Extension of Linear Response Regime in Weak-Value Amplification Technique

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The achievable precision of parameter estimation plays a significant role in evaluating a strategy of metrology. In practice, one may employ approximations in a theoretical model development for simplicity, which, however, will cause systematic error and lead to a loss of precision. We derive the error of maximum likelihood estimation in the weak-value amplification technique where the linear approximation of the coupling parameter is used. We show that this error is positively related to the coupling strength and can be effectively suppressed by improving the Fisher information. Considering the roles played by weak values and initial meter states in the weak-value amplification, we also point out that the estimation error can be decreased by several orders of magnitude by averaging the estimations resulted from different initial meter states or weak values. These results are finally illustrated in a numerical example where an extended linear response regime to the parameter is observed.

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Measurements of tiny physical quantities, such as strength of magnetic field and temperature, play a crucial role in advancement of physics.[1,2] However, due to inevitable errors in real quantum information schemes, estimation precision is upper-bounded with limited quantum resources.[3,4] Generally, noise has two different detrimental effects. The random noise affects the estimation by fluctuations around the real value, which can be reduced by increasing the repeating number of experiments; while some systematic errors, such as theoretical approximations, may bias the parameter estimation, leading to a bad performance of precision metrology. This kind of estimation error is irrelevant to the number of experiments, thus one needs to develop novel approaches for the purpose of weakening its influence on precision measurements.

Among the precision measurement protocols, the weak-value amplification (WVA) proposed by Aharonov, Albert, and Vaidman (AAV)[5,6] has attracted wide attention due to its potential applications in detecting tiny effects.[7–9] With the help of a postselection process, the average location of meter wave packet can experience a shift rate that goes far beyond the eigenvalue spectrum of the system observable.[10,11] As a result, the small coupling parameter is amplified into a detectable region. To date, this amplifying effect has been employed in many scenarios, such as the Kerr phase[12] and time-delay estimation,[13] longitudinal phase shift,[14] high-precision temperature,[15,16] and velocity measurement.[17] We note that although WVA also suffers from noise, it still presents metrological advantages over the conventional measurement (CM) scheme if systematic imperfections exist, such as the detector saturation[18] and jitter.[19,20]

In practice, as the parameter to be estimated is small, the conventional analysis of WVA usually assumes a linear approximation with respect to the coupling strength in order to simplify the model and thus data processing.[11,21] This treatment works well in most cases, but when the parameter becomes larger or the initial and postselection states are chosen to be nearly orthogonal, the linear approximation is not accurate anymore, which may limit further applications in high-resolution experiments. One solution for this is to take some higher-order effects into account, then the analysis can be proved to be valid for both weak and strong measurement regimes, while the expense one has to pay is more complicated calculations and lower experimental efficiency.[22–25] In this Letter, we carry out the estimation-error evaluation of linear approximation in WVA. By employing the maximum likelihood estimation (MLE) strategy,[13,26] the bias between the estimated and true values is analytically derived. We find that the bias can be effectively suppressed if one prepares the meter in different states or chooses weak values with opposite signs. In addition, we prove the consistency of the bias expression between estimate strategies based on MLE and meter shift under a special case, which actually provides us a feasible experimental scheme to test the validity of our method. It should be mentioned that although our consideration is restricted to the bias induced by only the quadratic term, the numerical example indicates that our proposal still realizes the reduction of error by several orders of magnitude.

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Theory of the WVA Scheme. Consider a two-level system (TLS) and a meter which are initially prepared in states $|\psi_i\rangle$ and $|\phi_i\rangle$, respectively. In the standard WVA scheme, they are firstly entangled by the interaction Hamiltonian modeled by

$$H = gA \otimes \Omega\delta(t - t_0), \tag{1}$$

where the operators A and Ω represent the observables of TLS and meter; g is a parameter which characterizes the coupling strength. The δ function indicates that this interaction is instantaneous at time t_0. After this interaction, the combined system evolves to state

$$|\Psi\rangle = \exp(-igA \otimes \Omega)|\psi_i\rangle|\phi_i\rangle. \tag{2}$$

This expression can be transformed by expanding the exponential function with respect to the parameter g,

$$|\Psi\rangle = \sum_{k=0}^{\infty} \frac{1}{k!}(-i)^k g^k A^k \Omega^k |\psi_i\rangle|\phi_i\rangle. \tag{3}$$

If the TLS is post-selected to the quantum state $|\psi_f\rangle$ with successful probability $P = |\langle\psi_f|\Psi\rangle|^2$, the meter will collapse to a normalized state expressed as

$$|\phi_{\rm p}\rangle = \frac{|\phi\rangle}{\sqrt{r_{\rm p}(g)}} = \frac{1}{\sqrt{r_{\rm p}(g)}} \sum_{k=0}^{\infty} \frac{1}{k!}(-i)^k g^k (A^k)_{\rm w} \Omega^k |\phi_i\rangle, \tag{4}$$

where $(A^k)_{\rm w} = \langle\psi_f|A^k|\psi_i\rangle/\langle\psi_f|\psi_i\rangle$ is defined as the generalized weak value of operator A^k. For simplicity, we assume $A = \sigma_i (i = x, y, z)$ is a Pauli operator in this work, thus we have $(A^{2k})_{\rm w} = 1$ and $(A^{2k+1})_{\rm w} = A_{\rm w}$. Considering the purpose of the WVA-based precision metrology, we usually desire $A_{\rm w}$ to be much larger than the maximum eigenvalue of A, i.e., $A_{\rm w} \gg 1$. In Eq. (4), $r_{\rm p}(g) = P/|\langle\psi_f|\psi_i\rangle|^2 = \langle\phi|\phi\rangle$ that shows the relative post-selection probability compared to the case where g is extremely small is given by

$$r_{\rm p}(g) = \sum_{k,l=0}^{\infty} \frac{1}{k!l!} i^{l-k} g^{k+l} (A^k)_{\rm w} (A^l)^*_{\rm w} \langle\Omega^{k+l}\rangle, \tag{5}$$

where $\langle\Omega^{k+l}\rangle = \langle\phi_i|\Omega^{k+l}|\phi_i\rangle$, and for simplicity, we denote $\langle\phi_i| \cdot |\phi_i\rangle$ as $\langle\cdot\rangle$ for short throughout this study.

We note that Eq. (4) is derived without any approximation thus it brings no error when used for estimating the parameter g. However, this treatment may lead to limited practical application due to the much more complicated calculations introduced by higher orders of g. Therefore, in most studies that are relevant to WVA, one prefers to ignore the higher orders of g and consider only the linear term, i.e., $|\phi_{\rm p}\rangle \approx \exp(-igA_{\rm w}\Omega)|\phi_i\rangle$. This approximation simplifies the model efficiently and still provides an appropriate estimation of g with very high precision when g is extremely small. However, as g grows, the nonlinear terms start to play significant roles, and as a result, the estimation of g deviates from the true value (marked as g_0), yielding a biased estimation. We next investigate the properties of this estimation error based on the MLE method. After that, several proposals aiming to decrease this error will be provided.

Analysis on the Estimation Error. In MLE, the information of g is extracted from a g-dependent probability distribution. Suppose a meter observable M with eigenvalues $\{m\}$ and corresponding eigenvectors $\{|\varphi_m\rangle\}$, i.e., $M = \int m|\varphi_m\rangle\langle\varphi_m|dm$. If we measure the orthonormal basis $\{|\varphi_m\rangle\}$ on the post-selected meter state $|\phi_{\rm p}\rangle$, the probability distribution with respect to m is

$$\begin{aligned} p(m, g) &= |\langle\varphi_m|\phi_{\rm p}\rangle|^2 \\ &= \frac{1}{r_{\rm p}(g)} \sum_{k,l=0}^{\infty} \frac{1}{k!l!} i^{l-k} g^{k+l} (A^k)_{\rm w} (A^l)^*_{\rm w} \\ &\quad \cdot \langle\varphi_m|\Omega^k \rho_i \Omega^l|\varphi_m\rangle, \end{aligned} \tag{6}$$

with $\rho_i = |\phi_i\rangle\langle\phi_i|$. Generally, by summing all the terms that are relevant to g^n, $p(m, g)$ can be rewritten as

$$p(m, g) = \sum_{n=0}^{\infty} d_n(m) g^n, \tag{7}$$

of which the first few coefficients are listed as follows:

$$d_0(m) = \langle\varphi_m|\rho_i|\varphi_m\rangle = \rho_i(m), \tag{8}$$

$$d_1(m) = 2\rho_i(m)\big[{\rm Im}[A_{\rm w}\Omega_{\rm w}(m)] - \langle\Omega\rangle{\rm Im}[A_{\rm w}]\big], \tag{9}$$

$$\begin{aligned} d_2(m) \approx \rho_i(m)\big[&-4{\rm Re}[A_{\rm w}]{\rm Im}[A_{\rm w}]{\rm Im}[\Omega_{\rm w}(m)]\langle\Omega\rangle \\ &+ |A_{\rm w}|^2(|\Omega_{\rm w}(m)|^2 - \langle\Omega^2\rangle) + 4{\rm Im}^2[A_{\rm w}](\langle\Omega\rangle^2 \\ &- \langle\Omega\rangle{\rm Re}[\Omega_{\rm w}(m)])\big]. \end{aligned} \tag{10}$$

A notable point here is that the terms associated with $(A^2)_{\rm w}$ are ignored in Eq. (10) because we have assumed $(A^2)_{\rm w} = 1 \ll A_{\rm w}$ above. $\Omega_{\rm w}(m)$ is the weak value of operator Ω, defined as

$$\Omega_{\rm w}(m) = \frac{\langle\varphi_m|\Omega|\phi_i\rangle}{\langle\varphi_m|\phi_i\rangle} = \frac{\langle\varphi_m|\Omega\rho_i|\varphi_m\rangle}{\langle\varphi_m|\rho_i|\varphi_m\rangle}. \tag{11}$$

Suppose that we repeat the experiments and observe the result m in a total of C_m times. Then we can construct the log-likelihood function as $L(g) = \int dmC_m \ln p_{\rm t}(m, g)$ where $p_{\rm t}(m, g)$ is the target probability distribution we used to achieve an estimate of g. In the decoherence-free case, as the total number of experiments $C = \int dmC_m$ increases, C_m in principle converges to $Cp(m, g_0)$. Then the log-likelihood function changes to $L(g) = C\int dmp(m, g_0)\ln p(m, g)$, and the MLE of g can be achieved by solving the likelihood equation $\partial L(g)/\partial g = 0$, which yields

$$\int dmp(m, g_0) \frac{\partial_g p_{\rm t}(m, g)}{p_{\rm t}(m, g)} = 0. \tag{12}$$

The purpose of MLE is to find the most appropriate g conditioned on the target function $p_{\rm t}(m, g)$ and the experimental data. Therefore, if $p_{\rm t}(m, g) = p(m, g)$, it is obvious that $g = g_0$ is the solution to Eq. (12) since $\int dmp(m, g) = 1$, which leads to an unbiased MLE. In our case, due to the approximation about the nonlinear terms of g, i.e., $p_{\rm t}(m, g) = d_0(m) + d_1(m)g \neq p(m, g)$, the estimation result g will slightly deviate from g_0. If we write $g = g_0 + \delta g$, then our goal is to explore the properties of this small quantity δg.

As the practical probability distribution can be expressed as $p(m,g) = p_{\rm t}(m,g) + q(m,g)$ with $q(m,g) = \sum_{n=2}^{\infty} d_n(m)g^n$ and $\int dm q(m,g) = 0$, Eq. (12) becomes

$$\int dm(p_{\rm t}(m,g_0) + q(m,g_0))\frac{\partial_g p_{\rm t}(m,g)}{p_{\rm t}(m,g)} = 0. \tag{13}$$

If we expand $\partial_g p_{\rm t}(m,g)/p_{\rm t}(m,g)$ at $g = g_0$ and only keep the terms up to the first order of δg, then we obtain

$$\int dm\Big[\frac{q(m,g_0)}{p_{\rm t}(m,g_0)}\partial_g p_{\rm t}(m,g)|_{g=g_0} - \delta g\frac{[\partial_g p_{\rm t}(m,g)]^2|_{g=g_0}}{p_{\rm t}(m,g_0)}\Big] = 0. \tag{14}$$

Here, we have utilized the relation $\int dm[\partial_g p_{\rm t}(m,g)]|_{g=g_0} = \int dm[\partial_g^2 p_{\rm t}(m,g)]|_{g=g_0} = 0$ and ignored the terms that contain $q(m,g_0)\delta g$ because both δg and $q(m,g_0)$ are very small. Then we get the first-order estimation error [26]

$$\delta g = \frac{\int dm q(m,g_0)\partial_g \ln p_{\rm t}(m,g)|_{g=g_0}}{F(g_0)}, \tag{15}$$

where $F(g_0) = \int dm \frac{1}{p_{\rm t}(m,g_0)}[\partial_g p_{\rm t}(m,g)]^2|_{g=g_0}$ represents the Fisher information (FI) of distribution $p_{\rm t}(m,g)$ at $g = g_0$. We note that CFI is always positive and reveals the amount of information about g contained in $p_{\rm t}(m,g)$. Substituting $p_{\rm t}(m,g)$ and $q(m,g)$ into Eq. (15), we finally obtain one of the key results of this study as follows:

$$\begin{aligned}\delta g &= \int dm \frac{\sum_{n=2}^{\infty} d_1(m)d_n(m)g_0^n}{d_0(m)+d_1(m)g_0}\frac{1}{F(g_0)} \\ &\approx \int dm \frac{d_1(m)d_2(m)g_0^2}{d_0(m)}\frac{1}{F(g_0)},\end{aligned} \tag{16}$$

with $F(g_0) \approx \int dm d_1^2(m)/d_0(m)$. We emphasize that Eq. (16) is irrelevant to C or the post-selection probability of success, which means that this error cannot be suppressed by increasing the repeating number of experiment, as the random noise is usually treated. Unsurprisingly, this result is consistent with the Cramer–Rao bound,

$$\langle(\delta g)^2\rangle \geq \frac{1}{CPF(g_0)} + \langle\delta g\rangle^2, \tag{17}$$

which shows the minimum achievable standard deviation of the estimator g based on analyzing state $|\phi_{\rm p}\rangle$ when repeating C independent WVA operations. Fortunately, if we analyze Eq. (16) carefully, we may get some insight into the effect of weak value $A_{\rm w}$ on δg and see that the δg with $A_{\rm w}$ is actually the opposite of that with $-A_{\rm w}$. This implies that the estimation error can be operationally suppressed by dividing the experiments into two groups. One is carried out by setting the weak value as $A_{\rm w}$, and for the other it is $-A_{\rm w}$. Then by computing the average between two set of data, δg can be finally suppressed or even nearly eliminated, therefore the parameter region of linear response in WVA is significantly extended. Because the FI $F(g_0)$ keeps the same for both groups of experiments, as one can easily check, the total random noise still scales as C^{-1}.

A Special Case of $\Omega = M$. Generally, there is freedom on the choice of observable M for a specific purpose of experiments or the convenience of experimental setup. However, in the standard WVA scheme, the FI is proved to depend on the selection of M for a certain weak value $A_{\rm w}$ and will reach its maximum (quantum Fisher information, shortly QFI) only if a specific match between Ω and M is realized. Take $\Omega = p$ for example, if $A_{\rm w}$ is completely real, the classical FI equals QFI when $M = x$. However, if the weak value $A_{\rm w}$ is purely imaginary, the condition for achieving QFI is $M = p$. To simplify the model, we here restrict the observable M to be the same as Ω to see the properties of δg. In this case, FI has a form of

$$F(g_0) = 4{\rm Im}^2[A_{\rm w}]{\rm Var}_i(\Omega), \tag{18}$$

where ${\rm Var}_i(\Omega)$ is the variance of Ω in the state $|\phi_i\rangle$. Then Eq. (16) becomes

$$\delta g = g_0^2\frac{|A_{\rm w}|^2(\langle\Omega^3\rangle - \langle\Omega^2\rangle\langle\Omega\rangle) + 4{\rm Im}^2[A_{\rm w}](\langle\Omega\rangle^3 - \langle\Omega^2\rangle\langle\Omega\rangle)}{2{\rm Im}[A_{\rm w}]{\rm Var}_i(\Omega)}. \tag{19}$$

When C is fixed, the Cramer–Rao bound tells us that the random noise [the first term on the right side of Eq. (17)] is inversely proportional to the average FI $PF(g_0)$ in each run, which means that, in principle, we are able to reduce this kind of error effectively if the quantity $PF(g_0)$ is maximized. Considering the definition of $A_{\rm w}$ and the linear approximation of post-selection probability P, one will find that this goal can be achieved by setting $A_{\rm w}/i \in \mathbb{R}$, or expressed as $|A_{\rm w}|^2 = {\rm Im}^2[A_{\rm w}]$. Then Eq. (19) can be simplified to

$$\delta g = g_0^2\frac{{\rm Im}[A_{\rm w}](\langle\Omega^3\rangle - 5\langle\Omega^2\rangle\langle\Omega\rangle + 4\langle\Omega\rangle^3)}{2{\rm Var}_i(\Omega)}. \tag{20}$$

This core result suggests other two potential methods to suppress the estimation error. On the one hand, because δg is inversely proportional to ${\rm Var}_i(\Omega)$, one can achieve a better estimation if a meter state with higher variance ${\rm Var}_i(\Omega)$ is initially prepared. For example, the quantum squeezed state with an optimized squeeze angle usually behaves better than the coherent state. It should be pointed out that this method also leads to a reduction of the random error because the average FI in our case can be written as $4{\rm Var}_i(\Omega)$. On the other hand, Eq. (20) indicates that, if we can find two initial meter states of which the expect values $\langle\Omega\rangle$ are the opposites of each other, the estimation error can be offset based on the similar experimental procedure as the proposal about utilizing $\pm A_{\rm w}$. One may find that $\langle\Omega\rangle = 0$ can also lead to the elimination of estimate error, which actually refers to a special case of the method illustrated above.

In fact, the above results can be derived by calculating the shift of wave packet as well. Generally, the average position of $|\phi_{\rm p}\rangle$ in the measured value of M is given by $\langle\phi_{\rm p}|M|\phi_{\rm p}\rangle$, or expressed as

$$\langle M\rangle_{\rm p} = \int mp(m,g)dm. \tag{21}$$

Then the estimation error due to the linear approximation can be written as $\delta g = \frac{\langle M\rangle_{\rm p} - \langle M\rangle_i}{\int m d_1(m)dm} - g_0$. Substituting Eq. (7) into this expression yields

$$\delta g \approx \frac{\int m d_2(m)dm}{\int m d_1(m)dm}g_0^2. \tag{22}$$

Its expansion based on Eqs. (8)–(10) is given by

$$\delta g = g_0^2 \frac{-i2\mathrm{Re}[A_\mathrm{w}]\mathrm{Im}[A_\mathrm{w}]\langle\Omega\rangle\langle[\Omega,M]\rangle + |A_\mathrm{w}|^2(\langle\Omega M\Omega\rangle - \langle\Omega^2\rangle\langle M\rangle) + 2\mathrm{Im}^2[A_\mathrm{w}](2\langle\Omega\rangle^2\langle M\rangle - \langle\Omega\rangle\langle\{\Omega,M\}\rangle)}{i\mathrm{Re}[A_\mathrm{w}]\langle[\Omega,M]\rangle + \mathrm{Im}[A_\mathrm{w}](\langle\{\Omega,M\}\rangle - 2\langle\Omega\rangle\langle M\rangle)}. \quad (23)$$

When $\Omega = M$ and A_w is purely imaginary, the above result can be simplified to Eq. (20). This implies that, even for the widely used parameter estimation based on meter shift, the results derived in this study still work well, which exactly shows applicability of our method in precision metrology.

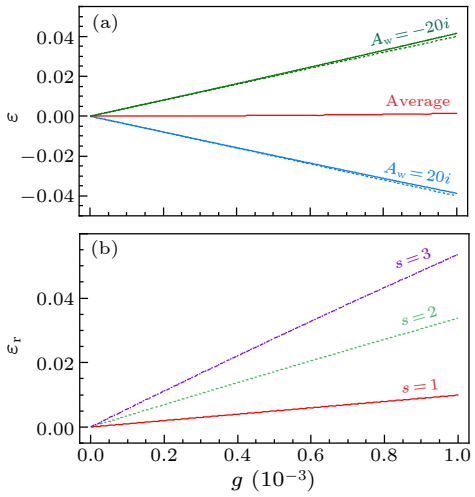

Fig. 1. (a) The relative estimation error as a function of g when $s = 2$. As the legend shows, the green and blue (dashed) lines represent the simulation results with $\pm A_\mathrm{w}$, and the average of them are given by the red line. It should be mentioned that the dashed lines are the prediction of Eq. (20) which ignores the terms $d_n(m)$ with $n \geq 3$, while the exact numerical solutions without approximation are shown by the solid lines for comparison. (b) The error reduction of our method for $s = 1$, 2, 3 and other parameters are set the same as in (a).

A Numerical Example. To illustrate the above results, we assume $\Omega = M = p$, the momentum operator defined by $p = i(a^\dagger - a)/\sqrt{2}$, and the meter is initially prepared in a coherent state $|se^{i\lambda}\rangle$ with $s, \lambda \in \mathbb{R}$. As shown in Fig. 1, we display a numerical result to compare the relative estimation error (REE) $\epsilon = \delta g/g_0$ between the cases with and without the $\pm A_\mathrm{w}$ related averaging procedures. From Fig. 1(a), we find that the REE increases rapidly as g grows for both $\pm A_\mathrm{w}$, and due to the approximation employed in Eq. (16), the expected REE (dashed lines) diverges slightly from the exact simulation results (blue and green solid curves) when g becomes larger, which results in the remained amount of error (red curve) even if our averaging method is used. However, if we consider the error reduction $\epsilon_r = (\epsilon(A_\mathrm{w}) + \epsilon(-A_\mathrm{w}))/(2\epsilon(A_\mathrm{w}))$, i.e., the ratio between REE after and before the averaging processing, we can see that our method still decreases the estimation error by several orders of magnitude [see Fig. 1(b)]. Therefore, we conclude that our method provides an efficient approach for reducing the estimation error in weak measurements. Note that in Fig. 1(b) the remained error seems to be larger when s or g grows. The reason is that, in our method, our consideration is restricted to the bias induced by only the quadratic term. Therefore, the remained error mainly results from the higher orders of g, i.e., g^n, $n = 3, 4, \ldots$. As g grows, these terms start to play crucial roles in the interaction process, which finally leads to an increment of the remained estimate error.

Figure 2 shows the optimization of linear response regime of parameter g in WVA technique. Inspired by the fact that the REEs derived from states $|\alpha\rangle$ and $|-\alpha\rangle$ have the ability to cancel each other to some extent, we here simulate the shift of meter wave packet for both the cases and then provide the averaged result. It is clear that this averaging treatment greatly reduces the difference between the linear term and processed data, which indicates the desired larger linear response regime of parameter estimation.

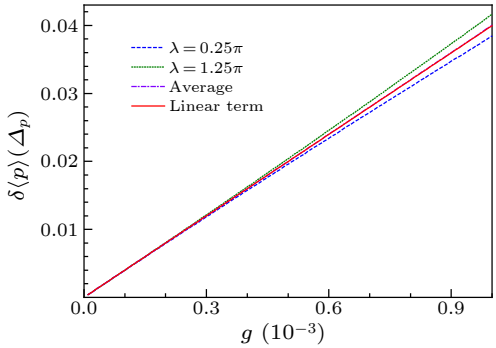

Fig. 2. The average shift of meter wave packet in p-space with unit $\Delta_\mathrm{p} = \mathrm{Var}_i(p)$ as a function of g. We note that the averaged result (purple dot-dashed curve) is almost coincided with the result derived from only the linear term (red solid line). Other parameters used in simulation are $s = 2$ and $A_\mathrm{w} = 20i$.

In summary, although the WVA technique has been widely employed in detecting tiny physical quantities, estimation error may occur when the theoretical model is inappropriately simplified. In the present study, we mainly consider the estimation error caused by the negligence of nonlinear terms of coupling parameter, just like the treatment in most studies. By introducing the MLE method, we find that this error is positively associated with the coupling strength, and most importantly, one can effectively suppress this by several orders of magnitude if the initial meter state or weak value is optimized. Aside from the estimation error mentioned above, an alternative factor that influences the precision of metrology is random error, which can be reduced by increasing the FI in estimation process and approaches 0 if $C \to \infty$. We point out that the FI in our optimization strategies can be increased or at least remain unchanged, therefore the overall precision of parameter estimation is guaranteed.

In addition, we note that, because no assumption about the operators used in Hamiltonian [Eq. (1)] is made during the theoretical model development, this method can be employed in many experiments associated with pre-

cision measurement. For example, the numerical simulation we illustrate above can be realized in the trapped ion system, where the Hamiltonian describes the coupling between spin and phonon states. A detailed description of experiments can be found in Ref. [11].

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