Higher Dimensional Camassa–Holm Equations

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Utilizing some conservation laws of the (1+1)-dimensional Camassa–Holm (CH) equation and/or its reciprocal forms, some (n+1)-dimensional CH equations for $n \geq 1$ are constructed by a modified deformation algorithm. The Lax integrability can be proven by applying the same deformation algorithm to the Lax pair of the (1+1)-dimensional CH equation. A novel type of peakon solution is implicitly given and expressed by the LambertW function.

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There are many effective methods to simplify a complex system to a simple one, such as limiting method taking some arbitrary parameters to some special limits, and symmetric method reducing the dimension or order of differential equations. It is much more difficult to explore the laws of complex systems from results of a simple system. Fortunately, in many cases, it is also possible to transform results of a simple system into a complex system. For instance, we can deform some particular solutions of the sine-Gordon equation (integrable only for $D = 1$)

$$\Box\phi = m_1 \sin\phi, \quad \Box \equiv \partial_t^2 - \sum_{i=1}^{D} \partial_{x_i}^2$$

to those of the nonintegrable double sine-Gordon equation [1,2]

$$\Box\phi = m_1 \sin\phi + m_2 \sin(2\phi).$$

For another example, we can use the Miura type transformations related to the Korteweg–de Vries (KdV) equation, the modified KdV equation and the Schwartz KdV equation, and we can get the (1+1)- and (2+1)-dimensional integrable sine-Gordon equations and Tzitzeica equations from the (0+1)-dimensional Riccati equation. [3]

Recently, in order to obtain more high-dimensional integrable systems in some traditional integrable meanings, we proposed a deformation algorithm, which enables arbitrary lower-dimensional integrable systems to be deformed to high-dimensional ones. [4]

Deformation Algorithm. For a general (1+1)-dimensional integrable local evolution system

$$\begin{aligned} &u_t = F(u, u_x, \ldots, u_{xn}), \quad u_{xn} = \partial_x^n u, \\ &u = (u_1, u_2, \ldots, u_m), \end{aligned} \tag{1}$$

if there exist some conservation laws

$$\begin{aligned} &\rho_{it} = J_{ix}, \quad i = 1, 2, \ldots, D-1, \quad \rho_i = \rho_i(u), \\ &J_i = J_i(u, u_x, \ldots, u_{xN}), \end{aligned} \tag{2}$$

where the conserved densities ρ_i are dependent only on the field u while the flows J_i can be field derivative dependent, then the deformed $(D+1)$-dimensional system

$$\hat{T}u = F(u, \hat{L}u, \ldots, \hat{L}^n u) \tag{3}$$

is integrable with the deformation operators

$$\hat{L} \equiv \partial_x + \sum_{i=1}^{D-1} \rho_i \partial_{x_i}, \ \hat{T} \equiv \partial_t + \sum_{i=1}^{D-1} \bar{J}_i \partial_{x_i} \tag{4}$$

and the deformed flows

$$\bar{J}_i = J_i|_{u_{x_j} \to \hat{L}^j u, j=1,2,\ldots,N}.$$

The correctness of the deformation algorithm (or the deformation conjecture) has been checked in Ref. [4] for almost all the known local integrable evolution systems. In fact, the conjecture has been proved by Dan-da Zhang and Casati at Ningbo University (private communication).

In theoretical physics, to find some completely integrable models themselves is physically significant even if they are not directly derived from physical systems. For instance, the nonlocal nonlinear Schrödinger equation proposed by Ablowitz and Musslimani in Ref. [5] was not derived from real physical problems but proposed only for integrability. Now, the study on nonlocal integrable systems has become a hot topic related to multiplace physics. [6] In Ref. [4], we specifically found higher-dimensional deformations from the (1+1)-dimensional KdV equation and the (1+1)-dimensional Ablowitz–Kaup–Newell–Segue (AKNS) system. It is found that the new high-dimensional integrable systems obtained from the deformation algorithm possess completely different structures and properties from the traditional integrable system. Although the new models possess elegant properties such as the existence of the Lax pair and the infinitely many symmetries, the traditional research methods of integrable systems can no longer be successfully applied because the original models and their several reciprocal systems are included in the same models.

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The Camassa–Holm (CH) equation

$$m_t + 2\,mu_x + um_x = 0, \quad m \equiv u - u_{xx}, \tag{5}$$

or equivalently in the conserved form

$$u_t = \left(u_{xt} + uu_{xx} + \frac{1}{2}u_x^2 - \frac{3}{2}u^2\right)_x, \tag{6}$$

was deduced using Hamiltonian methods in the study of shallow water regime.[7] As an integrable model, Eq. (6) appeared earlier in a list by Fuchssteiner and Fokas.[8] The CH equation also arises in the investigation of propagation of axially symmetric waves in hyper-elastic rods[9,10] and its high-frequency limit models nematic liquid crystals.[11,12] Because the CH equation can be applicable in various physical fields, to find higher-dimensional integrable CH systems may be useful in these related higher-dimensional physical problems.

It is clear that the deformation algorithm requires that the lower-dimensional equation should be a local evolution form and the conserved density should be field derivative independent. The CH Eq. (6) does not satisfy these conditions. In this Letter, we try to deform the CH Eq. (6) to higher-dimensional ones by modifying the deformation algorithm of Ref. [4].

Modified Deformation Algorithm. For a general (1+1)-dimensional integrable system

$$\begin{aligned} &F(u, u_x, u_t, \ldots, u_{xn,tp}) = 0, \quad u_{xn,tp} = \partial_x^n \partial_t^p u, \\ &u = (u_1, u_2, \ldots, u_m), \end{aligned} \tag{7}$$

if there exist some conservation laws

$$\begin{aligned} &\rho_{it} = J_{ix}, \quad i = 1, 2, \ldots, D-1, \quad \rho_i = \rho_i(u, u_x, \ldots, u_{xk,tq}), \\ &J_i = J_i(u, u_x, \ldots, u_{xN,tM}), \end{aligned} \tag{8}$$

then the deformed $(D+1)$-dimensional system

$$\begin{aligned} &F(u, \hat{L}u, \hat{T}u, \ldots, \hat{L}^n\hat{T}^p u) = 0, \\ &\rho_i = \rho_i(u, \hat{L}u, \ldots, \hat{L}^k\hat{T}^q u), \\ &J_i = J_i(u, \hat{L}u, \ldots, \hat{L}^N\hat{T}^M u), \end{aligned} \tag{9}$$

is integrable with the deformation operators

$$\hat{L} \equiv \partial_x + \sum_{i=1}^{D-1} \rho_i \partial_{x_i}, \quad \hat{T} \equiv \partial_t + \sum_{i=1}^{D-1} J_i \partial_{x_i}. \tag{10}$$

The (2+1)-Dimensional CH Equation and Its Lax Integrability. From the CH Eq. (6), we know that u is a conserved density with conserved flow $J_1 = u_{xt} + uu_{xx} + \frac{1}{2}u_x^2 - \frac{3}{2}u^2$. Thus, applying the modified deformation algorithm to the CH Eq. (6), we obtain a (2+1)-dimensional CH equation in the form

$$\begin{aligned} &\hat{T}_1 u = \hat{L}_1\, J, \\ &J = \hat{L}_1\hat{T}_1 u + u\hat{L}_1^2 u + \frac{1}{2}(\hat{L}_1 u)^2 - \frac{3}{2}u^2, \\ &\hat{L}_1 \equiv \partial_x + u\partial_y, \quad \hat{T}_1 \equiv \partial_t + J\partial_y, \end{aligned} \tag{11}$$

i.e.,

$$\begin{aligned} u_t &= J_x + uJ_y - Ju_y, \\ J &= (\partial_x + u\partial_y)(\partial_t + J\partial_y)u + u(\partial_x + u\partial_y)^2 u \\ &\quad + \frac{1}{2}(u_x + uu_y)^2 - \frac{3}{2}u^2. \end{aligned} \tag{12}$$

With the help of the conjugate operator of $\hat{L}_1$, $\tilde{L}_1 = -\partial_x - \partial_y u$, the (2+1)-dimensional CH Eq. (12) can be equivalently rewritten as

$$\begin{aligned} &u_t = \hat{L}_1\, J - Ju_y, \quad \hat{L}_1 = \partial_x + u\partial_y, \quad \tilde{L}_1 = -\partial_x - \partial_y u, \\ &\tilde{L}_1 u_t + J(1 + \tilde{L}_1 u_y) + u\tilde{L}_1 u_x + u^2\tilde{L}_1 u_y \\ &+ \frac{1}{2}u_x\tilde{L}_1 u - \frac{1}{2}u^2(u_y^2 - 3) = 0. \end{aligned} \tag{13}$$

It is known that the CH Eq. (6) is Lax integrable with (weak) Lax pair,

$$\begin{aligned} &M\psi = 0, \quad M \equiv \partial_x^2 - \frac{1}{4}(4\lambda m + 1), \quad m = u - u_{xx}, \\ &N\psi = 0, \quad N \equiv \partial_t - \frac{1}{2}(\lambda^{-1} - 2u)\partial_x - \frac{1}{2}u_x. \end{aligned} \tag{14}$$

The compatibility condition

$$\psi_{xxt} = \psi_{txx}|_{\{M\psi=0, N\psi=0\}},$$

or equivalently

$$[M, N]\psi|_{\{M\psi=0, N\psi=0\}} = (MN - NM)\psi|_{\{M\psi=0, N\psi=0\}} = 0$$

is just the usual CH Eq. (6). It should be mentioned that $[M, N] = 0$ does not lead to the CH Eq. (6), which means (14) is a weak Lax pair of the CH Eq. (6) but not a strong Lax pair.

To prove the Lax integrability of the (2+1)-dimensional CH Eq. (12), we can directly apply the deformation algorithm to the Lax pair (14) of the usual CH Eq. (6). The result reads

$$\begin{aligned} &\hat{M}\psi = 0, \quad \hat{M} \equiv (\partial_x + u\partial_y)^2 - \frac{1}{4}(4\lambda m + 1), \\ &m = u - (\partial_x + u\partial_y)^2 u, \\ &\hat{N}\psi = 0, \\ &\hat{N} \equiv \partial_t + J\partial_y - \frac{1}{2}(\lambda^{-1} - 2u)(\partial_x + u\partial_y) - \frac{1}{2}(u_x + uu_y). \end{aligned} \tag{15}$$

Similar to the (1+1)-dimensional case, $[\hat{M}, \hat{N}] = 0$ cannot yield the (2+1)-dimensional CH Eq. (12). However, the weak condition

$$[\hat{M}, \hat{N}]\psi|_{\{\hat{M}\psi=0, \hat{N}\psi=0\}} = 0$$

is just the (2+1)-dimensional CH Eq. (12).

The (1+1)-Dimensional Reciprocal CH Equation and Its (2+1)-Dimensional Deformation. It is clear that the original (1+1)-dimensional CH equation is still a special reduction of the (2+1)-dimensional CH equation. It is interesting that the (2+1)-dimensional CH Eq. (12) contains its (1+1)-dimensional reciprocal form

$$\begin{aligned} &u_t = uJ_y - Ju_y, \\ &J = u(uJ_y)_y + u^2(uu_y)_y + \frac{1}{2}u^2u_y^2 - \frac{3}{2}u^2, \end{aligned} \tag{16}$$

with the Lax pair

$$\begin{aligned} &(u\psi_y)_y - \frac{1}{4}\{4\lambda[1 - (uu_y)_y] + u^{-1}\}\psi = 0, \\ &\psi_t + J\psi_y - \frac{1}{2}(\lambda^{-1} - 2u)(\psi_x + u\psi_y) \\ &- \frac{1}{2}(u_x + uu_y)\psi = 0. \end{aligned} \tag{17}$$

It is straightforward to find that the reciprocal CH Eq. (16) possesses the conservation law

$$(u^{-1}) = -(Ju^{-1})_y. \tag{18}$$

Thus, applying the modified deformation algorithm to the reciprocal CH Eq. (16) with the help of the conservation law we can reach

$$\begin{aligned}
&\hat{T}_2 u = u\hat{L}_2 J - J\hat{L}_2 u, \quad \hat{L}_2 \equiv \partial_y + u^{-1}\partial_x,\\
&\hat{T}_2 \equiv \partial_t - u^{-1}J\partial_x,\\
&J = u\hat{L}_2(u\hat{L}_2 J) + u^2\hat{L}_2(u\hat{L}_2 u) + \frac{1}{2}u^2(\hat{L}_2 u)^2 - \frac{3}{2}u^2.
\end{aligned} \tag{19}$$

It is not difficult to find that two formally "different" expressions (11) and (19) are completely same as (12).

Traveling Wave Solutions of (2+1)-Dimensional CH Eq. (12) and a New Type of Peakon Solution. It is well known that the (1+1)-dimensional CH Eq. (6) possesses a peakon solution. Now, it is interesting for us to know if there is a possible peakon solution for the (2+1)-dimensional CH Eq. (13). One can prove that the traveling wave solutions of Eq. (13) can be written as

$$J = -c_1 u + C, \quad C = -(c_1 k_1 + \omega)k_2^{-1}, \tag{20}$$

$$u = U(X), \quad X = k_1 x + k_2 y + \omega t, \tag{21}$$

$$\begin{aligned}
&2(U - c_1)(k_1 + k_2 U + k_1)^2 U_{XX} + (k_1 + k_2 U)\\
&\cdot (3k_2 U - 2c_1 k_2 + k_1)U_X^2 + 2c_1 U - 2C - 3U^2 = 0,
\end{aligned} \tag{22}$$

where c_1, k_1, k_2 and ω are arbitrary constants.

The general solution of Eq. (22) can be expressed by the following elliptic integral

$$\int^U \frac{(k_1 + k_2\theta)\sqrt{\theta - c_1}\mathrm{d}\theta}{\sqrt{\theta^3 - c_1\theta^2 + 2C\theta + c_2}} = \pm(X - X_0) \tag{23}$$

with two arbitrary integral constants c_2 and X_0.

It is interesting that the traveling wave Eq. (22) allows a weak solution, a completely new type of peakon solution

$$u = \frac{k_1}{k_2}\mathrm{LambertW}(\eta), \quad \eta = \frac{k_2}{k_1}\exp\Big(-\Big|\frac{X - X_0}{k_1}\Big|\Big) \tag{24}$$

after taking $C = 0$. The LamberW function in the peakon solution is defined as the principle branch of

$$\mathrm{LambertW}(\eta)\exp[\mathrm{LambertW}(\eta)] = \eta. \tag{25}$$

Figure 1 displays the peakon structure of Eq. (24) with the fixed parameters $k_1 = 1$, $k_2 = 0.5$ and $X_0 = 0$.

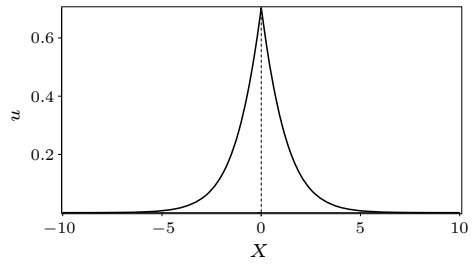

Fig. 1. Peakon solution (24) of the (2+1)-dimensional CH equation with the parameter selections $k_1 = 1$, $k_2 = 0.5$ and $X_0 = 0$.

The (4+1)-Dimensional CH Equation and Its Lower-Dimensional Reductions. For the CH Eq. (6), there are infinitely many conservation laws, three of them read

$$\rho_{it} = J_{ix}, \tag{26}$$

$$\rho_1 = u, \quad J_1 = u_{xt} + uu_{xx} + \frac{1}{2}u_x^2 - \frac{3}{2}u^2, \tag{27}$$

$$\rho_2 = u^2 - u_x^2, \quad J_2 = 2u^2 u_{xx} + 2uu_{xt} - 2u^3, \tag{28}$$

$$\rho_3^2 = u - u_{xx}, \quad J_3 = -u\rho_3. \tag{29}$$

Applying the modified deformation algorithm and the conservation laws (26)–(29) to the (1+1)-dimensional CH Eq. (6), we obtain an integrable (4+1)-dimensional CH system,

$$\begin{aligned}
&u_t = J_{1x} + uJ_{1y} + \rho_2 J_{1z} + \rho_3 J_{1\xi} - J_1 u_y - J_2 u_z + u\rho_3 u_\xi,\\
&J_1 = \hat{L}\hat{T}u + u\hat{L}^2 u + \frac{1}{2}(\hat{L}u)^2 - \frac{3}{2}u^2, \quad \rho_2 = u^2 - (\hat{L}u)^2,\\
&J_2 = 2u^2\hat{L}^2 u + 2u\hat{L}\hat{T}u - 2u^3, \quad \rho_3^2 = u - \hat{L}^2 u,
\end{aligned} \tag{30}$$

where the spacetime deformation operators are defined by

$$\begin{aligned}
\hat{L} &= \partial_x + u\partial_y + \rho_2\partial_z + \rho_3\partial_\xi,\\
\hat{T} &= \partial_t + J_1\partial_y + J_2\partial_z - u\rho_3\partial_\xi.
\end{aligned} \tag{31}$$

The Lax pair of the (4+1)-dimensional CH Eq. (30) possesses the form

$$\bar{M}\psi = 0, \quad \bar{N}\psi = 0 \tag{32}$$

with

$$\begin{aligned}
\bar{M} &\equiv (\partial_x + u\partial_y + \rho_2\partial_z + \rho_3\partial_\xi)^2 - \frac{1}{4}(4\lambda m + 1),\\
m &= u - (\partial_x + u\partial_y + \rho_2\partial_z + \rho_3\partial_\xi)^2 u,\\
\bar{N} &\equiv \partial_t + J_1\partial_y + J_2\partial_z - u\rho_3\partial_\xi - \frac{1}{2}(\lambda^{-1} - 2u)\\
&\quad \cdot (\partial_x + u\partial_y + \rho_2\partial_z + \rho_3\partial_\xi)\\
&\quad - \frac{1}{2}(u_x + uu_y + \rho_2 u_z + \rho_3 u_\xi).
\end{aligned} \tag{33}$$

The compatibility condition

$$[\bar{M}, \bar{N}]\psi|_{\{\bar{M}\psi=0,\,\bar{N}\psi=0\}} = 0$$

is a generalization of the (4+1)-dimensional CH Eq. (30) in the form ($\rho_1 \equiv u, J_3 \equiv -u\rho_3$),

$$\begin{aligned}
&\rho_{it} + J_1\rho_{iy} + J_2\rho_{iz} - u\rho_3\rho_{i\xi} = J_{ix} + uJ_{iy} + \rho_2 J_{iz} + \rho_3 J_{i\xi}, \quad i = 1, 2, 3,\\
&m_t + J_1 m_y + J_2 m_z - u\rho_3 m_\xi + 2m(u_x + uu_y + \rho_2 u_z + \rho_3 u_\xi)\\
&+ u(m_x + um_y + \rho_2 m_z + \rho_3 m_\xi) = 0,\\
&m = u - (\partial_x + u\partial_y + \rho_2\partial_z + \rho_3\partial_\xi)^2 u.
\end{aligned} \tag{34}$$

It should be mentioned that the system (34) is an undetermined system because there are only five equations for six independent variables $\rho_1 = u$, ρ_2, ρ_3, J_1, J_2, and m. In the situation, the weak Lax pairs lead to undetermined integrable systems, appeared in many cases. For instance, motivated by a class of infinitesimal Bäcklund transformations originally introduced in a gas dynamics context by Loewner in 1952, Konopelchenko and Rogers (LKR) were led to construct an undetermined master (2+1)-dimensional soliton system [13–15] with only seven equations but with eight arbitrary matrix functions. The comprehensive nature of the latter is made evident by the

fact that hierarchies of such so-called undetermined LKR systems have been shown to be compatible with generic multi-component Kadomtsev–Petviashvili and modified Kadomtsev–Petviashvili hierarchies.[16] Notable reductions of the LKR system include integrable (2+1)-dimensional versions of the sine-Gordon equation,[17] the principal chiral field model, the Bruschi–Ragnisco system[18] as well as Ernst–Weyl type equations.[19]

The system (34) is equivalent to Eq. (30) only for an additional condition, say,

$$J_1 = \hat{L}^2 J_1 + u\hat{L}^2 u + \frac{1}{2}(\hat{L}u)^2 - \frac{3}{2}u^2, \tag{35}$$

is added in (34), where $\hat{L}$ is defined in (31).

The (4+1)-dimensional CH system (30) and/or its equivalent form (34) with (35) is quite complicated in its expanded form. Here, we just list some special reductions.

In addition to the usual (1+1)-dimensional CH Eq. (6) [(30) with $u_y = u_z = u_\xi = 0$], the first type of reciprocal CH Eq. (16) [(30) with $u_x = u_z = u_\xi = 0$] and the (2+1)-dimensional CH Eq. (12) [(30) with $u_z = u_\xi = 0$], there are some other (n+1)-dimensional integrable reductions for $n = 1$, 2, and 3.

The second type of the (1+1)-dimensional reciprocal CH equation can be obtained by requiring the model (30) to be only z dependent,

$$\begin{aligned}
&u_t = \rho_2 J_{1z} - u(\rho_2 + 2J_1)u_z,\\
&(u\rho_2^3 u_{zz} + 2u^4 - u^2\rho_2 - \frac{3}{2}\rho_2^2)J_1 + \frac{1}{2}\rho_2^3 u_{zt} + u\rho_2^4 u_{zz}\\
&+ u\rho_2^3 u_z\rho_{2z} + uu_z u_t\rho_2^2 + \frac{1}{4}\rho_2(2u^2+\rho_2)(2u^2-3\rho_2) = 0,\\
&\rho_2 = u^2 - \rho_2^2 u_z^2.
\end{aligned} \tag{36}$$

If we require that the model is only ξ dependent, then the third type of the (1+1)-dimensional reciprocal CH equation can be read off from (30),

$$\begin{aligned}
&u_t = \rho_3\rho_{3\xi}u_{\xi t} + \rho_3^2 u_{\xi\xi t} + \rho_3 u_\xi(\rho_3^2 - 3u),\\
&\rho_3^2 = u - \rho_3^2 u_{\xi\xi} - \rho_3\rho_{3\xi}u_\xi.
\end{aligned} \tag{37}$$

In addition to the (2+1)-dimensional integrable CH Eq. (12), one can find at least other five (2+1)-dimensional integrable reduction systems from (30) in the spacetime $\{x,z,t\}$, $\{x,\xi,t\}$, $\{y,z,t\}$, $\{y,\xi,t\}$, and $\{z,\xi,t\}$, respectively. Here, we write down only a relatively simple one in the spacetime $\{x,\xi,t\}$,

$$\begin{aligned}
&u_t = \rho_3 uu_\xi + J_{1x} + \rho_3 J_{1\xi},\\
&J_1 = \frac{1}{2}u_x^2 - \frac{3}{2}u^2 + uu_{xx} + u_{xt}\\
&\quad - \frac{1}{2}\rho_3^2 u_\xi^2 + \rho_3 uu_{x\xi} + \rho_3 u_{\xi t},\\
&\rho_3^2 = u - u_{xx} - 2\rho_3 u_{x\xi} - \rho_3^2 u_{\xi\xi} - \rho_{3x}u_\xi - \rho_3\rho_{3\xi}u_\xi.
\end{aligned} \tag{38}$$

There are at least four (3+1)-dimensional integrable reductions of Eq. (30) in the spacetime $\{x,y,z,t\}$, $\{x,y,\xi,t\}$, $\{x,z,\xi,t\}$, and $\{y,z,\xi,t\}$, respectively. Here is the concrete form of one of them,

$$\begin{aligned}
&u_t = J_{1x} + uJ_{1y} - J_1 u_y + \rho_3 uu_\xi + \rho_3 J_{1\xi},\\
&J_1 = (\partial_x + u\partial_y + \rho_3\partial_\xi)^2 J_1 + (\partial_x + u\partial_y + \rho_3\partial_\xi)^2 u\\
&\quad + \frac{1}{2}(u_x + uu_y + \rho_3 u_\xi)^2 - \frac{3}{2}u^2,\\
&\rho_3^2 = u - (\partial_x + u\partial_y + \rho_3\partial_\xi)^2 u.
\end{aligned} \tag{39}$$

In summary, by means of the conservation laws of the (1+1)-dimensional CH equation, and/or its (1+1)-dimensional reciprocal CH equations, one can find various (n+1)-dimensional integrable CH systems with the help of the modified deformation algorithm. Applying the deformation algorithm to weak Lax pairs of the (1+1)-dimensional CH and/or reciprocal CH equation, one can get some undetermined integrable systems. To obtain determined integrable systems, one must deform strong Lax pairs or add some reasonable auxiliary conditions on the deformed weak Lax pairs.

The same idea can be applied to other peakon systems such as the Degasperis–Procesi equation, the Novikov equation, and the Fokas–Olver–Rosenau–Qiao equation.[20] In principle, the present deformation idea may be applied in differential-difference equations like the models discussed in literature.[21,22] However, the deformation algorithm should be modified and we have not yet found appropriate deformation algorithm for the discrete variables.

Because the original (1+1)-dimensional integrable systems and their several reciprocal forms are included in the same higher-dimensional models, to find some exact solutions of the higher-dimensional integrable systems of this study and those in Ref. [4] is quite difficult although the models are Lax and symmetry integrable. There are many papers and methods to find exact solutions for traditional lower- and higher-dimensional integrable and nonintegrable models.[23,24] However, these methods are difficult to directly apply to the deformed systems. For instance, the methods, e.g., in Ref. [23] based on Hirota's bilinear form, cannot be applied because there is no bilinear form for the deformed systems. In this study, only the travelling wave solutions of the (2+1)-dimensional CH Eq. (12) are obtained via a complicated elliptic integral. It is interesting that a completely new type of peakon solution (24) is described by the implicit LambertW function.

From the (1+1)-dimensional CH Eq. (5), one can see that a solution of the linear equation

$$m = u - u_{xx} = 0 \tag{40}$$

is trivially a solution of the nonlinear CH Eq. (5). The usual single peakon solution of the CH equation

$$u = c_1 \exp(-|x - c_2 t - x_0|) \tag{41}$$

is just a weak solution of the linear Eq. (40). As one of special local waves, peakon, in nonlinear systems, it is more reasonable to find some models where the peakons are really a solution of a nonlinear equation. The peakon solution provided in this study may be the first significant example.

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