New Painlevé Integrable (3+1)-Dimensional Combined pKP–BKP Equation: Lump and Multiple Soliton Solutions

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We introduce a new form of the Painlevé integrable (3+1)-dimensional combined potential Kadomtsev–Petviashvili equation incorporating the B-type Kadomtsev–Petviashvili equation (pKP–BKP equation). We perform the Painlevé analysis to emphasize the complete integrability of this new (3+1)-dimensional combined integrable equation. We formally derive multiple soliton solutions via employing the simplified Hirota bilinear method. Moreover, a variety of lump solutions are determined. We also develop two new (3+1)-dimensional pKP–BKP equations via deleting some terms from the original form of the combined pKP–BKP equation. We emphasize the Painlevé integrability of the newly developed equations, where multiple soliton solutions and lump solutions are derived as well. The derived solutions for all examined models are all depicted through Maple software.

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1. Introduction. The nonlinear evolution equations that appear in fluid dynamics, optical fibers, plasma dynamics and other fields have been investigated thoroughly in the literature. Integrable equations possess exact multiple soliton solutions. Solitons are stable, localized waves, with shapes, amplitudes and velocities that remain unchanged during the propagation process. Soliton interactions are elastic if the shapes, amplitudes and velocities of the solitons remain invariant before and after the interactions, as in the Korteweg–de Vries (KdV) equation.[1–12] However, inelastic wave interactions may exist where fission and fusion phenomena occur, as in Burgers equation. Furthermore, the elastic and inelastic interactions of some continuous integrable equations have been studied and discussed extensively.

The basic approaches to soliton solutions include the inverse scattering transform,[1–10] the Riemann–Hilbert technique,[11–20] the Darboux transformation,[16–25] and the Hirota direct method.[6–35] Significant solutions in mathematical physics, such as breather, complexion, lump and rogue wave solutions, are particular reductions of soliton solutions for different situations.

Many effective approaches[12–24] have been used to examine the complete integrability of nonlinear evolution equations, with elastic or nonelastic interactions, aiming to attain new results in scientific areas. Significant solutions in mathematical physics, such as breather, lump and rogue wave solutions, have been derived using a variety of efficient techniques, such as the algebraic–geometric method, the inverse scattering method, the Bäcklund transformation method, the Painlevé analysis, Lax integrability, the Darboux transformation method,[15–22] the Hirota bilinear method,[1–6,22–35] and other powerful schemes. In Ref. [33], solutions of the (2+1)-dimensional coupled partially nonlocal nonlinear Schrödinger equation are found by means of a coupled relation with the Darboux method. However, in Refs. [34,35], N-soliton solutions, the breathers and lumps for the (2+1)-dimensional Hirota–Satsuma–Ito equation, were obtained by using the Hirota bilinear method, as well as the complex conjugate parameter and the long wave limit method.

The Hirota bilinear method is an efficient, convenient and powerful technique to investigate integrable nonlinear science models, such as Bose–Einstein condensates, plasma physics, ferromagnetic chains, water waves, and nonlinear optical fibers. Therefore, it has attracted attention of a large number of researchers. Integrable equations can be transformed into Hirota bilinear forms by using dependent variable transformations. Also, computer symbolic systems such as Maple and Mathematica can be used to overcome the tedious computational strategies involved.

The potential Kadomtsev–Petviashvili (pKP) equation is given as

$$u_{xt} + 6u_x u_{xx} + u_{xxxx} + au_{yy} = 0, \tag{1}$$

obtained by u in in the KdV equation with u_x and integrating with respect to x once.

Moreover, the (2+1)-dimensional integrable B-type Kadomtsev–Petviashvili (BKP) equation reads

$$\begin{aligned}&(15(u_x)^3 + 15u_x u_{xxx} + u_{xxxxx})_x \\ &+ 5(u_{xxxy} + 3(u_x u_y)_x) + u_{xt} - u_{yy} = 0,\end{aligned} \tag{2}$$

which describes the interactions between exponentially localized structures and has been used as a model for shallow water waves in fluids and the electrostatic wave potential in plasmas.[1–18]

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Recently, a combined form of the pKP Eq. (1) and the BKP Eq. (2) given by Ma,[1] called the pKP–BKP equation, was given in the form

$$\begin{aligned} &a_1(15(u_x)^3+15u_xu_{xxx}+u_{xxxxx})_x\\ &+a_2(6u_xu_{xx}+u_{xxxx})+a_3(u_{xxxy}+3(u_xu_y)_x)\\ &+a_4u_{xx}+a_5u_{\mathrm{xt}}+a_6u_{yy}=0, \end{aligned} \tag{3}$$

where a_i are arbitrary constants and $a_5\neq 0$. This newly proposed equation was thoroughly studied in Refs. [1–5], with useful findings. In Ref. [1], the Hirota conditions for N-soliton solutions are studied and analyzed. A variety of soliton molecules were formed through solitons such as kinks, lumps, and breathers for the physical quantity $u(x,\,y,\,t)$ in Refs. [2–5]. Resonant multi-solitons, M–breather, M–lump and hybrid solutions of a combined pKP–BKP equation, were formally derived in Refs. [2–5]. Moreover, the lump waves and collisions among lumps and periodic waves, the collision between lump wave and single, double-kink soliton solutions, the collision among lump, periodic and single, double-kink soliton solutions, as well as periodic wave soliton solutions were achieved in Refs. [6–35]. The newly developed pKP–BKP Eq. (3) as undergone thorough and productive investigations, yielding many useful results.

In this Letter, we plan to extend Eq. (3) into a new (3+1)-dimensional pKP–BKP model with the objective of finding new results to advance the findings achieved in Refs. [1–5] and other works in the literature. The newly proposed (3+1)-dimensional pKP–BKP equation takes the form

$$\begin{aligned} &u_{\mathrm{xt}}+\alpha(15(u_x)^3+15u_xu_{xxx}+u_{xxxxx})_x\\ &+\beta(6u_xu_{xx}+u_{xxxx})+\gamma(u_{xxxy}+3(u_xu_y)_x)\\ &+au_{xx}+bu_{xy}+cu_{xz}+\mu u_{yy}=0, \end{aligned} \tag{4}$$

where some additional terms, namely bu_{xy} and cu_{xz}, are added to the pKP–BKP Eq. (3), a_5 is set equal to 1, and $u=u(x,\,y,\,z,\,t)$. The newly proposed pKP–BKP Eq. (4) is slightly different from Eq. (3) because of the newly added terms and the extension to (3+1) dimensions. The physical background of the governing Eq. (4) represents the pioneering work of combining nonlinear evolution equations to reveal nonlinear integrable equations, as discussed in Ref. [1].

The aims of this work are twofold. We aim first to perform Painlevé analysis to emphasize the complete integrability of the (3+1)-dimensional pKP–BKP model (4). Our analysis will extend Hirota's method to obtain the dispersion relation in order to derive multiple soliton solutions. We then continue to derive a variety of lump solutions with distinct physical structures. Moreover, we develop two more (3+1)-dimensional pKP–BKP equations by deleting some terms from the original Eq. (4). We will show that the newly developed equations are both Painlevé integrable and we derive multiple soliton and lump solutions for each developed model. The Painlevé analysis and the Hirota method are reliable techniques for handling nonlinear evolution equations.

2. New (3+1)-Dimensional pKP–BKP Equation. Here, we introduce a new (3+1)-dimensional pKP–BKP equation, given as

$$\begin{aligned} &u_{\mathrm{xt}}+\alpha(15(u_x)^3+15u_xu_{xxx}+u_{xxxxx})_x\\ &+\beta(6u_xu_{xx}+u_{xxxx})+\gamma(u_{xxxy}+3(u_xu_y)_x)\\ &+au_{xx}+bu_{xy}+cu_{xz}+\mu u_{yy}=0. \end{aligned} \tag{5}$$

We will start by using the Painlevé test to study the Painlevé integrability of this equation.

2.1 Painlevé Analysis and Integrability. There are many significant properties, such as Lax pairs, Hamiltonian structures, infinitely many symmetries, and infinite conservation laws, that can characterize the integrability of nonlinear evolution equations. We aim to study the Painlevé integrability of the pKP–BKP Eq. (5), thus we follow the Painlevé analysis method presented in Refs. [6–20] and in some of the references therein.

2.2 Painlevé Analysis. The Painlevé integrability of nonlinear partial differential equations (PDEs) can be examined using Painlevé analysis. It is important to know that the meaning of the Painlevé integrability of nonlinear PDEs is that the solution is single-valued in the vicinity of a movable singularity manifold. Weiss, Tabor, and Carnevale (WTC)[6] developed an algorithm (the WTC method) to study the compatibility criteria for Painlevé integrability. Using the Mathematica code for testing integrability shows that this equation is Painlevé integrable provided

$$\mu=-\frac{\gamma^2}{5\alpha},\quad \alpha\neq 0. \tag{6}$$

Note here that for $\gamma=0$, this part will be investigated independently later.

This in turn gives the pKP–BKP equation as

$$\begin{aligned} &u_{\mathrm{xt}}+\alpha(15(u_x)^3+15u_xu_{xxx}+u_{xxxxx})_x\\ &+\beta(6u_xu_{xx}+u_{xxxx})+\gamma(u_{xxxy}+3(u_xu_y)_x)\\ &+au_{xx}+bu_{xy}+cu_{xz}-\frac{\gamma^2}{5\alpha}u_{yy}=0. \end{aligned} \tag{7}$$

Equation (7) is assumed to have a solution that is a Laurent expansion about a singular manifold $\psi=\psi(x,\,y,\,z,\,t)$ as

$$u(x,\,y,\,z,\,t)=\sum_{k=0}^{\infty}u_k(x,\,y,\,z,\,t)\,\psi^{k-\gamma_1}. \tag{8}$$

The Painlevé test consists of three main steps:[4–20] (i) computing leading order and coefficients, (ii) determining resonant points, and (iii) verifying compatibility conditions. In what follows, we will examine each concept.

(i) *Leading-Order Behavior and Coefficients.* To obtain the leading-order behavior and coefficients, we substitute the ansatz

$$u(x,\,y,\,z,\,t)=u_0\phi^{\alpha_1} \tag{9}$$

into Eq. (5) to obtain two distinct cases, given as

$$\text{(i) } \alpha_1=-1,\ u_0=-2\phi_x,\quad \text{(ii) } \alpha_1=-1,\ u_0=-4\,\phi_x. \tag{10}$$

(ii) *Resonant Points.* We aim to determine the resonant points, which are those values of j at which it is possible to introduce arbitrary functions into the Laurent series

$$u(x,\, y,\, z,\, t) = \sum_{j=0}^{\infty} u_j\, \phi^{j+k}, \tag{11}$$

and it is single-valued in the neighborhood of singularity manifold ϕ. To achieve this aim, we insert

$$u(x,\, y,\, z,\, t) = u_0\phi^{-1} + u_j\phi^{j-1} \tag{12}$$

into Eq. (7), following the WTC analysis [5] and, by balancing the most dominant terms, we obtain: (i) the principal branch: $k = -1, 1, 2, 3, 6, 10$, (ii) the secondary branch: $k = -2, -1, 1, 5, 6, 12$, where each branch includes six resonance points due to the sixth order of the linear structure of Eq. (7).

(iii) *Verifying Compatibility Conditions.* To verify the compatibility conditions, we follow the works in Refs. [6–16]. For the principal branch (i), the resonance at $k = -1$ corresponds to the arbitrariness of singular manifold $\psi(x,\, y,\, z,\, t) = 0$. Moreover, the Painlevé compatibility works for levels $1, 2, 3, 6$, and 10.

For the secondary branch we ignore $k = -2$, because it is < -1. The Painlevé works for levels 1, 5, 6, and 12. Based on this, we conclude that the pKP–BKP Eq. (7) is Painlevé integrable.

2.3 Multiple-Soliton Solutions. Substituting

$$u(x,\, y,\, z,\, t) = e^{\theta_i},\ \theta_i = k_i x + r_i y + s_i z - c_i t \tag{13}$$

into the linear terms of Eq. (7) gives the dispersion relation by

$$c_i = \frac{\alpha k_i^6 + \beta k_i^4 + \gamma k_i^3 r_i + a k_i^2 + b k_i r_i + c k_i s_i - \frac{\gamma^2}{5\alpha}\, r_i^2}{k_i}, \tag{14}$$

and this gives

$$\theta_i = k_i x + r_i y + s_i z - \frac{\alpha k_i^6 + \beta k_i^4 + \gamma k_i^3 r_i + a k_i^2 + b k_i r_i + c k_i s_i - \frac{\gamma^2}{5\alpha} r_i^2}{k_i}\, t, \tag{15}$$

for $i = 1,\ 2, \ldots,\ N$. The multi-soliton solution of Eq. (7) is obtained by following Hirota's method and plugging the transformation

$$u(x,\, y,\, z,\, t) = 2(\ln f(x,\, y,\, z,\, t))_x = 2\frac{f_x}{f}, \tag{16}$$

where the auxiliary function $f(x,\, y,\, z,\, t)$ for the single-front wave solution [10–30] is given by

$$\begin{aligned} &f(x,\, y,\, z,\, t)\\ &= 1 + e^{\theta_1}\\ &= 1 + e^{k_1 x + r_1 y + s_1 z - \frac{\alpha k_1^6 + \beta k_1^4 + \gamma k_1^3 r_1 + a k_1^2 + b k_1 r_1 + c k_1 s_1 - \frac{\gamma^2}{5\alpha} r_1^2}{k_1} t}. \end{aligned} \tag{17}$$

Substituting Eq. (17) into Eq. (16) gives the single-front wave solution

$$u(x,\, y,\, z,\, t) = \frac{2k_1\, e^{k_1 x + r_1 y + s_1 z - \frac{\alpha k_1^6 + \beta k_1^4 + \gamma k_1^3 r_1 + a k_1^2 + b k_1 r_1 + c k_1 s_1 - \frac{\gamma^2}{5\alpha} r_1^2}{k_1} t}}{1 + e^{k_1 x + r_1 y + s_1 z - \frac{\alpha k_1^6 + \beta k_1^4 + \gamma k_1^3 r_1 + a k_1^2 + b k_1 r_1 + c k_1 s_1 - \frac{\gamma^2}{5\alpha} r_1^2}{k_1} t}}. \tag{18}$$

Figure 1 shows the single-front wave solution (18) where all parameters are substituted by 1 for each.

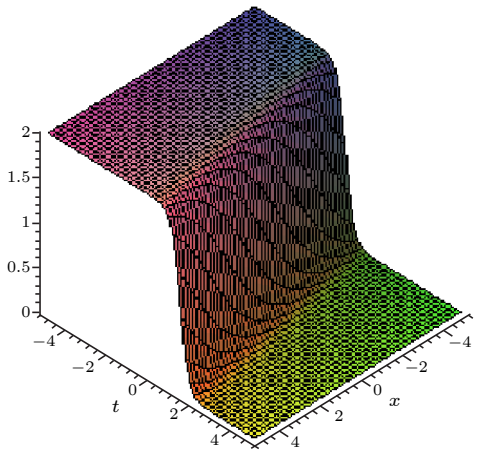

Fig. 1. Profile of the front wave solution for $y = 1$, $z = 1$.

For the two-soliton solutions we use the auxiliary function as

$$f(x,\, y,\, z,\, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{19}$$

where a_{12} is the phase shift of the interaction of solitons. To determine the phase shift a_{12}, we substitute Eq. (19) into Eq. (7) and solving for the phase shift a_{ij} we obtain

$$\begin{aligned} a_{ij} = \Big[&25\alpha^2 k_i^2 k_j^2 (k_i^2 - k_i k_j + k_j^2)(k_i - k_j)^2\\ &+ 15\alpha\beta k_i^2 k_j^2 (k_i - k_j)^2 + 5\alpha\gamma k_i k_j (k_i\\ &- k_j)(k_i^2 r_j + 2k_i k_j r_i - 2k_i k_j r_j - k_j^2 r_i)\\ &+ \gamma^2 (k_i r_j - k_j r_i)^2\Big] \Big/ \Big[25\alpha^2 k_i^2 k_j^2 (k_i^2 + k_i k_j + k_j^2)(k_i\\ &+ k_j)^j + 15\alpha\beta k_i^2 k_j^2 (k_i + k_j)^2 + 5\alpha\gamma k_i k_j (k_i\\ &+ k_j)(k_i^2 r_j + 2k_i k_j r_i + 2k_i k_j r_2 + k_j^2 r_i)\\ &+ \gamma^2 (k_i r_j - k_j r_i)^2\Big], \end{aligned} \tag{20}$$

for $1 < j \leq 3$. The result (20) shows that the phase shifts (20) depend on the parameters α, β and γ, and do not depend on the a, b and c coefficients of the terms u_{xx}, u_{xy} and u_{xz}, respectively. The two-soliton solutions are obtained by substituting Eqs. (19) and (20) into Eq. (16).

For the three-soliton solutions, we apply the auxiliary

function $f(x, y, z, t)$ as

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}. \tag{21}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{22}$$

The three-soliton solutions are obtained by substituting Eq. (21) into Eq. (16). This also shows that N-soliton solutions can be obtained for finite N, where $N \geq 1$.

2.4 Lump Solutions. It is well known that the lump solution is a special kind of rational function solution localized in all spatial directions. However, a soliton solution is an analytic solution that is exponentially localized in all directions in space of x, y and z and time t.[6,20–35] Lump solutions arise when surface tension dominates the shallow water surface, as in plasmas, optical media and other physical applications. The basis of symbolic computation method, the generalized positive quadratic function, is a powerful technique to study lump solutions. In this section, we derive a class of lump solutions for arbitrary values of the parameters. We first transform the (3+1)-dimensional pKP–BKP Eq. (7) into a bilinear equation in operator form, given as

$$\Big(D_xD_t + \alpha D_x^6 + \beta D_x^4 + \gamma D_x^3 D_y + aD_x^2 + bD_xD_y + cD_xD_z - \frac{\gamma^2}{5\alpha}D_y^2\Big) f\cdot f = 0, \tag{23}$$

where D_t, D_x, D_y, and D_z are the Hirota bilinear derivative operators. Equation (23) can be transformed to

$$\begin{aligned}&(ff_{\rm xt} - f_xf_t) + \alpha(ff_{xxxxxx} - 6f_xf_{xxxxx} + 15f_{xxxx}f_{xx} - 10(f_{xxx})^2) + \beta(ff_{xxxx} - 4f_{xxx}f_x + 3(f_{xx})^2)\\ &+ \gamma(ff_{xxxy} - 3f_xf_{xxy} + 3f_{xx}f_{xy} - f_{xxx}f_y) + a(ff_{xx} - f_xf_x) + b(ff_{xy} - f_xf_y) + c(ff_{xz} - f_xf_z)\\ &- \frac{\gamma^2}{5\alpha}(ff_{yy} - f_yf_y) = 0,\end{aligned} \tag{24}$$

obtained upon using

$$u(x, y, z, t) = 2(\ln f(x, y, z, t))_x. \tag{25}$$

To obtain the quadratic soliton solutions for the (3+1)-dimensional pKP–BKP Eq. (7), we set the following assumptions:

$$\begin{aligned}g &= a_1x + a_2y + a_3z + a_4t + a_5,\\ h &= a_6x + a_7y + a_8z + a_9t + a_{10},\\ f &= g^2 + h^2 + a_{11},\end{aligned} \tag{26}$$

where a_j, $1 \leq j \leq 11$, are real parameters that we will be formally determined. Substituting Eq. (26) into Eq. (24), we get a polynomial of the variables x, y, z, and t. To determine the parameters a_j, $1 \leq j \leq 11$, we build up a system of equations of the coefficients of t^2, xt, yt, zt, t, z^2, xz, yz, z, y^2, xy, y, x^2, x and the constant terms. By solving this system using Maple, we obtain the following selected sets of constraining equations on the various parameters, noting that other sets can be obtained:

Case 1. To determine the first set of lump solutions, we use the first set of selections of the parameters, and obtain

$$\begin{aligned}&a_1 = a_1,\ a_3 = a_3,\ a_5 = a_5,\ a_6 = a_6,\ a_9 = a_9,\\ &a_{10} = a_{10},\ a_{11} = a_{11} > 0,\\ &a_2 = -a_1,\ a_4 = \frac{1}{5}a_1 - a_3,\ a_7 = -a_6,\end{aligned} \tag{27}$$

where $a_{11} > 0$ and, for simplicity, we select $\alpha = \beta = \gamma = a = b = c = 1$. The obtained parameters (27) generate the class of positive quadratic function solutions upon substituting (27) into (26). This, in turn, gives a first class of lump solutions to the (3+1)-dimensional pKP–BKP equation by using $u = 2(\ln f(x, y, z, t))_x$ as follows.

For example, selecting

$$\begin{aligned}&a_1 = 1,\ a_3 = 1,\ a_5 = 1,\ a_6 = 2,\\ &a_9 = 1,\ a_{10} = 2,\ a_{11} = 1,\end{aligned} \tag{28}$$

and using Eq. (27) gives

$$a_2 = -1,\ a_4 = -\frac{4}{5},\ a_7 = -2,\ a_8 = -\frac{8}{5}, \tag{29}$$

which will give the lump solution

$$u(x, y, z, t) = \frac{2(10x - 10y - \frac{22}{5}z + \frac{32}{5}t + 10)}{(x - y + z - \frac{4}{5}t + 1)^2 + (2x - 2y - \frac{8}{5}z + 2t + 2)^2 + 1}. \tag{30}$$

Note that the obtained lump solutions $u(x, y, z, t) \to 0$ if and only if $g^2 + h^2 \to \infty$.

Figure 2 shows the lump solution (30) where all parameters are substituted by 1.

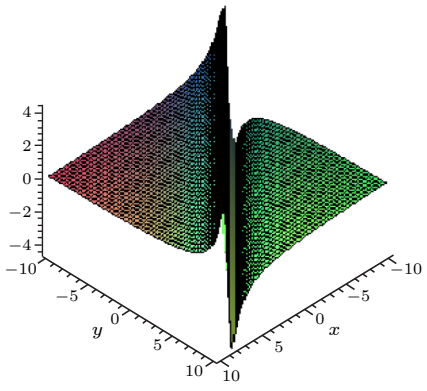

Fig. 2. Profile of the lump solution for $y = 1$, $z = 1$.

Case 2. In this case, we select

$$\begin{aligned}&a_1 = a_1,\ a_2 = a_2,\ a_3 = a_3,\ a_5 = a_5,\\ &a_6 = a_6,\ a_7 = a_7,\ a_9 = a_9,\ a_{10} = a_{10},\\ &a_4 = -\frac{5(a_1 + a_2 + a_3)(a_1^2 + a_6^2) - a_1(a_2^2 - a_7^2) - 2a_2a_6a_7}{5(a_1^2 + a_6^2)},\end{aligned}$$

$$a_8 = -\frac{5(a_6+a_7+a_9)(a_1^2+a_6^2)+a_6(a_2^2-a_7^2)-2a_1a_2a_7}{5(a_1^2+a_6^2)},$$
$$a_{11} = \frac{15(a_1^2+a_6^2)^2(a_1^2+a_1a_2+a_6^2+a_6a_7)}{(a_1a_7-a_2a_6)^2}, \tag{31}$$

which needs to satisfy the determinant condition

$$\Delta = (a_1a_7 - a_2a_6) = \begin{vmatrix} a_1 & a_2 \\ a_6 & a_7 \end{vmatrix} \neq 0, \tag{32}$$

to guarantee a well-defined function $f(x, y, z, t)$, its positiveness and the localization of $u(x, y, z, t)$ in all directions in the space, respectively. The obtained parameters (31) generate the class of positive quadratic function solutions upon substituting (31) in (26). This in turn will give a first class of lump solutions to the (3+1)-dimensional pKP–BKP equation by using $u = 2(\ln f(x, y, z, t))_x$.

For example, selecting

$$a_1 = 1,\ a_2 = 2,\ a_3 = 1,\ a_5 = 1,$$
$$a_6 = 2,\ a_7 = 1,\ a_9 = 1,\ a_{10} = 2, \tag{33}$$

and using Eq. (27) gives

$$a_4 = -\frac{89}{25},\ a_8 = -\frac{127}{25},\ a_{11} = 375, \tag{34}$$

and for simplicity we select $\alpha = \beta = \gamma = a = b = c = 1$, which will give the lump solution

$$\begin{aligned} &u(x, y, z, t) \\ &= \Big[2\Big(10x + 8y - \frac{458}{25}z + \frac{22}{25}t + 10\Big)\Big] \Big/ \Big[\Big(x + 2y + z \\ &\quad - \frac{89}{25}t + 1\Big)^2 + \Big(2t + 2x + y - \frac{127}{25}z + 2t + 2\Big)^2 + 375\Big]. \end{aligned} \tag{35}$$

3. New (3+1)-Dimensional pKP-BKP Equation: New Model for $\gamma = 0$. As stated earlier, we will examine the case of $\gamma = 0$; hence Eq. (7) becomes

$$\begin{aligned} &u_{xt} + \alpha(15(u_x)^3 + 15u_xu_{xxx} + u_{xxxxx})_x \\ &+ \beta(6u_xu_{xx} + u_{xxxx}) + au_{xx} + bu_{xy} + cu_{xz} = 0. \end{aligned} \tag{36}$$

Note that the analysis we will perform on Eq. (36) keeps it in this form. This is due to the fact that integrating this equation with respect to x will provide a non-integrable form. We will start by first using the Painlevé test to study Painlevé integrability of Eq. (36).

3.1 Painlevé Analysis and Integrability. Proceeding as presented in the above section, we obtain the same results derived earlier for Eq. (7) where $\gamma \neq 0$. Recall that by using the WTC analysis [6] and balancing the most dominant terms, we obtain the following two branches of resonance points: (i) the principal branch: $k = -1,\ 1,\ 2,\ 3,\ 6,\ 10$, (ii) the secondary branch: $k = -2,\ -1,\ 1,\ 5,\ 6,\ 12$, where each branch includes six resonance points due to the sixth order of the linear structure of Eq. (36). Using the same concept shows that Eq. (36) is Painlevé integrable, hence we do not go into further detail.

3.2 Multiple-Soliton Solutions: Case for $\gamma = 0$. Substituting

$$u(x, y, z, t) = e^{\theta_i},\quad \theta_i = k_ix + r_iy + s_iz - c_it, \tag{37}$$

into the linear terms of Eq. (36) yields the dispersion relation by

$$c_i = \alpha k_i^5 + \beta k_i^3 + ak_i + br_i + cs_i, \tag{38}$$

and this gives

$$\begin{aligned} \theta_i =& k_ix + r_iy + s_iz \\ &- (\alpha k_i^5 + \beta k_i^3 + ak_i + br_i + cs_i)t, \end{aligned} \tag{39}$$

for $i = 1,\ 2, \ldots,\ N$. The multi-front wave solutions of Eq. (36) are obtained by following Hirota's method and plugging the transformation

$$u(x, y, z, t) = 2(\ln f(x, y, z, t))_x = 2\frac{f_x}{f}, \tag{40}$$

where the auxiliary function $f(x, y, z, t)$ for the single-front wave solution [10–30] is given by

$$\begin{aligned} f(x, y, z, t) =& 1 + e^{\theta_1} \\ =& 1 + e^{k_1x + r_1y + s_1z - (\alpha k_1^5 + \beta k_1^3 + ak_1 + br_1 + cs_1)\,t}. \end{aligned} \tag{41}$$

Substituting Eq. (41) into Eq. (40) gives the single-front wave solution

$$u(x, y, z, t) = \frac{2k_1\, e^{k_1x + r_1y + s_1z - (\alpha k_1^5 + \beta k_1^3 + ak_1 + br_1 + cs_1)\,t}}{1 + e^{k_1x + r_1y + s_1z - (\alpha k_1^5 + \beta k_1^3 + ak_1 + br_1 + cs_1)t}}. \tag{42}$$

For the two-soliton solutions, we use the auxiliary function as

$$f(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2}, \tag{43}$$

where a_{12} is the phase shift of the interaction of solitons. To determine the phase shift a_{12}, we substitute Eq. (43) into Eq. (36), and solving for the phase shift a_{12}, after we generalize

$$a_{ij} = \frac{(k_i - k_j)^2(5\alpha(k_i^2 - k_ik_j + k_j^2) + 3\beta)}{(k_i + k_j)^2(5\alpha(k_i^2 + k_ik_j + k_j^2) + 3\beta)}. \tag{44}$$

This result shows that the phase shifts (44) depend on the parameters α and β and not on the a, b, and c coefficients of the terms u_{xx}, u_{xy}, and u_{xz}, respectively. The two-soliton solutions are obtained by substituting Eqs. (43) and (44) into Eq. (40).

For the three-soliton solutions, we proceed as presented earlier.

4. New (3+1)-Dimensional pKP–BKP Equation: New Model for $\beta = 0$. For $\beta = 0$, the (3+1)-dimensional pKP–BKP Eq. (7) can be rewritten as

$$\begin{aligned} &u_{xt} + \alpha(15(u_x)^3 + 15u_xu_{xxx} + u_{xxxxx})_x + \gamma(u_{xxxy} \\ &+ 3(u_xu_y)_x) + au_{xx} + bu_{xy} + cu_{xz} + \mu u_{yy} = 0, \end{aligned} \tag{45}$$

where two terms of the pKP equation are removed from the pKP–BKP equation. We will start by using the Painlevé test to study the Painlevé integrability of this equation.

4.1 Painlevé Analysis and Integrability. There are many significant properties, such as Lax pairs, Hamiltonian structures, infinitely many symmetries, and infinite conservation laws, that can characterize the integrability of nonlinear evolution equations. We aim to study the Painlevé integrability of pKP–BKP Eq. (45). We follow the Painlevé analysis method presented in Refs. [6–20] and in some of the references therein.

4.2 Painlevé Analysis. The Painlevé integrability of nonlinear PDEs can be examined using Painlevé analysis. It is important to know that the meaning of the Painlevé integrability of nonlinear PDEs is that the solution is single-valued in the vicinity of a movable singularity manifold. WTC[6] developed an algorithm (the WTC method) to study the compatibility criteria for Painlevé integrability. Using the Mathematica code for testing integrability shows that this equation is Painlevé integrable provided

$$\mu = -\frac{\gamma^2}{5\alpha}, \quad \alpha\gamma \neq 0. \tag{46}$$

This in turn gives the new model of the pKP–BKP equation, for $\beta = 0$, as

$$\begin{aligned} &u_{xt} + \alpha(15(u_x)^3 + 15u_x u_{xxx} + u_{xxxxx})_x \\ &+ \gamma(u_{xxxy} + 3(u_x u_y)_x) + a u_{xx} + b u_{xy} + c u_{xz} \\ &- \frac{\gamma^2}{5\alpha}\, u_{yy} = 0. \end{aligned} \tag{47}$$

Equation (47) is assumed to have a solution that is a Laurent expansion about a singular manifold $\psi = \psi(x,\, y,\, z,\, t)$ as

$$u(x,\, y,\, z,\, t) = \sum_{k=0}^{\infty} u_k(x,\, y,\, z,\, t)\, \psi^{k-\gamma_1}. \tag{48}$$

Following the WTC analysis,[6] and balancing the most dominant terms, we obtain the following two branches of resonance points, namely: (i) the principal branch: $k = -1,\, 1,\, 2,\, 3,\, 6,\, 10$, (ii) the secondary branch: $k = -2,\, -1,\, 1,\, 5,\, 6,\, 12$, where each branch includes six resonance points due to the sixth order of the linear structure of Eq. (47). The obtained results in this case are the same as those obtained for the standard Eq. (7). As a result, we conclude that the pKP–BKP Eq. (47) is Painlevé integrable.

4.3 Multiple-Soliton Solutions. Substituting

$$u(x,\, y,\, z,\, t) = e^{\theta_i}, \theta_i = k_i x + r_i y + s_i z - c_i t \tag{49}$$

into the linear terms of Eq. (47) yields the dispersion relation by

$$c_i = \frac{\alpha k_i^6 + \gamma k_i^3 r_i + a k_i^2 + b k_i r_i + c k_i s_i - \frac{\gamma^2}{5\alpha} r_i^2}{k_i}, \; k_i \neq 0, \tag{50}$$

and this gives

$$\begin{aligned} \theta_i =& k_i x + r_i y + s_i z \\ &- \frac{\alpha k_i^6 + \gamma k_i^3 r_i + a k_i^2 + b k_i r_i + c k_i s_i - \frac{\gamma^2}{5\alpha} r_i^2}{k_i}\, t, \end{aligned} \tag{51}$$

for $i = 1,\, 2, \ldots,\, N$. The multi-soliton solutions of Eq. (47) are obtained by following Hirota's method and plugging the transformation

$$u(x,\, y,\, z,\, t) = 2(\ln f(x,\, y,\, z,\, t))_x = 2\frac{f_x}{f}, \tag{52}$$

where the auxiliary function $f(x,\, y,\, z,\, t)$ for the single-front wave solution[10–30] is given by

$$\begin{aligned} &f(x,\, y,\, z,\, t) \\ &= 1 + e^{\theta_1} \\ &= 1 + e^{k_1 x + r_1 y + s_1 z - \frac{\alpha k_1^6 + \gamma k_1^3 r_1 + a k_1^2 + b k_1 r_1 + c k_1 s_1 - \frac{\gamma^2}{5\alpha} r_1^2}{k_1} t}. \end{aligned} \tag{53}$$

Substituting Eq. (53) into Eq. (52) gives the single-front wave solution

$$\begin{aligned} &u(x,\, y,\, z,\, t) \\ &= \frac{2k_1\, e^{k_1 x + r_1 y + s_1 z - \frac{\alpha k_1^6 + \gamma k_1^3 r_1 + a k_1^2 + b k_1 r_1 + c k_1 s_1 - \frac{\gamma^2}{5\alpha} r_1^2}{k_1} t}}{1 + e^{k_1 x + r_1 y + s_1 z - \frac{\alpha k_1^6 + \gamma k_1^3 r_1 + a k_1^2 + b k_1 r_1 + c k_1 s_1 - \frac{\gamma^2}{5\alpha} r_1^2}{k_1} t}}. \end{aligned} \tag{54}$$

For the two-soliton solutions we set

$$f(x,\, y,\, z,\, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{55}$$

where a_{12} is the phase shift of the interaction of solitons. To determine the phase shift a_{12}, we substitute Eq. (55) into Eq. (47), solving for the phase shift a_{12}, and hence we obtain

$$a_{ij} = \frac{25\alpha^2 k_i^2 k_j^2 (k_i^2 - k_i k_j + k_j^2)(k_i - k_j)^2 + 5\alpha\gamma k_i k_j (k_i - k_j)(k_i^2 r_j + 2k_i k_j r_i - 2k_i k_j r_j - k_j^2 r_i) + \gamma^2 (k_i r_j - k_j r_i)^2}{25\alpha^2 k_i^2 k_j^2 (k_i^2 + k_i k_j + k_j^2)(k_i + k_j)^j + 5\alpha\gamma k_i k_j (k_i + k_j)(k_i^2 r_j + 2k_i k_j r_i + 2k_i k_j r_2 + k_j^2 r_i) + \gamma^2 (k_i r_j - k_j r_i)^2}. \tag{56}$$

This result shows that the phase shifts (56) depend on the parameters α and γ and not on the coefficients a, b and c of the terms u_{xx}, u_{xy} and u_{xz}, respectively. The two-soliton solutions are obtained by substituting Eqs. (56) and (55) into Eq. (52).

For the three-soliton solutions, we proceed as presented above to derive the three-soliton solutions.

4.4 Lump Solutions. Proceeding as before, we first transform the (3+1)-dimensional pKP–BKP Eq. (47) into a bilinear equation in operator form given as

$$\begin{aligned} &\Big(D_x D_t + \alpha D_x^6 + \gamma D_x^3 D_y + a D_x^2 + b D_x D_y + c D_x D_z \\ &- \frac{\gamma^2}{5\alpha}\, D_y^2\Big) f \cdot f = 0, \end{aligned} \tag{57}$$

where D_t, D_x, D_y, and D_z are Hirota's bilinear derivative operators. Equation (57) can be transformed to

$$(ff_{\rm xt} - f_x f_t) + \alpha(ff_{xxxxxx} - 6f_x f_{xxxxx} + 15f_{xxxx}f_{xx} - 10(f_{xxx})^2) + \gamma(ff_{xxxy} - 3f_x f_{xxy} + 3f_{xx}f_{xy} - f_{xxx}f_y) + a(ff_{xx} - f_x f_x) + b(ff_{xy} - f_x f_y) + c(ff_{xz} - f_x f_z) - \frac{\gamma^2}{5\alpha}(ff_{yy} - f_y f_y) = 0, \tag{58}$$

obtained upon using

$$u(x,\, y,\, z,\, t) = 2(\ln f(x,\, y,\, z,\, t))_x. \tag{59}$$

To obtain the quadratic soliton solutions for the (3+1)-dimensional pKP–BKP Eq. (47), we set the following assumptions:

$$\begin{aligned} g &= a_1 x + a_2 y + a_3 z + a_4 t + a_5, \\ h &= a_6 x + a_7 y + a_8 z + a_9 t + a_{10}, \\ f &= g^2 + h^2 + a_{11}. \end{aligned} \tag{60}$$

Proceeding as before, we find the following two cases.

Case 1. To determine the first set of lump solutions, we use the first set of selections of the parameters, and obtain

$$\begin{aligned} &a_1 = a_1,\ a_4 = a_4,\ a_5 = a_5,\ a_6 = 0,\ a_7 = a_7,\ a_8 = a_8, \\ &a_9 = 0,\ a_{10} = 0,\ a_2 = \frac{5a_1(a_7 + a_8)}{2a_7}, \\ &a_3 = -\frac{45a_1^2a_7^2 - 25a_1^2a_8^2 + 20a_1a_4a_7^2 + 4a_7^4}{20a_1a_7^2}, \end{aligned}$$

$$a_{11} = \frac{75a_1^4(a_7 + a_8)}{2a_7^3},\ a_7a_8 > 0,\ a_1 \neq 0, \tag{61}$$

where $a_{11} > 0$, $a_7a_8 > 0$ and $a_6 = a_9 = a_{10} = 0$. For simplicity, we select $\alpha = \gamma = a = b = c = 1$. The obtained parameters (61) generate the class of positive quadratic function solutions upon substituting (61) in (60). This in turn will give a first class of lump solutions to the (3+1)-dimensional pKP–BKP equation by using $u = 2(\ln f(x,\, y,\, z,\, t))_x$.

For example, selecting

$$\begin{aligned} &a_1 = 1,\ a_4 = 2,\ a_5 = 1,\ a_6 = 0,\ a_7 = 1,\ a_8 = 2, \\ &a_9 = 0,\ a_{10} = 0, \end{aligned} \tag{62}$$

and using Eq. (61) give

$$a_2 = \frac{15}{2},\ a_3 = \frac{11}{20},\ a_{11} = \frac{225}{2}, \tag{63}$$

which will give the lump solution

$$u(x,\, y,\, z,\, t) = \frac{2(2x + 15y + \frac{11}{10}z + 4t + 2)}{(x + \frac{15}{2}y + \frac{11}{20}z + 2t + 1)^2 + (y + 2z)^2 + \frac{225}{2}}. \tag{64}$$

Case 2. In this case, we select

$$\begin{aligned} &a_1 = a_1,\ a_2 = a_2,\ a_5 = a_5,\ a_6 = a_6,\ a_7 = a_7,\ a_8 = a_8, \\ &a_3 = \frac{5a_1^3(a_7 + a_8) - a_1^2a_2(5a_6 + 2a_7) + a_1a_6(2a_2^2 + 5a_6a_7) + a_1a_6(5a_6a_8 - 2a_7^2) - a_2a_6^2(5a_6 - 2a_7)}{5a_6(a_1^2 + a_6^2)}, \\ &a_4 = -\frac{a_1(5a_1^2(a_6 + a_7) + a_1(5a_1a_8 - 2a_2a_7) + a_6(a_2^2 + 5a_6^2) + a_6a_7(5a_6 - a_7) + 5a_6^2a_8)}{5a_6(a_1^2 + a_6^2)}, \\ &a_9 = a_4a_6,\ a_{10} = \frac{a_5a_6}{a_1},\ a_{11} = \frac{15(a_1^2 + a_6^2)^2(a_1a_2 + a_6a_7)}{(a_1a_7 - a_2a_6)^2}. \end{aligned} \tag{65}$$

For $a_1 \neq 0$, $a_1a_7 - a_2a_6 \neq 0$, it needs to satisfy the determinant condition to guarantee a well-defined function $f(x,\, y,\, z,\, t)$, its positiveness and the localization of $u(x,\, y,\, z,\, t)$ in all directions in the space, respectively. The obtained parameters (65) generate the class of positive quadratic function solutions upon substituting (65) in (60). This in turn will give a first class of lump solutions to the (3+1)-dimensional pKP–BKP equation by using $u = 2(\ln f(x,\, y,\, z,\, t))_x$.

5. Discussion. In this work, we have introduced a (3+1)-dimensional combined pKP–BKP equation, confirmed its Painlevé integrability and derived N-soliton solutions. Moreover, we derive a set of lump solutions with distinct parameters. We are able to develop from this equation to (3+1)-dimensional combined pKP–BKP models. The analysis confirms the integrability of each of the newly developed equations and, thus, the integrability of the new developed equations is retrieved. Moreover, we also derive lump solutions for each of the two developed models. The last two integrable models presented in this work show the construction of integrable models from an integrable model through deleting some terms from the original equation and not through extending or combining integrable equations.

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