Ground State and Its Topological Properties of Three-Dimensional Spin-Orbit Coupled Degenerate Fermi Gases

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Three-dimensional (3D) degenerate Fermi gases in the presence of spin-orbit coupling, inducing various kinds of physical findings and phenomena, have attracted tremendous attention in these years. We investigate the 3D spin-orbit coupled degenerate Fermi gases in theory and first present the analytic expression of their ground state. Our study provides an innovative perspective into understanding of the topological properties of 3D unconventional superconductors, and describes the topological phase transitions in trivial and topological phase areas. Further, such a system is provided with a richer set of Cooper pairings than traditional superconductors. The dual Cooper pairs with same spin directions emerge and exhibit peculiar behaviors, leading to topological phase transitions. Our study and discussion can be generalized to some other types of unconventional superconductors and areas of optical lattices.

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In recent years, ultracold Fermi gases with tunable atom interaction through Feshbach resonance have attracted tremendous attention, for their important use as ideal platform that can emulate many interesting physical phenomena.[1–7] Such phenomena range from Bardeen–Cooper–Schrieffer (BCS) superfluid, Bose–Einstein condensate (BEC) of molecules to vortices, and have been evidently observed.[8–23] In addition, the system can be introduced with other forms of physical effects, and a variety of intriguing findings are predicted and later confirmed in experiments.[24–40]

The spin–orbit coupling (SOC), as it has the orbital degrees of freedom, can interact with the atomic pseudospin, giving rise to intriguing physics and applications. Notably, the SOC for ultracold atoms were explored in both theory and experiment,[24,41–45] and the concern in this kind of interaction was induced since it offered an innovative way for investigating the SOC physics using degenerate cold Fermi gases.[46–48] Especially, the study of the BCS-BES crossover physics from the interaction between SOC and 3D degenerate Fermi gases (i.e., 3D spin-orbit coupled degenerate Fermi gases) was conducted.[12,13,49] The topological phase transitions of 3D superfluid state were predicted and realized experimentally,[14–16,50,51] and these works have gained great interests.[40,52] However, this type of topological phase transitions has not been explained up to date, and the system of 3D spin-orbit coupled degenerate Fermi gases still requires further study and discussion.

In this Letter, we investigate the 3D degenerate Fermi gases in the presence of SOC and Zeeman field, and deduce their analytic expression of ground state firstly. Moreover, our work provides an innovative perspective into understanding of the topological properties of a 3D unconventional superconductors, and studies the topological phase transitions in trivial and topological phase areas. Specifically in real space, when the system lies in the area of the trivial phase, the variables that represent entire four Cooper pairing strengths oscillate and decay exponentially; while in the topological area, the newly emerged variables for the strengths of dual Cooper pairs induced by SOC decay in an algebraic way, and the external magnetic field modulates the dual Cooper pairs, leading to the topological phase transitions. Finally, we anticipate our study, and the discussion can be generalized to some other types of unconventional superconductors and areas of optical lattices.

Hamiltonian. The system we consider is a 3D degenerate Fermi gas introduced with a Rashba-type SOC in the xy plane and a perpendicular Zeeman field along the z direction.[12] In experiment, the 2D degenerate Fermi gas can be realized using a one-dimensional deep optical lattice,[27,41,53–56] and the Rashba SOC with Zeeman

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field can be realized using the adiabatic atoms.[57,58] The Hamiltonian for this system can be written as ($\hbar = k_{\rm B} = 1$)

$$H = H_0 + H_{\rm int}, \tag{1}$$

where the single particle Hamiltonian reads

$$H_0 = \sum_{\boldsymbol{k}\gamma\gamma'} c^\dagger_{\boldsymbol{k}\gamma}[\xi_{\boldsymbol{k}} I + \alpha(k_y\sigma_x - k_x\sigma_y) + \Gamma\sigma_z]_{\gamma\gamma'} c_{\boldsymbol{k}\gamma'}. \tag{2}$$

Here, $\gamma = \uparrow, \downarrow$, $\xi_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} - \mu$ is the reduced particle energy, μ is the chemical potential, $\epsilon_{\boldsymbol{k}} = \frac{k_x^2 + k_y^2}{2m}$ is the free particle energy, $\xi_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} - \mu$ is the reduced particle energy, Γ is the strength of the Zeeman field, α is the Rashba SOC strength, $\pi = \alpha(-ik_x + k_y)$, I is the 2×2 unit matrix, σ_i is the Pauli matrix, and $c_{\boldsymbol{k}\gamma}$ is the annihilation operator.

In the mean-field approximation, the s-wave pair potential has the following form:

$$\Delta = g\sum_{\boldsymbol{k}} \langle c_{\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}\uparrow}\rangle \tag{3}$$

and the interaction term is obtained as follows:

$$H_{\rm int} = -\frac{1}{g}\Delta^2 + \Delta\sum_{\boldsymbol{k}} (c_{\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}\uparrow} + c^\dagger_{-\boldsymbol{k}\uparrow} c^\dagger_{\boldsymbol{k}\downarrow}). \tag{4}$$

We ignore the constant term. Under the Nambu spinor basis, we can obtain such an expression:

$$\Psi_{\boldsymbol{k}} = (c_{\boldsymbol{k}\uparrow},\, c_{\boldsymbol{k}\downarrow},\, c^\dagger_{-\boldsymbol{k}\downarrow},\, -c^\dagger_{-\boldsymbol{k}\uparrow})^{\rm T}. \tag{5}$$

The Hamiltonian is $H = \sum_k \Psi_k^\dagger H_k \Psi_k$, where the Hamiltonian H_k is

$$H_k = \begin{pmatrix} \xi_k + \Gamma & \pi^\dagger & \Delta & 0 \\ \pi & \xi_k - \Gamma & 0 & \Delta \\ \Delta & 0 & -\xi_k + \Gamma & -\pi^\dagger \\ 0 & \Delta & -\pi & -\xi_k - \Gamma \end{pmatrix}. \tag{6}$$

The quasiparticle excitation energy Λ_k and its relation of the eigenvalue equation of H_k read

$$\Psi_k^\dagger H_k \Psi_k = \Psi_k^\dagger U U^\dagger H_k U U^\dagger \Psi_k = \beta_k^\dagger \Lambda_k \beta_k, \tag{7}$$

$$\Lambda_k = \begin{pmatrix} \kappa_+ & 0 & 0 & 0 \\ 0 & \kappa_- & 0 & 0 \\ 0 & 0 & -\kappa_- & 0 \\ 0 & 0 & 0 & -\kappa_+ \end{pmatrix}, \tag{8}$$

$$\kappa_+ = \sqrt{E_{\rm f} + 2E_0}, \quad \kappa_- = \sqrt{E_{\rm f} - 2E_0},$$

where

$$E_{\rm f} = k^2\alpha^2 + \Gamma^2 + \Delta^2 + \xi_k^2,$$
$$E_0 = \sqrt{\Gamma^2\Delta^2 + \xi_k^2(k^2\alpha^2 + \Gamma^2)}. \tag{9}$$

When $\alpha \to 0$, $\Gamma \to 0$, and $E_0 = 0$, the system can be explained by the standard BCS theory.

BCS Ground State. Now, we start to form the ground state of the 3D spin-orbit coupled degenerate Fermi gases. Based on the BCS theory, we consider the following traditional Hamiltonian:

$$\mathcal{H}_{\rm BCS} = \sum_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} (c^\dagger_{\boldsymbol{k}} c_{\boldsymbol{k}} + c^\dagger_{-\boldsymbol{k}} c_{-\boldsymbol{k}}) + \sum_{\boldsymbol{k},\boldsymbol{k}'} V_{\boldsymbol{k}\boldsymbol{k}'} c^\dagger_{\boldsymbol{k}'} c^\dagger_{-\boldsymbol{k}'} c_{-\boldsymbol{k}} c_{\boldsymbol{k}}, \tag{10}$$

where we have used the relations $V_{\boldsymbol{k}\boldsymbol{k}'} = V_{-\boldsymbol{k},-\boldsymbol{k}'}$ and $c^\dagger_{\boldsymbol{k}'} \equiv c^\dagger_{\boldsymbol{k}'\uparrow}; c^\dagger_{-\boldsymbol{k}'} \equiv c^\dagger_{-\boldsymbol{k}'\downarrow}$, etc. According to the mean-field approach, we introduce the following quantities:

$$c_{-\boldsymbol{k}} c_{\boldsymbol{k}} = \langle c_{-\boldsymbol{k}} c_{\boldsymbol{k}}\rangle + (c_{-\boldsymbol{k}} c_{\boldsymbol{k}} - \langle c_{-\boldsymbol{k}} c_{\boldsymbol{k}}\rangle),$$
$$c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}} = \langle c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}}\rangle + (c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}} - \langle c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}}\rangle). \tag{11}$$

In the normal state, $\langle c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}}\rangle = \langle c_{-\boldsymbol{k}} c_{\boldsymbol{k}}\rangle = 0$, they are the expectation numbers of operators that affect the number of particles.

Introducing the chemical potential to ensure system to satisfy non-conservation of the particle number, the BCS Hamiltonian becomes

$$\mathcal{H}_{\rm BCS} = \sum_{\boldsymbol{k}} (\varepsilon_{\boldsymbol{k}} - \mu)(c^\dagger_{\boldsymbol{k}} c_{\boldsymbol{k}} + c^\dagger_{-\boldsymbol{k}} c_{-\boldsymbol{k}}) + \sum_{\boldsymbol{k},\boldsymbol{k}'} V_{\boldsymbol{k}\boldsymbol{k}'} (c^\dagger_{\boldsymbol{k}'} c^\dagger_{-\boldsymbol{k}'} \phi_{\boldsymbol{k}} + \phi^*_{\boldsymbol{k}'} c_{-\boldsymbol{k}} c_{\boldsymbol{k}} - \phi^*_{\boldsymbol{k}'} \phi_{\boldsymbol{k}}). \tag{12}$$

Diagonalize it through the Bogoliubov transformation:

$$\gamma_{\boldsymbol{k}} = u_{\boldsymbol{k}} c_{\boldsymbol{k}} - v_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}}, \qquad \gamma_{-\boldsymbol{k}} = u_{\boldsymbol{k}} c_{-\boldsymbol{k}} + v_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}},$$
$$\gamma^\dagger_{\boldsymbol{k}} = u_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} - v_{\boldsymbol{k}} c_{-\boldsymbol{k}}, \qquad \gamma^\dagger_{-\boldsymbol{k}} = u_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}} + v_{\boldsymbol{k}} c_{\boldsymbol{k}}. \tag{13}$$

The ground state of the Bogoliubov quasiparticles $|\Psi_0\rangle$ is just the vacuum state for the new operators

$$\gamma_{\boldsymbol{k}}|\Psi_0\rangle = \gamma_{-\boldsymbol{k}}|\Psi_0\rangle = 0, \quad \langle\Psi_0|\gamma^\dagger_{-\boldsymbol{k}} = \langle\Psi_0|\gamma^\dagger_{\boldsymbol{k}} = 0. \tag{14}$$

By the product of all the $u_k v_k$, we can obtain

$$|\Psi_0\rangle = \Big[\prod_{\boldsymbol{k}} (1 + z_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}})\Big]|\emptyset\rangle = \exp\Big[\sum_{\boldsymbol{k}} z_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}}\Big]|\emptyset\rangle. \tag{15}$$

Ground State with SOC. According to the BCS theory, when the system is in the presence of SOC, the ground state should be the vacuum state of the quasi-particle annihilation operator:

$$\alpha_{-k\uparrow}|\Psi_0\rangle = \alpha_{k\uparrow}|\Psi_0\rangle = \alpha_{k\downarrow}|\Psi_0\rangle = \alpha_{-k\downarrow}|\Psi_0\rangle = 0. \tag{16}$$

Then we assume the form of ground state as follows:

$$|\Psi_0\rangle = \prod_k \alpha_{-k\uparrow}\alpha_{k\uparrow}\alpha_{-k\downarrow}\alpha_{k\downarrow}|\emptyset\rangle, \tag{17}$$

where $|0\rangle$ is the vacuum state of the particle annihilation operator, expressing the entire quasi-particle operators in the ground state:

$$|\Psi_0\rangle = \Big[\prod_k a^*_k + b^*_{1k} c^\dagger_{k\uparrow} c^\dagger_{-k\uparrow} + b^*_{2k} c^\dagger_{k\downarrow} c^\dagger_{-k\downarrow} + b^*_{3k} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + b^*_{4k} c^\dagger_{k\downarrow} c^\dagger_{-k\uparrow} + b^*_{5k} c^\dagger_{k\uparrow} c^\dagger_{-k\uparrow} c^\dagger_{k\downarrow} c^\dagger_{-k\downarrow}\Big]|\emptyset\rangle, \tag{18}$$

with

$$(\alpha^+_{k\uparrow}\ \alpha^+_k\ \alpha_{-k\downarrow}\ \alpha_{-k\uparrow}) = (c^+_{k\uparrow}\ c^+_{k\downarrow}\ c_{-k\downarrow}\ -c_{-k\uparrow})U. \tag{19}$$

U is the unitary matrix. Then, we can combine these relations with Eq. (16) and give the expressions of variables as

$$
\begin{aligned}
a_k^* &= \frac{2\Gamma^2 A - 4\xi_k^2 B}{N(k_x^2+k_y^2)\alpha^2(\Gamma-\xi_k)^2},\\
b_{1k}^* &= \frac{2i[C_1+\xi_k(B_1+B_2+2\xi_k B)]}{N(k_x+ik_y)\alpha\Delta(\Gamma-\xi_k)^2},\\
b_{2k}^* &= -\frac{2i[C_2+\xi_k(B_1-B_2-2\xi_k B)]}{N(k_x-ik_y)\alpha\Delta(\Gamma-\xi_k)^2},\\
b_{3k}^* &= 2\frac{D_1+\xi_k(D_2+(E_1+E_2-2\xi_k)\xi_k B)}{N(k_x^2+k_y^2)\alpha^2\Delta(\Gamma-\xi_k)^2},\\
b_{4k}^* &= -b_{3k}^*,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
A &= 2\Gamma^2(E_1E_2 - E_\mathrm{f} + 2\Delta^2),\\
B &= -E_\mathrm{f} + \Delta^2 + \xi_k^2,\\
B_1 &= E_1E_2\Gamma - E_\mathrm{f}\Gamma,\\
B_2 &= E_1E_2\Gamma - E_\mathrm{f}\Gamma,\\
C_1 &= \Gamma(E_0(E_1-E_2) - 2\Gamma\Delta^2),\\
C_2 &= \Gamma(E_0(E_1-E_2) + 2\Gamma\Delta^2),\\
D_1 &= \Gamma^2(E_0(E_1-E_2) - (E_1+E_2)\Delta^2,\\
D_2 &= \Gamma^2(E_1E_2 - E_\mathrm{f} + 2\Delta^2).
\end{aligned}
\tag{21}
$$

These variables satisfy the relation

$$
b_{1k}^* \cdot b_{2k}^* - b_{3k}^* \cdot b_{4k}^* = b_{5k}^*. \tag{22}
$$

There are no terms with more than two fermion operators except for $c_{k\uparrow}^\dagger c_{-k\uparrow}^\dagger c_{k\downarrow}^\dagger c_{-k\downarrow}^\dagger$, and the exponential form of the ground state wave function is:

$$
\begin{aligned}
&|\Psi_0\rangle\\
&= C\prod_k e^{b_{1k}c_{k\uparrow}^\dagger c_{-k\uparrow}^\dagger + b_{2k}c_{k\downarrow}^\dagger c_{-k\downarrow}^\dagger + b_{3k}c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + b_{4k}c_{k\downarrow}^\dagger c_{-k\uparrow}^\dagger}|\emptyset\rangle.
\end{aligned}
\tag{23}
$$

The expressions of variables are rewritten through transformation (replacing k_x, k_y with k, θ) as

$$
\begin{aligned}
b_{1k} &= \frac{ike^{-i\theta}\alpha\{C_1+\xi_k[B_1+B_2+2\xi_k B]\}}{\Delta[D_2-2\xi_k^2 B]},\\
b_{2k} &= \frac{-ike^{i\theta}\alpha\{C_2+\xi_k[B_1-B_2-2\xi_k B]\}}{\Delta[D_2-2\xi_k^2 B]},\\
b_{3k} &= \frac{D_1+\xi_k[D_2+(E_1+E_2-2\xi_k)\xi_k B]}{\Delta[D_2-2\xi_k^2 B]},\\
b_{4k} &= -b_{3k}.
\end{aligned}
\tag{24}
$$

Here, the analytic expressions for the ground state of 3D spin-orbit coupled degenerate Fermi gases are presented in Eqs. (23) and (24), and these expressions have never reported before.

From the expressions, we can observe the two newly emerged fermion operators that denote two Cooper pairs with same spin directions; b_1 and b_2 represent the strengths of these pairs respectively; while in Eq. (13), z_k represents the chance of forming a Cooper pair for a fermion with momentum k. The ground state variables in Eq. (24) correspond to the probabilities of the Cooper pairs being occupied. However, we note that the essence of such discoveries still remains unknown.[12]

As a result, it is natural to explore the physics of the ground state with SOC; how these variables behave in real space, and how they are related with the topological phase transitions in particular.

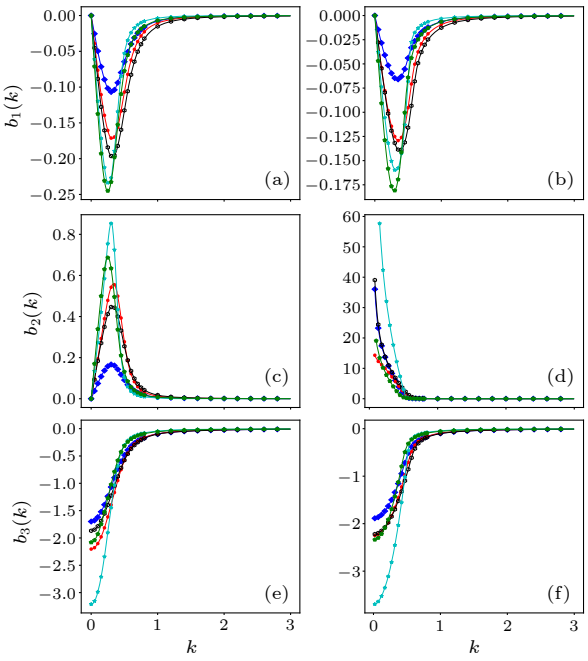

Fig. 1. The variables of ground state in momentum space shown in (a)–(f). The left column depicts the transition of trivial phase ($\sqrt{\xi_{\boldsymbol{k}}^2+\Delta^2} > \Gamma$) and right column denotes that of the topological phase ($\sqrt{\xi_{\boldsymbol{k}}^2+\Delta^2} < \Gamma$), and solid lines of various colors represent different parameters.

Discussion for Topological Property. The properties of the variables $b_1(k)$–$b_4(k)$ in momentum space are shown in Fig. 1. In order to study the properties of the variables b_1–b_4 in real space and to know what happens to the variables as $x \to \infty$, we perform the Fourier transform of all the variables:

$$
\begin{aligned}
b_i(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b_i(k)dk_x dk_y\\
&= \int_0^{+\infty}\int_0^{2\pi} b_i(k)e^{-ikx\cos(\theta)}kd\theta dk,
\end{aligned}
\tag{25}
$$

where $i = 1, 2, 3, 4$, $k = \sqrt{k_x^2+k_y^2}$ denotes the value of momentum, and θ is the angle between the coordinates k_x and k_y.

According to the results, the traditional BCS theory does not apply to such a system of ground state, though both relate to Cooper pairings. The traditional BCS theory assumes that there are $(\boldsymbol{k}, \uparrow)$ and $(-\boldsymbol{k}, \downarrow)$ pairings locked near the Fermi surface, i.e., spin $\uparrow$ and momentum $\boldsymbol{k}$ ($k > 0$), and spin $\downarrow$ and momentum $-\boldsymbol{k}$. However, due to the SOC effect, this locking effect no longer exists, and we have $(\boldsymbol{k}, \uparrow) \leftrightarrow (-\boldsymbol{k}, \uparrow)$, $(\boldsymbol{k}, \downarrow) \leftrightarrow (-\boldsymbol{k}, \downarrow)$, $(\boldsymbol{k}, \uparrow) \leftrightarrow (-\boldsymbol{k}, \downarrow)$, $(\boldsymbol{k}, \downarrow) \leftrightarrow (-\boldsymbol{k}, \uparrow)$. In the presence of SOC, even strong Zeeman field fails to break the original Cooper pairs, instead to add a dual Cooper pairs with same spin directions.

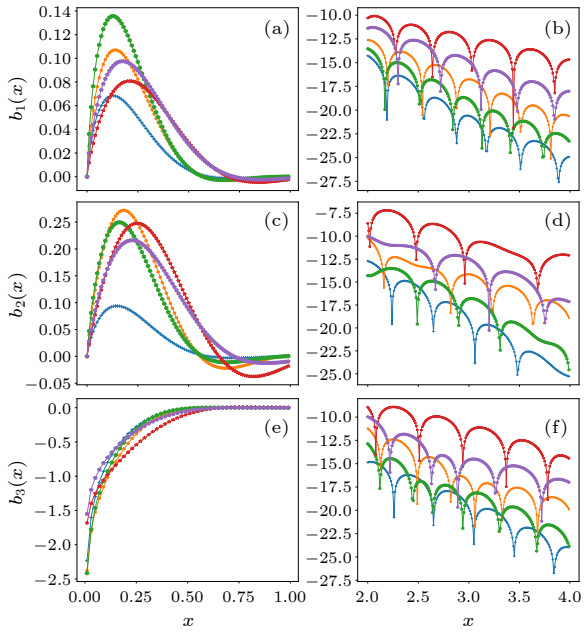

Fig. 2. The variables for ground state of the trivial phase region in real space shown in (a)–(f). The left column is the trivial phase ($\sqrt{\xi_{\boldsymbol{k}}^2+\Delta^2} > \Gamma$); right column shows the asymptotic behavior of $b_i(x)$, as $x \to \infty$, all variables decay exponentially, and solid lines of various colors represent different parameters.

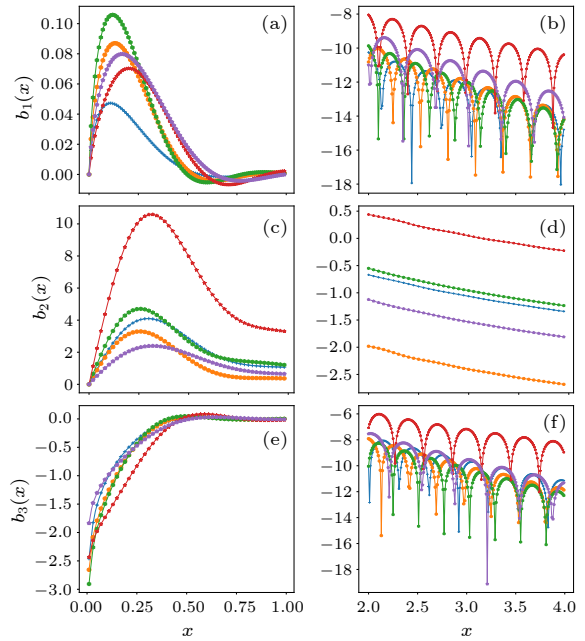

Fig. 3. The variables for ground state of the topological area in real space, shown in (a)–(f). The left column denotes the topological phase ($\sqrt{\xi_{\boldsymbol{k}}^2+\Delta^2} > \Gamma$); right column shows the asymptotic behavior of $b_i(x)$ as $x \to \infty$, all the variables are exponentially decaying; however, $b_2(x)$ is decaying when the external magnetic field $\Gamma > 0$, and solid lines of various colors represent different parameters.

For the ground-state wave function, variables b_1–b_4 correspond to the strength of four Cooper pairings mentioned above, and some variables exhibit different behaviors in the trivial and topological regions. Specifically, in real space, when the system is in area of the trivial phase, the variables in general show an oscillating and exponential decaying with x (Fig. 2); while in topological area, variables $b_3(x)$ and $b_4(x)$ still exhibit the exponential decaying, $b_1(x)$ ($\Gamma > 0$) or $b_2(x)$ ($\Gamma < 0$) exhibits an algebraic decaying (Fig. 3).

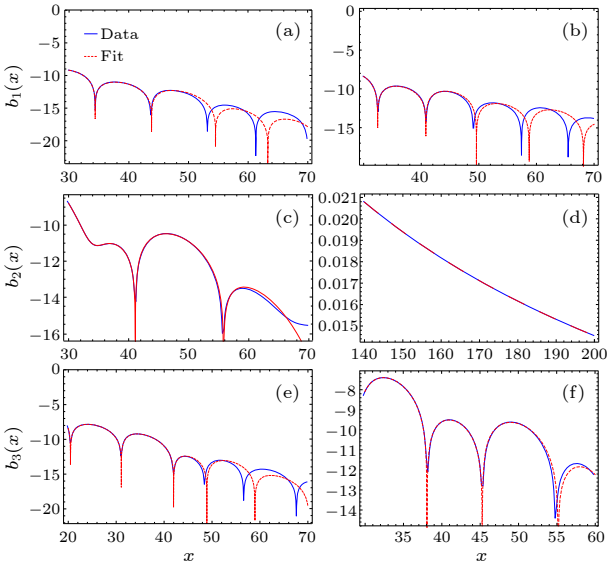

Fig. 4. The fittings of variables in real space shown in (a)–(f). The solid blue line denotes data, the red dashed curve represents fitted functions, and the horizontal and vertical coordinates are taken as algebraic absolute values. The left column shows the transition in the trivial phase, where $m = 0.1$, $\Gamma = 0.2$, $\Delta = 0.4$, $\mu = 0.2$, $\alpha = 0.9$; the right column is topological phase, where $m = 0.1$, $\Gamma = 0.5$, $\Delta = 0.4$, $\mu = 0.2$, $\alpha = 0.9$. The fitted functions are described as follows: (a) $[0.072\cos(0.34x - 3.85) + 0.022\cos(0.14x - 0.79)]\exp(-0.21x)$, (b) $[0.27\cos(0.32x - 5.59) + 0.31\cos(0.32x - 2.68)]\exp(-0.16x)$, (c) $[0.14\cos(0.33x - 4.21) + 0.24\cos(0.17x - 1.43)]\exp(-0.20x)$, (d) $2.91/x$, (e) $[0.10\cos(0.33x - 5.7) + 0.046\cos(0.12x - 5.29)]\exp(-0.20x)$, (f) $[0.059\cos(0.37x - 6.15) + 0.017\cos(0.06x - 3.26)]\exp(-0.14x)$.

As algebraic decaying is slower than exponential decaying, Cooper pairs with same spin direction [$b_1(x)$ and $b_2(x)$] decay slower than those with opposite directions [$b_3(x)$ and $b_4(x)$], thus the former pairs gradually predominate in this area.

In a word, this discussion explains how the topological phase emerges from a novel way. In the presence of external Zeeman field and the SOC, the system is provided with a richer set of Cooper pairs than the traditional one, and the added Cooper pairs induce the topological phase transition.

In summary, we have investigated the 3D degenerate Fermi gas system in the presence of SOC and Zeeman field, and first presented the analytic expression for its ground state. Further, our study has provided an innovative perspective into understanding of the topological properties

of a 3D unconventional superconductors, and explained the topological phase transitions in trivial and topological phase areas. Specifically in real space, when the system lies in the area of the trivial phase, the entire four Cooper pairing strengths oscillate and decay exponentially; while in the topological area, the variables of dual Cooper pairs induced by SOC decay in the algebraic way, and the Zeeman field also modulates the dual Cooper pairs, leading to the topological phase transitions. Finally, we anticipate that our study and discussion can be generalized to some other types of unconventional superconductors and areas of optical lattices.

Appendix. We provide a detailed derivation for variables $b_1(k)$ and $b_2(k)$. As shown in Fig. (1), all the variables $b_i(k)$ have a similar property: as $k \to 0$, there exists an evident peak. When $\Gamma > 0$, the Taylor series for $b_2(k)$ near $k = 0$ has the approximate form:

$$b_2(k) \approx C\frac{e^{i\theta}}{k}, \tag{26}$$

where

$$C = \frac{2i\Gamma(-\Gamma^2 + \Delta^2 + \mu^2)}{\alpha\Delta(\Gamma - \mu)}. \tag{27}$$

The Fourier transform of the above expression could be obtained as follows:

$$b_2(x) = \int_0^{\epsilon}\int_0^{2\pi} k\Big(C\frac{e^{i\theta}}{k}\Big)d\theta dk, \tag{28}$$

Here $\epsilon \to 0$, and $b_2(x)$ can be deduced in the following form:

$$\begin{aligned} b_2(x) &= \int_0^{\epsilon}\int_0^{2\pi} k\Big(C\frac{e^{i\theta}}{k}e^{ikx\cos(\theta)}\Big)d\theta dk \\ &\approx \int_0^{\infty}\int_0^{2\pi} kb_2(k)d\theta dk = \frac{2\pi iC(1 - J_0(kx))}{x}\Big|_0^{\infty}, \end{aligned} \tag{29}$$

where J_0 is the first class of zero-order Bessel functions:

$$J_0(0) = 1, \quad J_0(\epsilon x) = J_0(\infty) = 0. \tag{30}$$

The analytical expression of $b_2(x)$ with $\Gamma > 0$ could be written as

$$b_2(x) = \frac{4\pi\Gamma(\Gamma^2 - \mu^2 - \Delta^2)}{\alpha\Delta(\Gamma - \mu)x}. \tag{31}$$

Similarly, the expression of $b_1(x)$ with $\Gamma < 0$ can also be derived,

$$\begin{aligned} &x \to \infty, \quad \Gamma > 0, \quad b_2(x) = \frac{4\pi\Gamma(\Gamma^2 - \mu^2 - \Delta^2)}{\alpha\Delta(\Gamma - \mu)x}, \\ &x \to \infty, \quad \Gamma < 0, \quad b_1(x) = \frac{4\pi\Gamma(-\Gamma^2 + \mu^2 + \Delta^2)}{\alpha\Delta(-\Gamma - \mu)x}. \end{aligned} \tag{32}$$

As a result, The analytical expressions for $b_1(x)$ and $b_2(x)$ are presented in a plainer form, and the above expressions contribute to a better understanding of such variables.

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