Dark Korteweg–De Vrise System and Its Higher-Dimensional Deformations

Si-Yu Zhu(祝思妤)[1], De-Xing Kong(孔德兴)[1], and Sen-Yue Lou(楼森岳)[2*]

[1]*Zhejiang Qiushi Institute for Mathematical Medicine, Hangzhou 311121, China*
[2]*School of Physical Science and Technology, Ningbo University, Ningbo 315211, China*

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The new dimensional deformation approach is proposed to generate higher-dimensional analogues of integrable systems. An arbitrary (K+1)-dimensional integrable Korteweg–de Vries (KdV) system, as an example, exhibiting symmetry, is illustrated to arise from a reconstructed deformation procedure, starting with a general symmetry integrable (1+1)-dimensional dark KdV system and its conservation laws. Physically, the dark equation systems may be related to dark matter physics. To describe nonlinear physics, both linear and nonlinear dispersions should be considered. In the original lower-dimensional integrable systems, only liner or nonlinear dispersion is included. The deformation algorithm naturally makes the model also include the linear dispersion and nonlinear dispersion.

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There is now a growing body of evidence that the evolution of the Universe may be dominated by dark energy.[1] The dark energy could be of relevance in an unknown way for the emergence of life in our Universe. In fact, some of the recent estimates the emergence of life in the Universe to be correlated with the onset of the dominance of dark energy.[2] Moreover, the motion of stars can not be accounted for unless it is assumed that most of the mass in the Universe is carried on by a dark matter.[3] The similar concept of dark equations is first proposed by Kupershmidt.[4] In Ref. [4], the general classification on the (1+1)-dimensional two-component integrable dark Korteweg–de Vrise (KdV) systems has been given. The authors of Ref. [5] have obtained the full classification on the (1+1)-dimensional two-component integrable dark modified KdV equations. If these dark issues are assumed such that they do exist, one can systematically discover them and study their properties, some of which turn out to be strange and others mysterious.[4]

The dark equations can also be considered as integrable couplings.[5–7] Integrable couplings with different dynamical systems turn up over and over again in the study of nonlinear phenomena, such as the Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem,[8,9] Kaup–Newell (KN) spectral problem,[10,11] Wadati–Konno–Ichikawa (WKI) spectral problem,[12] and Kadomtsev–Petviashvili(KP) equation.[13] In addition, they are associated with some very fantastic mathematics.

On the other hand, to find some integrable systems is always an important and difficult topic in theoretical physics. Back to papers,[14–18] it has been shown that the deformation algorithm is naturally fitted for constructing certain classes of higher-dimensional integrable systems that contain functions of several variables from lower-dimensional integrable ones.

Deformation Algorithm. For a general (1+1)-dimensional integrable local evolution system, we have

$$u_t = F(u, u_x, \ldots, u_{xn}), \quad u_{xn} = \partial_x^n u, \\ u = (u_1, u_2, \ldots, u_m), \tag{1}$$

if there exist some conservation laws,

$$\rho_{it} = J_{ix}, \quad i = 1, 2, \ldots, D-1, \quad \rho_i = \rho_i(u), \\ J_i = J_i(u, u_x, \ldots, u_{\mathrm{xN}}), \tag{2}$$

where the conserved densities ρ_i are dependent only on the field u while the flows J_i can be field derivative dependent, then the deformed (D+1)-dimensional system

$$\hat{T}u = F(u, \hat{L}u, \ldots, \hat{L}^n u) \tag{3}$$

is integrable with the deformation operators

$$\hat{L} \equiv \partial_x + \sum_{i=1}^{D-1} \rho_i \partial_{x_i}, \quad \hat{T} \equiv \partial_t + \sum_{i=1}^{D-1} \bar{J}_i \partial_{x_i}, \tag{4}$$

and the deformed flows

$$\bar{J}_i = J_i\Big|_{u_{x_j} \to \hat{L}^j u,\ j=1,2,\ldots,N}. \tag{5}$$

In this Letter, in order to find more high-dimensional integrable systems, we reconstruct the conservation laws (2) in the deformation algorithm to the new form

$$\mathcal{P}_{kt} = \mathcal{J}_{kx}, \quad k = 1, 2, \ldots, K-1 \tag{6}$$

for arbitrary positive integer K with

$$\mathcal{P}_k = \sum_{i=1}^{D-1} c_{ki}\rho_i, \ \mathcal{J}_k = \sum_{i=1}^{D-1} c_{ki} J_i, \ k = 1, 2, \ldots, K-1, \tag{7}$$

where c_{ki}, $i = 1, 2, \ldots, D-1$, $k = 1, 2, \ldots, K-1$, are arbitrary constants.

*Corresponding author. Email: lousenyue@nbu.edu.cn
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The basis for reconstructing Eq. (6) is the operators ∂_x and ∂_t that are linear operators. Thus, any linear combinations of conserved densities is still a conserved density. In fact, one can readily check that Eq. (6) is correct by substituting Eqs. (7) and (2) into Eq. (6).

Applying the new form (6) of the conservation laws in the deformation algorithm, we can obtain a $(K+1)$-dimensional integrable system (3) where the deformed operators $\hat{L}$ and $\hat{T}$ should be replaced by

$$\hat{\mathcal{L}} \equiv \partial_x + \sum_{k=1}^{K-1} \mathcal{P}_k \partial_{x_k}, \quad \hat{\mathcal{T}} \equiv \partial_t + \sum_{k=1}^{K-1} \bar{\mathcal{J}}_k \partial_{x_k} \tag{8}$$

with the new deformed flows

$$\bar{\mathcal{J}}_k = \mathcal{J}_k\Big|_{u_{x_j} \to \hat{L}^j u,\ j=1,2,\ldots,N}. \tag{9}$$

The correctness of the deformation algorithm has been strictly proved by Casati and Zhang at Ningbo University.[19] It is found that the new high-dimensional integrable systems obtained from the deformation algorithm possess completely different structures and properties from the traditional integrable systems.

In this study, we try to propose a novel generalized (1+1)-dimensional multi-component dark KdV system and then extend it to higher dimensions via the deformation algorithm.

One can prove that the following (1+1)-dimensional multi-component dark KdV system

$$\begin{cases} u_t = (u_{xx} + 3u^2)_x, \\ v_t = (v_{xx} + 6uv)_x, \\ w_t = (4w_{xx} + 6uw)_x - 3u_x w, \\ p_t = (p_{xx} + 3up)_x, \\ q_t = (-2q_{xx} - 6uq)_x, \end{cases} \quad \text{or} \quad U_t \equiv \begin{pmatrix} u \\ v \\ w \\ p \\ q \end{pmatrix}_t = \begin{pmatrix} (u_{xx} + 3u^2)_x \\ (v_{xx} + 6uv)_x \\ (4w_{xx} + 6uw)_x - 3u_x w \\ (p_{xx} + 3up)_x \\ (-2q_{xx} - 6uq)_x \end{pmatrix} \tag{10}$$

is integrable because of the existence of the higher-order generalized symmetry U_τ with

$$U_\tau \equiv \begin{pmatrix} u \\ v \\ w \\ p \\ q \end{pmatrix}_\tau = \begin{pmatrix} (u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3)_x \\ (v_{xxxx} + 10vu_{xx} + 10uv_{xx} + 10u_x v_x + 30u^2 v)_x \\ (16w_{xxxx} + 40uw_{xx} + 20u_x w_x + 30u_{xx} w + 30u^2 w)_x - 15w(u^2 + u_{xx})_x \\ (p_{xxxx} + 5up_{xx} + 5u_x p_x + 5p(2u^2 + u_{xx}))_x \\ (-4q_{xxxx} - 20(uq_x)_x - 10q(u^2 + u_{xx}))_x \end{pmatrix}. \tag{11}$$

In other words, the compatibility condition

$$U_{t\tau} = U_{\tau t} \tag{12}$$

is satisfied identically.

It is obvious that four known (1+1)-dimensional two-component dark KdV systems obtained in Ref. [4], i.e.,

$$\begin{aligned} u_t &= (u_{xx} + 3u^2)_x, \\ v_t &= (v_{xx} + 6uv)_x, \end{aligned} \tag{13}$$

$$\begin{aligned} u_t &= (u_{xx} + 3u^2)_x, \\ w_t &= (4w_{xx} + 6uw)_x - 3u_x w, \end{aligned} \tag{14}$$

$$\begin{aligned} u_t &= (u_{xx} + 3u^2)_x, \\ p_t &= (p_{xx} + 3up)_x, \end{aligned} \tag{15}$$

and

$$\begin{aligned} u_t &= (u_{xx} + 3u^2)_x, \\ q_t &= (-2q_{xx} - 6uq)_x, \end{aligned} \tag{16}$$

are just the special cases of Eq. (10).

The dark KdV system (13) can be obtained from some different ways, for instance, from the bosonization procedure[20] of the supersymmetric boson KdV equation

$$V_t = (V_{xx} + 3V^2)_x, \quad V = u + \theta\xi, \quad \xi = \eta v, \tag{17}$$

where $V = V(x,t,\theta)$ is a supersymmetric boson field, θ is the Grassmann variable in super-space $\{x,t,\theta\}$, $\xi = \xi(x,t)$ is the fermion partner field of the boson field $u = u(x,t)$, η is a Grassmann constant which is used to the bosonization procedure such that the fermion field ξ can be expressed with the help of the boson field $v = v(x,t)$.

The dark KdV system (14) can be obtained from the bosonization procedure, $\xi = \eta w$, for the super-integrable KdV equation[21]

$$\begin{aligned} u_t &= (u_{xx} + 3u^2 + 3\xi\xi_x)_x, \\ \xi_t &= (4\xi_{xx} + 6u\xi)_x - 3u_x\xi, \end{aligned} \tag{18}$$

where $\xi = \xi(x,t)$ is a fermion field, u is a boson field, η is a Grassmann constant which is used to the bosonization procedure such that the fermion field ξ can be expressed in terms of the boson field $w = w(x,t)$.

The dark KdV system (15) can also be obtained from the bosonization procedure,[22] $\xi = \eta p$, for the supersymmetric KdV equation[23] in the component form

$$u_t = (u_{xx} + 3u^2 + 3\xi\xi_x)_x, \quad \xi_t = (\xi_{xx} + 3u\xi)_x, \tag{19}$$

where $\xi = \xi(x,t)$ is a fermion field, u is a boson field, η is a Grassmann constant which is used to the bosonization procedure such that the fermion field ξ can be expressed in terms of the boson field $p = p(x,t)$.

The fourth dark KdV system (16) is called a secrete KdV equation[21] which has not yet found from fermion related systems.

From the dark KdV system (10), it is not difficult to find six conservation laws

$$\rho_{it} = J_{ix}, \quad i = 1,2,\ldots,6 \tag{20}$$

with the field derivative independent conserved densities

$$\rho_1 = u,\ \rho_2 = u^2,\ \rho_3 = v,\ \rho_4 = w^2,\ \rho_5 = p,\ \rho_6 = q, \quad (21)$$

and the field derivative dependent conserved flows

$$\begin{aligned} &J_1 = u_{xx} + 3u^2,\ J_2 = 2uu_{xx} - u_x^2 + 4u^3,\ J_3 = v_{xx} + 6uv,\\ &J_4 = 2ww_{xx} - w_x^2 + 6uw^2,\ J_5 = p_{xx} + 3up,\\ &J_6 = -2q_{xx} - 6uq. \end{aligned} \quad (22)$$

Reconstructing the conservation laws (20) with Eqs. (21) and (22), we have

$$\mathcal{P}_{kt} = \mathcal{J}_{kx}, \quad k = 1, 2, \ldots, K-1 \quad (23)$$

with

$$\mathcal{P}_k = \sum_{i=1}^{6} c_{ki}\rho_i,\ \mathcal{J}_k = \sum_{i=1}^{6} c_{ki}J_i,\ k = 1, 2, \ldots, K-1, \quad (24)$$

where c_{ki}, $i = 1, 2, \ldots, 6$, $k = 1, 2, \ldots, K-1$, are arbitrary constants, $\{\rho_i, J_i\}$, $i = 1, 2, \ldots, 6$, are expressed in Eqs. (21) and (22).

Applying the conservation laws (23) in the deformation algorithm for the dark KdV system (10), we obtain a $(K+1)$-dimensional integrable KdV system

$$\hat{\mathcal{T}}U \equiv \hat{\mathcal{T}}\begin{pmatrix} u\\ v\\ w\\ p\\ q \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{L}}(\hat{\mathcal{L}}^2 u + 3u^2)\\ \hat{\mathcal{L}}(\hat{\mathcal{L}}^2 v 6uv)\\ \hat{\mathcal{L}}(4\hat{\mathcal{L}}^2 w + 6uw) - 3w\hat{\mathcal{L}}u\\ \hat{\mathcal{L}}(\hat{\mathcal{L}}^2 p + 3up)\\ \hat{\mathcal{L}}(-2\hat{\mathcal{L}}q - 6uq) \end{pmatrix}, \quad (25)$$

where

$$\hat{\mathcal{L}} \equiv \partial_x + \sum_{k=1}^{K-1} \mathcal{P}_k\partial_{x_k}, \quad \hat{\mathcal{T}} \equiv \partial_t + \sum_{k=1}^{K-1} \bar{\mathcal{J}}_k\partial_{x_k} \quad (26)$$

with $\mathcal{P}, \mathcal{J}$ given by Eq. (24) and the deformed flow defined by

$$\bar{\mathcal{J}}_k = \mathcal{J}_k\Big|_{u_{x_j}\to\hat{L}^j u, j=1,2,\ldots,N}. \quad (27)$$

To check the integrability of the $(K+1)$-dimensional KdV Eq. (25), we have to fix the definition of the integrability under some special meanings. Here, the integrability is concerned in the sense of symmetry integrability. The investigation of the symmetry integrability of Eq. (25) can be carried out by acting the deformation algorithm on the higher order symmetry flow Eq. (11).

It is straightforward to prove that the fifth order symmetry flow Eq. (11) possesses the same conserved densities (21) as those of the dark KdV Eq. (10) but with the different conserved flows as follows:

$$\begin{aligned} G_1 &= u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3,\\ G_2 &= 2uu_{xxxx} - 2u_xu_{xxx} + u_{xx}^2 + 20u^2u_{xx} + 15u^4,\\ G_3 &= v_{xxxx} + 10vu_{xx} + 10uv_{xx} + 10u_xv_x + 30u^2v,\\ G_4 &= 32(ww_{xxxx} - w_xw_{xxx}) + 16w_{xx}^2 + 80wuw_{xx}\\ &\quad + 40w_x(wu_x - uw_x) + 30w^2(u_{xx} + u^2),\\ G_5 &= p_{xxxx} + 5up_{xx} + 5u_xp_x + 5p(2u^2 + u_{xx}),\\ G_6 &= -4q_{xxxx} - 20(uq_x)_x - 10q(u^2 + u_{xx}). \end{aligned} \quad (28)$$

Thus, we can introduce the deformed operator $\hat{\tau}$,

$$\hat{\tau} \equiv \partial_\tau + \sum_{k=1}^{K-1} \bar{\mathcal{G}}_k\partial_{x_k}, \quad (29)$$

where $\mathcal{G}_k$ is defined as

$$\mathcal{G}_k = \sum_{i=1}^{6} c_{ki}G_i, \quad k = 1, 2, \ldots, K-1, \quad (30)$$

and $\bar{\mathcal{G}}_x$ are the deformed forms of $\mathcal{G}_k$,

$$\bar{\mathcal{G}}_k = \mathcal{G}_k\Big|_{u_{x_j}\to\hat{L}^j u, j=1,2,\ldots,N}. \quad (31)$$

Repeating the foregoing deformed operators $\hat{\tau}$ and $\hat{\mathcal{L}}$ to the fifth-order dark KdV system (11), we can reach

$$\hat{\tau}U \equiv \hat{\tau}\begin{pmatrix} u\\ v\\ w\\ p\\ q \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{L}}(\hat{\mathcal{L}}^4 u + 10u\hat{\mathcal{L}}^2 u + 5(\hat{\mathcal{L}}u)^2 + 10u^3)\\ \hat{\mathcal{L}}(\hat{\mathcal{L}}^4 v + 10v\hat{\mathcal{L}}^2 u + 10u\hat{\mathcal{L}}^2 v + 10(\hat{\mathcal{L}}u)\hat{\mathcal{L}}v + 30u^2 v)\\ \hat{\mathcal{L}}(16\hat{\mathcal{L}}^4 w + 40u\hat{\mathcal{L}}^2 w + 20(\hat{\mathcal{L}}u)\hat{\mathcal{L}}^4 w + 30w\hat{\mathcal{L}}^2 u + 30u^2 w) - 15w\hat{\mathcal{L}}(u^2 + \hat{\mathcal{L}}^2 u)\\ \hat{\mathcal{L}}(\hat{\mathcal{L}}^4 p + 5u\hat{\mathcal{L}}^2 p + 5(\hat{\mathcal{L}}u)\hat{\mathcal{L}}p + 5p(2u^2 + \hat{\mathcal{L}}^2 u))\\ \hat{\mathcal{L}}(-4\hat{\mathcal{L}}^4 q - 20\hat{\mathcal{L}}(u\hat{\mathcal{L}}q) - 10q(u^2 + \hat{\mathcal{L}}^2 u)) \end{pmatrix}. \quad (32)$$

After finishing some tedious calculations for the $(K+1)$-dimensional KdV system (25) and the $(K+1)$-dimensional fifth order KdV system (32), one can check that the compatibility condition

$$U_{t\tau} = U_{\tau t}$$

is identically satisfied, which means that Eq. (25) is symmetry integrable.

Because of the complexity, we do not discuss the details of the general $(K+1)$-dimensional KdV system (25). A simple (2+1)-dimensional form of Eq. (25) for $K = 2$ is of consideration and

$$P \equiv \mathcal{P}_1 = \sum_{i=1}^{6} c_i\rho_i = c_1u + c_2u^2 + c_3v + c_4w^2 + c_5p + c_6q. \quad (33)$$

In this special case, the deformation operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{T}}$ are simplified to

$$\begin{aligned} &\hat{\mathcal{L}} = \partial_x + P\partial_y,\\ &P = c_1u + c_2u^2 + c_3v + c_4w^2 + c_5p + c_6q, \end{aligned} \quad (34)$$

$$\begin{aligned} &\hat{\mathcal{T}} = \partial_x + J\partial_y,\\ &J \equiv \mathcal{J}_1 = c_1\bar{J}_1 + c_2\bar{J}_2 + c_3\bar{J}_3 + c_4\bar{J}_4 + c_5\bar{J}_5 + c_6\bar{J}_6, \end{aligned} \quad (35)$$

with

$$\begin{aligned}
&\bar{J}_1 = (\partial_x + P\partial_y)^2 u + 3u^2,\\
&\bar{J}_2 = 2u(\partial_x + P\partial_y)^2 u - (u_x + Pu_y)^2 + 4u^3,\\
&\bar{J}_3 = (\partial_x + P\partial_y)^2 v + 6uv,\\
&\bar{J}_4 = 2w(\partial_x + P\partial_y)^2 w - (w_x + Pw_y)^2 + 6uw^2,\\
&\bar{J}_5 = (\partial_x + P\partial_y)^2 p + 3up,\\
&\bar{J}_6 = -2(\partial_x + P\partial_y)^2 q - 6uq. \qquad (36)
\end{aligned}$$

By using the simplification (33)–(36), the deformed KdV system becomes

$$\begin{aligned}
u_t =& u_y R + 3Q(Pu_{yy} + u_{xy}) + u_{yyy}P^3\\
&+ 3P(u_{xyy}P + u_{xxy} + 2uu_y) + (u_{xx} + 3u^2)_x,\\
v_t =& v_y R + 3Q(Pv_{yy} + v_{xy}) + v_{yyy}P^3\\
&+ 3P[Pv_{xyy} + (v_{xx} + 2vu)_y] + (u_{xx} + 6uv)_x,\\
w_t =& w_y R + 3Q(Pw_{yy} + w_{xy}) + w_{yyy}P^3\\
&+ 3P[w_{xyy}P + (w_{xx} + uw)_y + uw_y] + w_{xxx}\\
&+ 3u_x w + 6uw_x,\\
p_t =& p_y R + 3Q(Pp_{yy} + p_{xy}) + p_{yyy}P^3\\
&+ 3P[p_{xyy}P + (p_{xx} + pu)_y] + (p_{xx} + 3up)_x,\\
q_t =& -(2R + 3J)q_y - 6Q(Pq_{yy} + q_{xy}) - 2q_{yyy}P^3\\
&- 6P[Pq_{xyy} + (q_{xx} + qu)_y] - 2(q_{xx} + 3uq)_x, \qquad (37)
\end{aligned}$$

where

$$\begin{aligned}
&Q \equiv P_x + PP_y,\\
&R \equiv P_{yy}P^2 + P_y^2 P + 2P_{xy}P + P_x P_y + P_{xx} - J, \qquad (38)
\end{aligned}$$

and P and J are fixed by Eqs. (34) and (35) with Eq. (36).

Furthermore, if we further restrict in a two-component case for $P = u + v$, i.e., $c_1 = c_3 = 1, c_2 = c_4 = c_5 = c_6 = w = p = q = 0$ in Eq. (34), then Eq. (37) is reduced to

$$\begin{aligned}
P_t =& \Big[P_{xx} - 3u(u - 2P) + 3PP_{xy} + \frac{3}{2}P^2 P_{yy}\Big]_x\\
&+ \Big[P^3 P_{yy} - 3Pu(u - 2P) + \frac{3}{2}P^2 P_{xy}\Big]_y\\
&+ 6u(u - 2P)P_y,\\
u_t =& \Big(u_{xx} + 3u^2 + 3Pu_{xy} + \frac{3}{2}P^2 u_{yy}\Big)_x\\
&+ \Big(P^3 u_{yy} + u^3 + \frac{3}{2}P^2 u_{xy}\Big)_y. \qquad (39)
\end{aligned}$$

From the (2+1)-dimensional KdV system (39), it is easy to find that both the linear dispersion effect (related to the terms u_{xxx} and P_{xxx}) and nonlinear effects (related to the terms $P^3 u_{yyy}$, $P^2 u_{yyx}$, Pu_{xxy}, PP_{xxy}, $P^2 P_{xyy}$, and $P^3 P_{yyy}$) are included. However, in the usual KdV equation and the dark KdV systems like Eq. (10) there exist only linear dispersions.

Though the models proposed in this study are integrable, to find some exact solutions is still very difficult. Here, we just write down a special exact traveling wave solutions of Eq. (39) in the form

$$u = P = U(X), \quad X = kx + \kappa y + \omega t + X_0 \qquad (40)$$

with arbitrary constants k, κ, ω, and X_0.

After substituting Eq. (40) into Eq. (39) and integrating three times, we have

$$\begin{aligned}
&\int^{U(X)} \kappa^{3/2} z_1 \cdot \Big[2k^3 + k\kappa\omega - c_0\kappa^2 - 2(\kappa\omega + 3k^2)z_1\\
&\qquad + (2k + c_1\kappa^3)z_1^2 - 2z_1^3\Big]^{-1/2} dz\\
&\quad = \pm X + c_2, \qquad (41)\\
&z_1 \equiv \kappa z + k,
\end{aligned}$$

with three arbitrary integration constants c_0, c_1, and c_2.

If the constants c_0 and c_1 are fixed as

$$c_0 = 0, \quad c_1 = k^{-1}\kappa^{-3}(4k^2 + \kappa\omega),$$

then the travelling wave solution becomes the soliton solutions in the implicit form

$$U(X) = \frac{\omega}{2k}\mathrm{sech}^2\Big[\frac{\sqrt{k\omega}}{2k^3}\big[kX - \kappa\sqrt{k\omega - 2k^2 U(X)}\big]\Big]. \qquad (42)$$

It is clear and reasonable that the soliton solution (42) will reduce to the usual KdV soliton with the vanishing deformation parameter κ.

It should be emphasized that the deformed (2+1)-dimensional KdV system (37) and the (K+1)-dimensional KdV system (25) are coupled for all fields u, v, w, p and q though the field u is decoupled to all other fields in the original dark KdV system (10). Such a system (37), however, exists in two important special cases. To this point, we have assumed that two variables x and y have an equal status. Suppose now that all fields are y-independent, the model (37) reduces to the original dark KdV system (10).

In addition, the (2+1)-dimensional model (37) turns into a special coupled reciprocal link of Eq. (10) where one of the variables x is singled out by the condition $\partial_x\{u, v, w, p, q\} = 0$,

$$\begin{aligned}
u_t =& u_y(P_{yy}P^2 + P_y^2 P - J) + 3P^2 P_y u_{yy}\\
&+ u_{yyy}P^3 + 6Puu_y,\\
v_t =& v_y(P_{yy}P^2 + P_y^2 P - J) + 3P^2 P_y v_{yy}\\
&+ v_{yyy}P^3 + 6P(vu)_y,\\
w_t =& w_y(P_{yy}P^2 + P_y^2 P - J) + 3P^2 P_y w_{yy} + w_{yyy}P^3\\
&+ 3P(wu_y + 2uw_y),\\
p_t =& p_y(P_{yy}P^2 + P_y^2 P - J) + 3P^2 P_y p_{yy}\\
&+ p_{yyy}P^3 + 3P(pu)_y,\\
q_t =& -q_y(2P_{yy}P^2 + 2P_y^2 P + J) - 6P^2 P_y q_{yy}\\
&- 2q_{yyy}P^3 - 6P(qu)_y. \qquad (43)
\end{aligned}$$

Here, P is still given by Eq. (33) while J is simplified to

$$\begin{aligned}
J =& c_1[(P\partial_y)^2 u + 3u^2] + c_2[2u(P\partial_y)^2 u - (Pu_y)^2 + 4u^3]\\
&+ c_3[(P\partial_y)^2 v + 6uv] + c_4[2w(P\partial_y)^2 w - (Pw_y)^2 + 6uw^2]\\
&+ c_5[(P\partial_y)^2 p + 3up] - c_6[2(P\partial_y)^2 q + 6uq]. \qquad (44)
\end{aligned}$$

To see some simpler cases, we consider the reciprocal links of the two-component dark KdV systems (13)–(16). By

fixing $c_2 = c_4 = c_5 = c_6 = w = p = q = 0$, $c_1 = c$, and $c_3 = 1$, the system (43) becomes

$$u_t = (P^3 u_{yy} + P^2 P_y u_y + cu^3)_y - \frac{1}{2}(P^2 u_y^2)_y,$$
$$P_t = (P^3 P_{yy} + \frac{1}{2}P^2 P_y^2 + 3cPu^2)_y - 6Pu_y(2cu - P), \quad (45)$$

where P is related to v by $v = P - cu$. The coupled system (45) is a reciprocal link of the decoupled dark KdV system (13).

A coupled reciprocal link of the dark KdV system (15) is achieved as follows:

$$u_t = (P^3 u_{yy})_y + Pu_y[3u + (PP_y)_y],$$
$$P_t = P(P^2 P_{yy} + PP_y^2)_y + 3P^2 u_y, \quad (46)$$

by saying $c_2 = c_3 = c_4 = c_6 = v = w = q = 0$, $c_1 = c$, and $c_5 = 1$, where p can be obtained by $p = P - cu$.

A simple coupled reciprocal link of the dark KdV system (16) takes the form

$$u_t = (P^3 u_{yy} - 3cu^3)_y + u_y P[(PP_y)_y + 12u],$$
$$P_t = [-2P^3 P_{yy} + 3cP^2 (Pu_y)_y - P^2 P_y^2 - 9cPu^2]_y$$
$$- 3P[cP_y(Pu_y)_y - 2u_y(6cu - P)], \quad (47)$$

after the assumption $c_2 = c_3 = c_4 = c_5 = v = w = p = 0$, $c_1 = c$, and $c_6 = 1$ with an appropriate choice $q = P - cu$.

We conclude this study by several remarks on the higher-dimensional integrable systems described above.

The dark systems such as the dark KdV equations may be related to dark matter physics.[21] In this work, the dark KdV systems have been deformed to higher dimensions. Some new types of higher-dimensional dark systems can be derived.[24] For instance, related to the (2+1)-dimensional KdV system (39) with $P = u$ we can derive the following dark equation system

$$v_t = \Big(v_{xx} + 6uv + 3vu_{xy} + 3uv_{xy} + 3uvu_{yy} + \frac{3}{2}u^2 v_{yy}\Big)_x$$
$$+ 3u\Big(uvu_{yy} + \frac{u^2}{3}v_{yy} + uv + vu_{xy} + \frac{u}{2}v_{xy}\Big)_y,$$
$$u_t = \Big(u_{xx} + 3u^2 + 3uu_{xy} + \frac{3}{2}u^2 u_{yy}\Big)_x$$
$$+ \Big(u^3 u_{yy} + u^3 + \frac{3}{2}u^2 u_{xy}\Big)_y. \quad (48)$$

Using the conservation laws of the dark system (48), say, the conservation laws related to the conserved densities u and v, one can find more higher-dimensional integrable systems by means of the extended deformation algorithm for higher-dimensional integrable systems.[16] The deformation procedure to find higher-dimensional integrable systems and dark systems can be iteratively forward.

In Ref. [14], a deformation algorithm based on the utilization of conservation laws was proposed to construct higher-dimensional integrable systems. A new step or a natural extension is made in this study. It is shown that some changes applied to deformation operators make it possible to build classes of higher-dimensional integrable systems in a regular way, which is particularly effective in constructing nonlinear coupled higher-dimensional integrable systems. The path towards the implementation of this deformation is outlined by the *deformation algorithm*. Some examples of such higher-dimensional integrable systems are indeed produced.

A principal feature of all these higher-dimensional systems is that the original and their several reciprocal systems are included in the same models. One may query whether the higher-dimensional systems have the Painlevé property, which can be useful as a test for integrability.[25] They may fail to satisfy the Painlevé property because of the reciprocal systems, but such systems remain exactly integrable in the sense of the existence of higher order generalized symmetries.

We want it to be clear that the above idea and algorithm are powerful and broad beyond anyone of the particular examples used, and could lead to a wealth of future possibilities and explorations, both in theory and experiments.

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