forms of nonlinear evolution equations whose integrability aspects as well as several other features, such as bifurcation analysis, perturbation theory, conservation laws, prolongation structures and many such aspects, have been widely studied and they are being continuously worked upon for a variety of new and upcoming models. This trend is very commonly visible in various equations that are visible in nonlinear optics.[13–15] An example of these evolution models is the modified Korteweg–de Vries (mKdV) equation which has been demonstrated to describe the evolution of long waves in the critical case of vanishing quadratic nonlinearity.[16] Particularly, this equation is relevant for nonlinear waves in distributed Schottky barrier diode transmission lines[17] and internal waves in stratified fluids.[18] Moreover, this equation has also been applied in other settings such as soliton propagation in lattices,[19] meandering ocean currents,[20] the dynamics of traffic flow,[21–23] nonlinear Alfvén waves propagating in plasma,[24] and ion acoustic soliton experiments in plasmas.[25] Recent studies reveal that the mKdV equation also applies for describing the propagation of light pulses consisting of a few cycles in Kerr-type media beyond the slowly varying envelope approximation.[26–29] In addition to the above, this equation has gained further importance recently, mainly because of its effectiveness in modeling supercontinuum generation in optical fibers.[30] All these wide-ranging potential applications make the study of mKdV equation from different points of view a very important problem to understand various physical phenomena.

Rich families of explicit solutions have been reported so far for the mKdV equation. In particular, Wadati derived the exact N-soliton solutions of the mKdV equation by using the inverse-scattering transform scheme.[31] In addition, Kevrekidis *et al.*[32] obtained some classes of periodic solutions of this model. Another class of nonlinear wave solutions that conserve their energy during evolution -breathers (oscillatory wave packets) has also been found for this model (see, e.g., Refs. [33–35]). The focusing and defocusing mKdV equations with nonzero boundary conditions are studied for inverse scattering transforms with the matrix Riemann–Hilbert problem in Ref. [36].

Finding novel exact analytical solutions is an important direction of research in nonlinear wave dynamics because wave solutions are helpful for a better understanding of physical phenomena such as the stability of nonlinear

*Corresponding author. Email: kiwivlad1@gmail.com
© 2023 Chinese Physical Society and IOP Publishing Ltd

wave propagation. In the context, important results have been obtained with recent studies discussing the soliton dynamics within the framework of different nonlinear wave models. For example, Geng *et al.*[37] obtained the non-degenerate one- and two-soliton solutions of the nonlocal nonlinear Schrödinger equation by means of the nonstandard Hirota method. Furthermore, He *et al.*[38] derived multi-type soliton solutions of Gross–Pitaevskii equations describing spin-orbit coupled spin-1 Bose–Einstein condensates by applying the similarity transformation method. In addition, Wen *et al.*[39] utilized the physical neural network to obtain different types of vector solitons of the coupled mixed derivative nonlinear Schrö dinger equation. These findings indicate that the soliton propagation is one of the physically relevant phenomena associated with dynamical systems. It is worthy to mention that obtaining solutions in analytic form is relevant not only to determine certain important physical quantities and serve as diagnostics for simulations but also even for comparing experimental results with theory. Notably, while a closed form solution is vital to understand the nonlinear dynamical processes, it is not always possible to find it.

In this Letter, we predict the existence of three new types of nonlinear wave structures through discovery of physically important exact solutions of the mKdV model that includes the cubic nonlinearity term with either positive or negative sign. Significant classes of breather and interacting soliton pairs and periodic waves are presented for the first time. Remarkably, the breather and interacting periodic waves formation are observed in the case of a negative coefficient of the cubic nonlinear term while the pair of interacting soliton solution is formed when this coefficient is positive.

We start by considering the mKdV equation in standard dimensionless form as

$$u_t + u_{xxx} + 6\mu u^2 u_x = 0, \tag{1}$$

where $u(x,t)$ is the real function. The parameter $\mu = \pm 1$ denotes the type of nonlinearity, i.e., $+1$ for focusing type of nonlinearity and -1 for defocusing nonlinearity. The mKdV is a fully integrable equation which means that it has an infinite number of conserved invariants.[40]

Two soliton branches exist for the mKdV Eq. (1) in the case of a positive coefficient of the cubic nonlinear term (i.e., $\mu = 1$), they are defined as

$$u(x,t) = a + \frac{b^2}{\Lambda \cosh(b(x - x_0 - vt) + \psi_0) + 2a}, \tag{2}$$

where $\Lambda = \pm\sqrt{4a^2 + b^2}$, $v = 6a^2 + b^2$, and a, b, x_0 are the arbitrary constants. In the limit of $a \to 0$ the solution in Eq. (2) tends to the familiar sech-shaped soliton family which is known to be stable with respect to small perturbations (see, e.g., Ref. [41]). In the case of $\mu = 1$ there also exists the algebraic solitary wave solution as

$$u(x,t) = p - \frac{p}{p^2(x - x_0 - vt)^2 + 1/4}, \tag{3}$$

where $v = 6p^2$, and p, x_0 are the arbitrary constants. We note that the N-soliton solution (1) can be obtained using the Darboux transform:

$$u(x,t) = -i\frac{\partial}{\partial x} \ln \frac{W(\Psi_{1x}, \Psi_{2x}, \ldots, \Psi_{Nx})}{W(\Psi_1, \Psi_2, \ldots, \Psi_N)}, \tag{4}$$

where $\Psi_j = \exp(\Theta_j) + i\exp(-\Theta_j)$ and $\Theta_j = \nu_j(x - x_j) - 4\nu_j^3 t + i\theta_j/2$, and parameters ν_j are connected with eigenvalues λ_j of the appropriate inverse-scattering problem. Here W is the Wronskian for N eigenfunctions Ψ_j in the denominator and for their spatial derivatives Ψ_{jx} in the numerator. This formal expression has inner symmetries and the properties of determinants which can be used for construction of corresponding solutions.[42,43]

The breather's expression for $\mu = 1$ was obtained from inverse scattering transform in Ref. [16] and also in Ref. [43]. Note that the breather has much more complicated dynamics than soliton. Recently Slunyaev and Pelinovsky[44] also presented an explicit breather solution for the mKdV Eq. (1) with $\mu = 1$ as

$$u(x,t) = 2pq\Big(\frac{p\sinh\theta\sin\phi - q\cosh\theta\cos\phi}{p^2\sin^2\phi + sq^2\cosh^2\theta}\Big), \tag{5}$$

when $s = -1$. Here θ and ϕ, which control the wave envelope and the inner wave respectively, are given by

$$\theta = p(x - x_0) + p(3q^2 - p^2)t + \theta_0, \tag{6}$$

$$\phi = q(x - x_0) + q(q^2 - 3p^2)t + \phi_0, \tag{7}$$

where p, q and θ_0, ϕ_0, x_0 are the arbitrary real parameters. We have remarked that this breather solution still also exists for the case of $s = +1$ with the same relations of θ and ϕ as those given in Eqs. (6) and (7). Thus the mKdV Eq. (1) with the focusing type of nonlinearity possesses two exact breather solutions (5) corresponding to the values $s = \pm 1$.

In what follows, we report the analytical demonstration of a class of breather, interacting periodic and soliton solutions for the mKdV equation with either focusing or defocusing types of nonlinearity. This class of solutions follows through the Zakharov–Shabat or Ablowitz–Kaup–Newell–Segur inverse-scattering transform method.[45,46] The inverse-scattering transform method applied to mKdV Eq. (1) for focusing type of nonlinearity ($\mu = 1$) leads the initial-valued problem given by the system of equations as

$$\frac{\partial \Phi_1}{\partial x} = -u(x)\Phi_2 + \lambda\Phi_1, \tag{8}$$

$$\frac{\partial \Phi_2}{\partial x} = u(x)\Phi_1 - \lambda\Phi_2, \tag{9}$$

where λ is a complex-valued eigenvalue and $u(x)$ is an initial perturbation connected to Eq. (1). The discrete complex eigenvalues λ generally arise in quartets and correspond to localized solutions. This approach is based on the Gel'fand–Levitan–Marchenko integral equation which is separable and hence can be solved algebraically. In the case of defocusing nonlinearity ($\mu = -1$) the initial perturbation $u(x)$ in Eqs. (8) and (9) should be replaced to $iu(x)$.

Using the inverse scattering method we have found three types of exact solutions for the modified KdV Eq. (1) with $\mu = \pm 1$. These solutions are given in Eq. (10) (for $\mu = 1$), and Eqs. (15) and (20) (for $\mu = -1$). A first class

of localized waves in the form of a pair interacting solitons is obtained for Eq. (1) in the case of focusing cubic nonlinearity (i.e., $\mu = 1$) as

$$u(x,t) = 2pq\Big(\frac{p\sinh\theta\sinh\phi - q\cosh\theta\cosh\phi}{p^2\sinh^2\phi + sq^2\cosh^2\theta}\Big), \tag{10}$$

where $s = \pm 1$ and θ, ϕ are given by

$$\theta = p(x - x_0) - p(3q^2 + p^2)t + \theta_0, \tag{11}$$

$$\phi = q(x - x_0) - q(3p^2 + q^2)t + \phi_0. \tag{12}$$

Hereafter p, q, θ_0, ϕ_0, and x_0 are the arbitrary real parameters. The wave solution (10) represents a pair of interacting bipolar solitons whose dynamic features are delineated in Fig. 1 (with $\mu = 1$) for the case of $p = 1$, $q = 0.2$ and $s = 1$. Here we choose the initial soliton position x_0 and phases θ_0 and ϕ_0 equal to zeros. Figure 1 shows the overtaking interaction between two solitons of opposite polarities for $u(x,t)$ given in Eq. (10), from which we can see that the solitonic amplitudes and shapes have not changed after the interaction. Compared with the well-known sech-shaped solitary wave which is a single soliton structure, this novel mKdV soliton solution takes the form of a pair of interacting solitons, which can interact purely elastically and behave like independent solitons far from the meeting point. We note that the wave solution (10) with $s = -1$ also takes the shape of a pair of interacting solitons with opposite polarity.

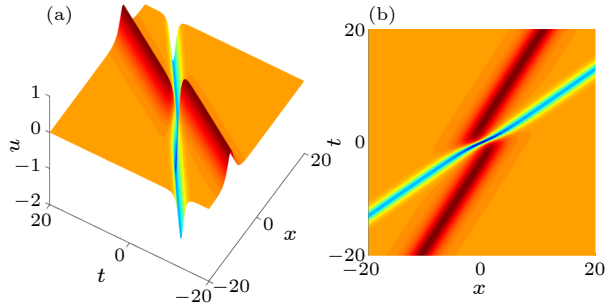

Fig. 1. (a) Evolution of the interacting soliton solution of the mKdV equation defined by Eq. (10) ($\mu = 1$) for $p = 1$, $q = 0.2$, and $s = 1$. (b) The corresponding contour plot of the interacting soliton solution in (a).

In the case of $q = -p$ and $s = 1$ the solution given in Eq. (10) has the form of modified soliton solution traveling with dimensionless velocity v. This modified soliton solution is given by

$$u(x,t) = \frac{4q\cosh(\theta_0 + \phi_0)}{\cosh(2q\xi + 2\phi_0) + \cosh(2q\xi - 2\theta_0)}, \tag{13}$$

where $\xi = (x - x_0) - vt$ and $v = 4q^2$.

In the special case of $\theta_0 + \phi_0 = 0$ the solution (13) reduces to soliton solution in Eq. (2) with $a = 0$, $b = 2q$ and $\psi_0 = 2\phi_0$.

In the case $q = p$ and $s = 1$ the solution given in Eq. (10) also has the form of modified soliton solution traveling with dimensionless velocity v. This modified soliton solution is given by

$$u(x,t) = \frac{-4q\cosh(\theta_0 - \phi_0)}{\cosh(2q\xi + 2\phi_0) + \cosh(2q\xi + 2\theta_0)}, \tag{14}$$

where $\xi = (x - x_0) - vt$ and $v = 4q^2$. In the special case of $\theta_0 - \phi_0 = 0$ the solution (14) reduces to soliton solution (2) with $a = 0$, $b = 2q$, and $\psi_0 = 2\phi_0$.

We have obtained another exact solution for the mKdV Eq. (1) with $\mu = -1$ in the form of an interacting periodic wave solution as

$$u(x,t) = 2pq\Big(\frac{p\sin\theta\sin\phi + q\cos\theta\cos\phi}{p^2\sin^2\phi - sq^2\cos^2\theta}\Big), \tag{15}$$

where $s = \pm 1$, θ and ϕ are

$$\theta = p(x - x_0) + p(3q^2 + p^2)t + \theta_0, \tag{16}$$

$$\phi = q(x - x_0) + q(3p^2 + q^2)t + \phi_0. \tag{17}$$

In the case of $q = -p$ and $s = -1$ the solution (15) takes the shape of periodic traveling wave,

$$u(x,t) = \frac{2p\cos(\theta_0 + \phi_0)}{\sin^2(p\xi - \phi_0) + \cos^2(p\xi + \theta_0)}, \tag{18}$$

where $\xi = (x - x_0) - vt$ and $v = -4p^2$. The denominator in (16) is zero only in the case when $\theta_0 + \phi_0 = \pi/2 + \pi n$ with $n = 0, \pm 1, \pm 2, \ldots$. If this denominator is zero the numerator in (16) is zero as well. The solution given in (18) is not singular when $\phi_0 + \theta_0 \neq \pi/2 + \pi n$ with $n = 0, \pm 1, \pm 2, \ldots$. In the special case of $\theta_0 + \phi_0 = \pi n$ this periodic wave solution reduces to constant solution $u(x,t) = (-1)^n 2p$.

In the case of $q = p$ and $s = -1$ the solution given by Eq. (15) also takes the shape of periodic traveling wave,

$$u(x,t) = \frac{2p\cos(\theta_0 - \phi_0)}{\sin^2(p\xi + \phi_0) + \cos^2(p\xi + \theta_0)}, \tag{19}$$

where $\xi = (x - x_0) - vt$ and $v = -4p^2$. The denominator in Eq. (19) is zero only in the case of $\theta_0 - \phi_0 = \pi/2 + \pi n$ with $n = 0, \pm 1, \pm 2, \ldots$. If this denominator is zero, the numerator in Eq. (19) will be zero as well. The solution given in Eq. (19) is not singular when $\theta_0 - \phi_0 \neq \pi/2 + \pi n$ with $n = 0, \pm 1, \pm 2, \ldots$. In the special case of $\theta_0 - \phi_0 = \pi n$ this periodic wave solution reduces to constant solution $u(x,t) = (-1)^n 2p$.

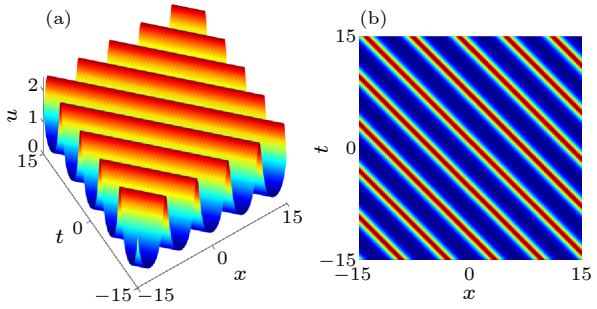

Fig. 2. (a) Evolution of periodic traveling wave solution of the mKdV equation defined by Eq. (18) ($\mu = -1$) for $p = 0.5$, $\phi_0 = 0$, and $\theta_0 = \pi/4$. (b) The corresponding contour plot of the periodic wave solution in (a).

Figure 2 depicts an example of the nonlinear wave solution (18) (with $\mu = -1$) for the case of $p = 0.5$, $\phi_0 = 0$, and $\theta_0 = \pi/4$. Here the initial parameter x_0 is chosen to be zero. We observe that the nonlinear waveform (16)

presents an oscillating behavior during the process of wave evolution.

We have also found that the mKdV Eq. (1) with the defocusing type of nonlinearity (i.e., $\mu = -1$) has an exact breather solution of the form

$$u(x,t) = 2pq\Big(\frac{p\sin\theta\sinh\phi + q\cos\theta\cosh\phi}{p^2\sinh^2\phi - sq^2\cos^2\theta}\Big), \tag{20}$$

where $s = \pm 1$, θ and ϕ are

$$\theta = p(x - x_0) + p(p^2 - 3q^2)t + \theta_0, \tag{21}$$

$$\phi = q(x - x_0) + q(3p^2 - q^2)t + \phi_0. \tag{22}$$

Thus the exact solutions (10), (15), and (20) to the mKdV equation with the focusing and defocusing types of nonlinearity are obtained via the inverse-scattering transform. Moreover, one may conclude that a physical system described by the mKdV equation could allow a breather evolution in either the focusing or the defocusing nonlinearity. A typical example of the singular breather profile for solution in Eq. (20) is shown Fig. 3 (with $\mu = -1$) for $p = 1$, $q = 0.98$, and $s = -1$. In this case, the initial parameters x_0, θ_0, and ϕ_0 are chosen to be zeros. It can be seen that the breather structure has nontrivial periodical behavior while evolving in (x,t) plane. We emphasize that this breather solution has periodic singularities in (x,t) plane under the condition $\phi(x,t) = 0$. Hence this singular solution does not have a direct application for physical problems. However, such solutions are important for the general theory of nonlinear systems.[42]

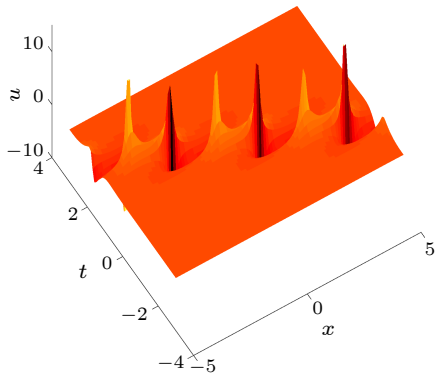

Fig. 3. Breather solution of the mKdV equation defined by Eq. (20) ($\mu = -1$) for $p = 1$, $q = 0.98$, and $s = -1$.

Finally, let us present a physical description of the theoretical results obtained above. The present study interestingly explores the existence of new types of localized and periodic waves of the mKdV equation describing the wave dynamics in many nonlinear systems. Especially, we demonstrate the existence of novel classes of interacting soliton pairs, periodic waves and breather solutions for the mKdV model with either focusing or defocusing type of nonlinearity. The novelty of the obtained wave structures lie essentially in their functional forms, which are different from the previously attained results. For example, we see that the denominator of the interacting soliton pair solution (10) involves the hyperbolic function $\sinh^2$ and a joint parameter s which takes both positive and negative values of unity ($s = \pm 1$), markedly different from the breather solution (5) presented in Ref. [44] whose denominator contains the trigonometric function $\sin^2$ and the sign minus. It is worth mentioning that the inclusion of the joint parameter s makes the obtained solution (10) more general in the sense that every solution is determined for each value of this parameter. Moreover, the new interacting periodic wave and breather wave solutions (15) and (20) obtained here for the mKdV model are firstly presented in this study. We note that our solutions are also general in a mathematical sense, as they are obtained without assuming parametric conditions for their existence.

It is also relevant to mention that the functional form of the interacting periodic wave and breathers wave solutions (15) and (20) are also different from those obtained very recently by Ma and Li[47] and previously by Kevrekidis *et al.*[32] as well as Alejo and Muñoz.[48] This illustrates the richness of nonlinear systems modeled by the mKdV equation, which is often measured by the variety of wave structures that the medium can support.

From the point of wave profile, Fig. 1 clearly shows that the pair interacting soliton structure (10) contains two localized waves of opposite polarity in which neither the wave amplitude nor shape change after the interaction. This interesting profile is different from the shape of the known sech solitary wave, which has only one soliton structure. Moreover, one can see from Fig. 2 that the profile of the nonlinear wave (18) presents the periodic property and its amplitude remains constant during the process of wave evolution. Additionally, we observe that the wave profile of the breather wave (20) exhibits a nontrivial periodical behavior while evolving in (x,t) plane, as seen from Fig. 3. These wave structures can help one to well understand the physical phenomena and dynamical processes modeled by this model.

It is interesting to note that the obtained novel soliton and periodic waves have significant applications in fluid dynamics, plasma physics and applied sciences. The nonlinear wave solutions presented here could also be useful in some engineering applications, since the underlying model is relevant for many systems, for example, for describing nonlinear waves in distributed Schottky barrier diode transmission lines.[17]

In conclusion, we have obtained novel classes of spatially localized and periodic wave solutions for the mKdV equation which governs the nonlinear wave dynamics in many physical systems when there is polarity symmetry. Specifically, new types of breathers, interacting solitons and interacting periodic wave solutions are derived by employing the inverse scattering transform method. We have found that breather waves can be formed in either a focusing or a defocusing nonlinearity. It is also shown that the exact form of interacting soliton solutions arises in the case of focusing cubic nonlinearity whereas the interacting periodic wave solutions forms when the nonlinear medium is defocusing. Additionally, we have found that the existence of these new structures depend drastically on the type of nonlinearity of the nonlinear medium (a self-focusing or defocusing one). These results constitute the analytical demonstration of existence of new closed form solutions in

the form of a pair interacting solitons (10), interacting periodic waves (15) and breathers waves (20) to the mKdV equation. The wide applicability of the mKdV equation to describe evolution dynamics of nonlinear waves in a variety of different physical media allows the usefulness of our results to recognize various nonlinear phenomena and dynamical processes in these systems.

References

[1] Mahnke C and Mitschke F 2012 *Phys. Rev. A* **85** 033808
[2] Wang L H, Porsezian K, and He J S 2013 *Phys. Rev. E* **87** 053202
[3] Mežnaršič T, Arh T, Brence J, Pišljar J, Gosar K, Gosar V Z, Žitko R, Zupanič E, and Jeglič P 2019 *Phys. Rev. A* **99** 033625
[4] Marchukov O V, Malomed B A, Dunjko V, Ruhl J, Olshanii M, Hulet R G, and Yurovsky V A 2020 *Phys. Rev. Lett.* **125** 050405
[5] Johansson M, Sukhorukov A A, and Kivshar Y S 2009 *Phys. Rev. E* **80** 046604
[6] Kuznetsov E A, Rubenchik A M, and Zakharov V E 1986 *Phys. Rep.* **142** 103
[7] Lamb K G, Polukhina O, Talipova T, Pelinovsky E, Xiao W, and Kurkin A 2007 *Phys. Rev. E* **75** 046306
[8] Chabchoub A, Hoffmann N, Onorato M, and Akhmediev N 2012 *Phys. Rev. X* **2** 011015
[9] Chowdury A and Krolikowski W 2017 *Phys. Rev. E* **95** 062226
[10] Akhmediev N N and Korneev V I 1986 *Theor. Math. Phys.* **69** 1089
[11] Ma Y C 1979 *Stud. Appl. Math.* **60** 43
[12] Akhmediev N, Soto-Crespo J M, and Ankiewicz A 2009 *Phys. Lett. A* **373** 2137
[13] Zayed E M E, Shohib R M A, Alngar M E M, Biswas A, Ekici M, Khan S, Alzahrani A K, and Belic M R 2021 *Ukr. J. Phys. Opt.* **22** 38
[14] Adem A R, Ntsime B P, Biswas A, Khan S, Alzahrani A K, and Belic M R 2021 *Ukr. J. Phys. Opt.* **22** 83
[15] Biswas A, Edoki J, Guggilla P, Khan S, Alzahrani A K, and Belic M R 2021 *Ukr. J. Phys. Opt.* **22** 123
[16] Pelinovsky D and Grimshaw R 1997 *Phys. Lett. A* **229** 165
[17] Ziegler V, Dinkel J, Setzer C, Lonngren K E, and C 2001 *Solitons Fractals* **12** 1719
[18] Grimshaw R, Pelinovsky E, and Talipova T 1997 *Nonlinear Processes Geophys.* **4** 237
[19] Ono H 1992 *J. Phys. Soc. Jpn.* **61** 4336
[20] Ralph E A and Pratt L 1994 *J. Nonlinear Sci.* **4** 355
[21] Komatsu T S and Sasa S I 1995 *Phys. Rev. E* **52** 5574
[22] Ge H X, Dai S Q, Xue Y, and Dong L Y 2005 *Phys. Rev. E* **71** 066119
[23] Li Z P and Liu Y C 2006 *Eur. Phys. J. B* **53** 367
[24] Khater A H, El-Kalaawy O H, and Callebaut D K 1998 *Phys. Scr.* **58** 545
[25] Lonngren K E 1998 *Opt. Quantum Electron.* **30** 615
[26] Leblond H and Mihalache D 2009 *Phys. Rev. A* **79** 063835
[27] Leblond H and Mihalache D 2010 *J. Phys. A* **43** 375205
[28] Triki H, Leblond H, and Mihalache D 2012 *Opt. Commun.* **285** 3179
[29] Leblond H, Triki H, and Mihalache D 2013 *Rom. Rep. Phys.* **65** 925
[30] Leblond H, Grelu P, and Mihalache D 2014 *Phys. Rev. A* **90** 053816
[31] Wadati M 1972 *J. Phys. Soc. Jpn.* **32** 1681
[32] Kevrekidis P G, Khare A, Saxena A, and Herring G 2004 *J. Phys. A: Math. Gen.* **37** 10959
[33] Lamb J L 1980 *Elements of Soliton Theory* (New York: John Wiley & Sons)
[34] Grimshaw R, Pelinovsky E, Talipova T, Ruderman M, and Erdélyi R 2005 *Stud. Appl. Math.* **114** 189
[35] Ankiewicz A, Soto-Crespo J M, and Akhmediev N 2010 *Phys. Rev. E* **81** 046602
[36] Zhang G and Yan Z 2020 *Phys. D* **410** 132521
[37] Geng K L, Zhu B W, Cao Q H, Dai C Q, and Wang Y Y 2023 *Nonlinear Dyn.* **111** 16483
[38] He J T, Fang P P, and Lin J 2022 *Chin. Phys. Lett.* **39** 020301
[39] Wen X K, Jiang J H, Liu W, and Dai C Q 2023 *Nonlinear Dyn.* **111** 13343
[40] Miura R M, Gardner C S, and Kruskal M D 1968 *J. Math. Phys.* **9** 1204
[41] Pego R L and Weinstein M I 1992 *Philos. Trans. R. Sot. A* **340** 47
[42] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer-Verlag)
[43] Slyunyaev A V 2001 *J. Exp. Theor. Phys.* **92** 529
[44] Slunyaev A V and Pelinovsky E N 2016 *Phys. Rev. Lett.* **117** 214501
[45] Zakharov V E and Shabat A B 1972 *J. Exp. Theor. Phys.* **34** 62
[46] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[47] Ma Y L and Li B Q 2022 *Eur. Phys. J. Plus* **137** 861
[48] Alejo M A and Muñoz C 2013 *Commun. Math. Phys.* **324** 233