Entropy of Regular Black Holes in Einstein's Gravity

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(Received 15 August 2023; accepted manuscript online 20 October 2023)

We calculate the entropy of spherically symmetric regular black holes by the path-integral method in Einstein's gravity. This method provides evidence that the entropy of spherically symmetric regular black holes is proportional to a quarter of horizon area, indicating no violation of the entropy-area law.

DOI: 10.1088/0256-307X/40/12/120401

Research of thermodynamics of black holes (BHs) originates [1,2] from the entropy-area law, which establishes a proportional relationship between the entropy of a BH and the area of its outermost horizon. BH entropy has an interesting characteristic in that it is not solely determined by the properties of spacetime, but also influenced [3] by specific formulations of gravitational theories, where the former shows the geometric nature of entropy, while the latter represents the quantum nature of entropy because an action shows a quantum phase, i.e., one classical state corresponds to multiple degenerate quantum states. Through analyzing the geometric and quantum natures of entropy for regular black holes (RBHs) in Einstein's gravity, we shall verify the corresponding entropy-area law.

Before we verify whether the entropy-area law holds or not in RBHs governed by Einstein's gravity, it is important to note that deviations from the entropy-area law have already been observed in singular black holes (SBHs), which provides the background of our motivation for this study. We illustrate once such deviation that is closely related to the characteristic mentioned above. Given a metric of non-Ricci flat BHs (a manifold will be Ricci flat if its Ricci curvature vanishes; conversely, a non-Ricci flat manifold implies that the Ricci curvature does not vanish), the corresponding entropy cannot be determined uniquely because the spacetime metric can be solved from several gravitational theories. We take the Schwarzschild–de Sitter(dS)/anti-de Sitter (AdS) BH,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{1}$$

which is an SBH as an example, and shows that its entropy has three different formulations that correspond to three gravitational theories:

(1) Equation (1) is an exact solution [4] of Einstein's gravity, $\mathcal{L}_{\rm EH} \propto R - 2\Lambda$, which describes one of Einstein's manifolds with a non-vanishing cosmological constant Λ. In this case, the Wald entropy equals [5]

$$S_1 = \frac{A}{4G}, \tag{2}$$

which maintains the entropy-area law.

(2) Equation (1) is also the solution of the modified gravity $F(R)$, $\mathcal{L}_{\rm F} \propto R^2$. In this case, the Wald entropy takes [6,7] the form,

$$S_2 = \frac{A}{4G} \times 2\Lambda, \tag{3}$$

which breaks the entropy-area law.

(3) Equation (1) is still the solution [8,9] of the conformal gravity described by the Lagrangian density, $\mathcal{L}_{\rm cof} \propto W^{\mu\nu\alpha\beta}W_{\mu\nu\alpha\beta}$, because it leads to a vanishing Bach tensor, $B_{\alpha\beta} \equiv (2\nabla^\mu\nabla^\nu + R^{\mu\nu})W_{\mu\alpha\nu\beta} = 0$, where $W_{\mu\nu\alpha\beta}$ is the Weyl tensor. In this case, the corresponding Wald entropy reads

$$S_3 = \frac{A}{4G}\Big(\frac{1}{r_{\rm H}^2} - \frac{\Lambda}{3}\Big), \tag{4}$$

which still breaks the entropy-area law. Note that the first term is contributed [10] by the Gauss–Bonnet gravity, $\mathcal{L}_{\rm GB} \propto R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} + R^2 - 4R^{\mu\nu}R_{\mu\nu}$, and the second term comes from $-2R^2/3 + 2R^{\mu\nu}R_{\mu\nu}$, where the Ricci decomposition, $W^{\mu\nu\alpha\beta}W_{\mu\nu\alpha\beta} = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} + R^2/3 - 2R^{\mu\nu}R_{\mu\nu}$, has been applied in the derivation of Eq. (4).

According to the above discussions, we have shown that the Schwarzschild–dS/AdS BH as an SBH exhibits no inconsistency between its geometric and quantum properties within the framework of Einstein's gravity (see the first case). However, conflicts arise when considering modified and conformal gravity theories (see the second and third cases). For the latter situation, the entropy-area law can be restored by redefining, [7] which is a new Newtonian constant that is able to absorb additional factors. In this sense, the Newtonian constant depends on gravitational theories. To sum up, for SBHs, the entropy is dictated not only by the metric of spacetime (geometry), but also by the specific formulation of gravitational theory (quantum). The conflict between the geometric and quantum aspects can be resolved by redefining a Newtonian constant.

Now let us turn to RBHs, which were introduced with the intention of addressing the issue of ultraviolet incompleteness. RBHs are considered as quantum corrections to SBHs. For a more comprehensive understanding of RBHs, see, for instance, Ref. [11]. The concept of *regularity* implies that the spacetime of RBHs only contains [12] horizons

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as coordinate singularities and bypasses [13,14] the restriction of Penrose's singularity theorem due to there being no requirement for the strong energy condition. There are two non-equivalent definitions of regularity. One is that the curvature invariants are finite everywhere and the other is that the null and timelike geodesics are complete. In the present study, we use the first definition of regularity as the preferable one, which is related [15] to Markov's limiting curvature conjecture (this conjecture states that the curvature invariants must be uniformly restricted by a certain universal value) and is based on the concept that the singularities of spacetime should be judged by a complete set of curvature invariants for a given Riemann tensor. The elements in this set are dubbed Zakhary–Mcintosh (ZM) invariants. [16]

RBHs are not Einstein manifolds (this statement is obvious for spherically symmetric BHs because of Birkhoff's theorem, which states that any spherically symmetric solution of the vacuum Einstein equation must be the Schwarzschild metric; specifically, observing the algebraic properties of Ricci tensor R^{μ}_{ν}, one can find that the equations, $R^t_t = R^r_r$ and $R^t_t = R^{\theta}_{\theta}$, only lead to spherically symmetric SBHs), i.e., the Ricci tensor of RBHs is not proportional to the metric of RBHs. Therefore, the entropy of RBHs associated with different gravitational theories will vary. Let us clarify what is special for RBHs when compared to SBHs. Firstly, the conflict between the geometric and quantum aspects arises even in Einstein's gravity, leading to a violation of the entropy-area law. Secondly, the violation cannot be rectified by redefining the Newtonian constant, as was done for SBHs. Lastly, the entropy is not the only variable that deviates from its expected value in the thermodynamic formulas of RBHs. In fact, when an RBH's metric includes multiple parameters, such as mass and charge, several deviation terms will be present in the first law of thermodynamics, [17] indicating that there is no straightforward linear relationship between the thermodynamic and mechanical variables of RBHs.

The entropy problem of RBHs we are facing is that the entropy of any RBH calculated by Wald's formalism [5] or the semiclassical approach of Euclidean gravity [18] is not consistent with that obtained from the thermodynamic formula (although T and S are regarded as quantum variables, the term TdS does not contain the Planck constant; thus, $dM = TdS$ is referred to as a classical law), $dM = TdS$, where M is the Arnowitt–Deser–Misner (ADM) mass of RBHs. In other words, it does not resolve the conflict between the geometric and quantum aspects of RBHs to redefine the Newtonian constant. This problem is caused by the special logic in the field of RBHs, i.e., the metric of RBHs (geometric nature) is given by a phenomenological way first and then the action of gravitational theories (quantum nature dependent on the physical interpretation of RBHs) is provided. Owing to this problem, the entropy-area law was abandoned in some literature, [19–23] where the underlying motivation was to establish a well-defined first law of thermodynamics. In a previous treatment, the entropy was not an independent thermodynamic variable and it depended on the first law of thermodynamics. The entropy problem becomes more obvious if a BH contains more parameters. In order to obtain a well-defined first law of thermodynamics, more thermodynamic variables need to be redefined according to the first law of thermodynamics. For instance, if an RBH metric contains charge parameter Q, the corresponding electric potential must be redefined in order to obtain a well-defined first law of thermodynamics.

In this Letter, our main task is to confirm that the entropy of RBHs complies with the entropy-area law by investigating the geometric and quantum natures of entropy for RBHs in Einstein's gravity. In order to achieve this goal, we first categorize spherically symmetric RBHs using Segré classification. Subsequently, we calculate the entropy of each category of RBHs within the framework of Einstein's gravity. Our result shows that the entropy of RBHs should be proportional to a quarter of horizon area and that there is no violation of the entropy-area law.

Algebraic Property of Regular Black Holes. To discuss the entropy in terms of Euclidean gravity, we first classify RBHs by their algebraic property, which is also known [24] as the Segré classification (the Segré classification is an algebraic classification of symmetric tensors with two ranks; in the context of this study, it refers to the algebraic classification of the Einstein tensor G^{μ}_{ν}). A general metric with spherical symmetry has the following form (the maximum number of independent functions in a metric is 2, i.e., given a metric with three functions, $ds^2 = -f(r)dt^2 + g^{-1}(r)dr^2 + \rho^2(r)d\Omega^2$, one can always define a new variable, ξ via $(d\xi/dr)^2 = f/g$, such that the original metric becomes $ds^2 = -f(\xi)dt^2 + f^{-1}(\xi)d\xi^2 + \rho^2(\xi)d\Omega^2$):

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + \rho^2(r)d\Omega^2, \tag{5}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$, $r \in (-\infty, \infty)$, $\theta \in [0, \pi)$, and $\phi \in [0, 2\pi)$. The Euler characteristic of the geometry is two, which can be verified by carrying out the Euclidean integral of the Gauss–Bonnet term,

$$\chi = \frac{1}{32\pi^2}\int_0^{\beta} d\tau \int d\Omega \int_{r_{\rm H}}^{\infty} dr\sqrt{-g}\,\mathcal{L}_{\rm GB} = 2, \tag{6}$$

where we have used $f(r_{\rm H}) = 0$, $f(\infty) \sim 1$, $\rho'(\infty) \sim 1$, and

$$\mathcal{L}_{\rm GB} = \frac{1}{\rho^2}\Big\{4\rho'[2ff'\rho'' + \rho'(ff'' + f'^2)] - 4f''\Big\}. \tag{7}$$

The Segré classification can be realized by observing the components of Einstein's tensor G^{μ}_{ν},

$$G^t_t = \frac{f'\rho'}{\rho} + \frac{2f\rho''}{\rho} + \frac{f(\rho')^2}{\rho^2} - \frac{1}{\rho^2}, \tag{8a}$$

$$G^r_r = \frac{f'\rho'}{\rho} + \frac{f(\rho')^2}{\rho^2} - \frac{1}{\rho^2}, \tag{8b}$$

$$G^{\theta}{}_{\theta} = G^{\phi}{}_{\phi} = \frac{f''}{2} + \frac{f'\rho'}{\rho} + \frac{f\rho''}{\rho}. \tag{8c}$$

We can divide spherically symmetric RBHs into four classes based on the equality of the different components of Einstein's tensor:

$$[(1,\,1)(11)],\quad [1(11,\,1)],\quad [1,1(11)],\quad [1,(111)], \tag{9}$$

where symbol 1 corresponds to one component and the parentheses imply equal components; time and space components are distinguished by a comma. Although these four classes are derived from the nature of spacetime, they characterize the matter that can generate RBHs because of the connection between matter and spacetime via Einstein's equations.

The first class $[(1,\,1)(11)]$ implies the equality $G^t_t = G^r_r$, which provides the solution,

$$\rho(r) = c_2 r + c_1, \tag{10}$$

where c_1 and c_2 are integral constants. After an appropriate coordinate continuation (scaling and shift), we arrive at $\rho = r$. This implies that the RBHs in this class are determined only by $f(r)$. The condition of finite curvatures is shown by $1 - f \sim O(r^2)$ as r approaches zero, which indicates that $1 - f$ has no singularities in $\mathbb{R}$. This class of RBHs can be generated [25] by magnetic monopoles in nonlinear electrodynamics.

The second class $[1(11,\,1)]$ leads to $G^t{}_t = G^\theta_\theta$, from which we can represent f in terms of ρ,

$$f = c_2\rho^2 + \rho^2 \int \frac{c_1 - 2r}{\rho^4} dr. \tag{11}$$

From this formula we calculate the ZM invariants and then obtain the condition for the metric Eq. (5) to describe RBHs, i.e., the curvature invariants are finite if ρ has neither zeros nor poles in $\mathbb{R}$. Equivalently, such a condition can be derived if an alternative representation of the solution of $G^t_t = G^\theta_\theta$,

$$\rho^2 = c_2 f + f \int \frac{2r - c_1}{f^2} dr, \tag{12}$$

is applied. This class of RBHs can be interpreted in terms of nonlinear scalar fields [26] and spinor fields. [27] For more discussions on the Einstein–Dirac equations, see Bijan Saha's related works [28,29] and the references therein.

The algebra $[1, 1(11)]$ cannot lead to any equations used to restrict the metric components, which is different from the above-mentioned two situations. The conditions for finite curvatures are complicated and need to be discussed on a case-by-case basis. More specifically, ρ appears in the denominator of curvature invariants and thus all curvature invariants are finite if ρ has no zeros and f has no poles. However, if f has poles, ρ needs to be restricted according to the order of poles; see Ref. [11] and the references therein for a relevant discussion. This class of RBHs can be generated [30–32] by the combination of scalar (or spinor) fields with vector fields.

The last class $[1, (111)]$ gives $G^r_r = G^\theta_\theta$, where f and ρ can be separated by solving the differential equation

$$\xi'' + e^\xi = \left(\frac{f'}{f}\right)^2, \quad \text{with} \quad e^{-\xi} = -\frac{f\rho^2}{2}. \tag{13}$$

Thus, the ZM invariants can be represented via ξ, and the condition for the metric Eq. (5) to describe RBHs is determined by the poles of $\xi'' + e^\xi$. As a perfect fluid has the same algebra, $[1, (111)]$, it can be applied as a gravitational source, but it cannot be constructed by the combinations of scalar, spinor and vector fields under the assumption that scalar fields, spinor fields and the strength of vector fields are of the same symmetry as that of BH spacetime.

Euclidean Quantum Gravity. To calculate the entropy, we briefly introduce Euclidean quantum gravity with the following partition function, see, e.g., Ref. [33],

$$\mathcal{Z} = \int \mathcal{D}g\mathcal{D}\Phi\, e^{-\mathcal{I}_{\rm E}(g,\,\Phi)}, \tag{14}$$

where the Euclidean action $\mathcal{I}_{\rm E}(g,\,\Phi)$ is obtained from its real-time formulation with a Wick rotation, $t \to -i\tau$ and Φ is an abbreviation for all types of matter fields. The Euclidean action consists of four parts,

$$\mathcal{I}_{\rm E} = \mathcal{I}_{\rm EH} + \mathcal{I}_{\rm GHY} - \mathcal{I}_0 + \mathcal{I}_{\rm M}, \tag{15}$$

where the Euclidean version of Einstein–Hilbert action and the Gibbons–Hawking–York surface term take the following forms:

$$\mathcal{I}_{\rm EH} = -\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{g}\,R, \quad \mathcal{I}_{\rm GHY} = -\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{h}\,K, \tag{16}$$

$\mathcal{I}_0$ acts [34] as a term to cancel the infrared divergence from the surface term,

$$\mathcal{I}_0 = -\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{h}\,K_0, \tag{17}$$

and $\mathcal{I}_{\rm M}$ denotes the action of matter. Note that h stands for the induced metric on boundaries, K is the extrinsic curvature of the boundary $\partial\mathcal{M}$ and K_0 is the extrinsic curvature of $\partial\mathcal{M}$ embedded in flat spacetime. Two negative signs appear in $\mathcal{I}_{\rm GHY}$ and $\mathcal{I}_0$ because the boundaries $\partial\mathcal{M}$ are space-like in our following discussions.

Using the zero-loop approximation, [18] we can obtain

$$\mathcal{Z}_0 \approx e^{-\mathcal{I}_{\rm cl}}, \quad \mathcal{I}_{\rm cl} = \mathcal{I}_{\rm E}(\bar{g},\,\bar{\Phi}), \tag{18}$$

where the classical action $\mathcal{I}_{\rm cl}$ is calculated by substituting the classical solutions $\bar{g}$ and $\bar{\Phi}$ into the Euclidean action. Then, by relating the classical action to the thermodynamic free energy, $F = T\mathcal{I}_{\rm cl}$, we can derive the BH entropy through the thermodynamic relation, $S = \partial(T\ln\mathcal{Z})/\partial T$.

Next, we compute the entropy for the four classes of spherically symmetric RBHs.

Regular Black Holes with Algebra $[(1,\,1)(11)]$. For the first class of RBHs with algebra $[(1,\,1)(11)]$, the metric functions f and ρ take the following forms:

$$f = 1 - \frac{2M}{r}\sigma(r), \quad \rho = r. \tag{19}$$

This class of RBHs can be interpreted [35,36] by magnetic monopoles whose action can be cast in terms of nonlinear electrodynamics,

$$\mathcal{I}_M = \int d^4x\sqrt{g}\,\mathcal{L}(\mathcal{F}), \tag{20}$$

where $\mathcal{F}$ is the contraction of electromagnetic tensors, $\mathcal{F} \equiv F_{\mu\nu}F^{\mu\nu} = 2P/r^4$, P is the magnetic charge and the Lagrangian $\mathcal{L}(\mathcal{F})$ is given in terms of the Einstein equations as follows:

$$\mathcal{L}(\mathcal{F}) = \frac{M\sigma'(r)}{4\pi r^2}. \tag{21}$$

Therefore, we obtain the contributions to the classical action from the four parts,

$$\mathcal{I}_{\rm EH}+\mathcal{I}_{\rm M}=\frac{\beta M}{2}-\pi r_{\rm H}^2, \tag{22}$$

$$\mathcal{I}_{\rm GHY}=\frac{M}{2}\beta(r\sigma'+3\sigma)-r\beta\sim\frac{3M}{2}\beta-r\beta+O(r^{-1}), \tag{23}$$

$$\mathcal{I}_0=-r\beta\sqrt{1-\frac{2M\sigma}{r}}\sim-r\beta+M\beta+O(r^{-1}), \tag{24}$$

where $\beta\equiv 1/T$ and the last two asymptotic relations under $r\to\infty$ come from the assumption that this class of RBHs is asymptotic to Schwarzschild BHs at infinity, i.e., $\lim_{r\to\infty}\sigma=1$ and $\lim_{r\to\infty}r\sigma'=0$. Finally, combining the above results, we arrive at the classical action

$$\mathcal{I}_{\rm cl}=\beta M-\pi r_{\rm H}^2. \tag{25}$$

Thus, based on the canonical ensemble, $F=T\mathcal{I}_{\rm cl}=M-TS$, we directly deduce the entropy to $S=\pi r_{\rm H}^2$, which exhibits the well-known area law of BHs.

Alternatively, this class of RBHs can be interpreted by anisotropic fluids and the Euclidean action of anisotropic fluids reads[37,38]

$$\mathcal{I}_{\rm M}=\int d^4x\sqrt{g}(p_\perp+\varepsilon-p_r), \tag{26}$$

where the energy density ε, radial pressure p_r, and perpendicular pressure $p_\perp$ satisfy

$$\varepsilon=-p_r=\frac{M\sigma'(r)}{4\pi r^2},\quad p_\perp=-\frac{M\sigma''(r)}{8\pi r}. \tag{27}$$

Comparing Eqs. (26) and (27) with Eq. (21), we note that the energy density makes the same contribution as the Lagrangian $\mathcal{L}(\mathcal{F})$. Moreover, there is an additional contribution from $p_\perp-p_r=\varepsilon+p_\perp$, which is interpreted as chemical potential in the grand canonical ensemble. Namely, we have $\widetilde{F}=T\mathcal{I}_{\rm cl}=M-TS+\mu N$, where the product of chemical potential μ and particle number N takes the form

$$\mu N=\int d^4x\sqrt{g}(\varepsilon+p_\perp). \tag{28}$$

Obviously, the area law holds as well.

Next, we discuss the relationship between thermal pressure and Gliner pressure (the term *Gliner pressure* comes from the concept of the Gliner vacuum,[39] which refers to a type of matter vacuum that does not allow any preferable reference frame; this is distinct from the traditional vacuum with a vanishing energy-momentum tensor; the Gliner pressure is mathematically expressed as $P=T^r_r$, where T^r_r is the radial component of energy-momentum tensor) for the first class of RBHs. Considering the classical action Eq. (25), we calculate the thermal pressure $P_{\rm q}$ of spacetime with no matter,

$$P_{\rm q}=-\frac{\partial(\beta^{-1}(\mathcal{I}_{\rm cl}-\mathcal{I}_{\rm M}))}{\partial V_{\rm th}}=-\frac{\beta-4\pi r_{\rm H}}{8\pi\beta r_{\rm H}^2}, \tag{29}$$

where the mass M has been replaced[40] by the irreducible mass $r_{\rm H}/2$ and $V_{\rm th}$ is the volume of a sphere. In Ref. [41], the Gliner pressure was discussed in RBHs; here it is reasonable to ask whether the thermal pressure coincides with the Gliner pressure, $P_{\rm G}\equiv G^r_r/(8\pi)$, which is generated by the Gliner vacuum (or vacuum-like medium) at horizons. After calculating the Einstein tensor Eq. (8), we obtain the Gliner pressure from its r–r component,

$$P_{\rm G}=-\frac{M\sigma'(r_{\rm H})}{4\pi r_{\rm H}^2}, \tag{30}$$

while $\sigma'(r_{\rm H})$ can be represented via β,

$$\sigma'(r_{\rm H})=\frac{\beta-4\pi r_{\rm H}}{2\beta M}. \tag{31}$$

We verify that the Gliner pressure is exactly the thermal pressure after substituting Eq. (31) into Eq. (30). As a result, the thermal pressure equals the Gliner pressure if it is derived from a classical action, and the thermal volume $V_{\rm th}$ is regarded as the BH's volume, which is valid not only for RHBs, but also for SBHs, such as Reissner–Nordström–AdS BHs.

Regular Black Holes with Algebra [1(11, 1)]. For RBHs with algebra [1(11, 1)], the gravitational sources can be established[26,27] by phantom scalar fields and nonlinear spinor fields (the non-radial components of the energy-momentum tensor are trivial[29] without nonlinear terms in the action of spinor fields, such that they cannot generate RBHs). For phantom scalar fields, the Euclidean action reads (we have used the conventions of signs given by Ref. [33])

$$\mathcal{I}_{\rm M}=\int d^4x\sqrt{g}\Big[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+V(\phi)\Big], \tag{32}$$

where the scalar field and its potential are given by using the metric (5),

$$\phi'^2=\frac{\rho''}{4\pi\rho},\quad V(\phi)=\frac{(\rho^2 f')'}{16\pi\rho^2}. \tag{33}$$

If $\rho=\sqrt{r^2+b^2}$ is presumed,[26] where b is the regularization parameter, the integrability of the four parts in Eq. (15) is guaranteed. We then derive

$$\mathcal{I}_{\rm EH}+\mathcal{I}_{\rm M}=\frac{\beta M}{2}-\pi(r_{\rm H}^2+b^2), \tag{34}$$

$$\mathcal{I}_{\rm GHY}-\mathcal{I}_0=\frac{\beta M}{2}+O(r^{-1}). \tag{35}$$

Considering the above contributions, we have the classical action, $\mathcal{I}_{\rm cl}=\beta M-\pi(r_{\rm H}^2+b^2)$ and determine the entropy $S=\pi\rho_{\rm H}^2=\pi(r_{\rm H}^2+b^2)$.

Regular Black Holes with Algebra [1, 1(11)]. For RBHs with the algebraic structure [1, 1(11)], one can see Ref. [30], where the metric functions in Eq. (5) take the forms $f(r)=1-2M/\rho(r)$ and $\rho(r)=\sqrt{r^2+b^2}$. This class of RBHs can be generated[31] by phantom fields and nonlinear electromagnetic fields.

We start with the fixed $f(r)$, $f(r)=1-2M/\rho(r)$, but an arbitrary $\rho(r)$, and derive the components of the

Einstein tensor,

$$G^t_t = \frac{2(\rho-2M)\rho'' + (\rho')^2 - 1}{\rho^2}, \tag{36a}$$

$$G^r_r = \frac{(\rho')^2 - 1}{\rho^2}, \tag{36b}$$

$$G^\theta{}_\theta = G^\phi{}_\phi = \frac{(\rho - M)\rho''}{\rho^2}. \tag{36c}$$

We shall see that such an Einstein tensor includes the four classes of algebraic structures aforementioned by imposing various relations among the four components. Let us give a detailed discussion as follows.

The first class of algebras in Eq. (9), [(1, 1)(11)], cannot generate RBHs because the equation $G^t_t = G^r_r$ provides the solution (10), which is nothing but a Schwarzschild BH with some scaling and a shift of radial coordinates.

The second class of algebras in Eq. (9), [1(11, 1)], gives rise to $\rho(r) = \sqrt{(a+r)^2 + b^2} + 3M$, where a is constant, and the corresponding Kretschmann scalar has the denominator

$$[(a+r)^2 + b^2]^3\left[\sqrt{(a+r)^2+b^2} + 3M\right]^6, \tag{37}$$

which implies that the Kretschmann scalar has no singularities when $r \in \mathbb{R}$ as long as b is not zero. If this class of RBHs is interpreted by phantom scalar fields as discussed above, it is equivalent [26] to a phantom RBH and thus its entropy also obeys the area law.

The fourth class of algebras in Eq. (9), [1, (111)], provides a solution $\rho(r) = c_1^{-1}\sinh[c_1(c_2 + r)] + M$, where c_1 and c_2 are constants, and the denominator of Kretschmann scalars is $[\sinh(c_1(c_2 + r)) + c_1 M]^6$, i.e., the singularity is located at $r = -c_1^{-1}\sinh^{-1}(c_1 M) - c_2$ if $c_1 \neq 0$ and $c_1, c_2 \in \mathbb{R}$. The result means that this algebraic structure cannot generate RBHs.

Lastly, let us turn to the third class of algebras in Eq. (9), [1, 1(11)], we focus on now. We calculate the denominator of Kretschmann scalars and find that it is proportional to ρ^6. Thus, the generated BHs are regular if ρ as a function of r has neither zeros nor poles. The first example proposed in Ref. [30] takes $\rho = \sqrt{r^2 + b^2}$. Similarly, we point out that $\rho = \ln(r^2 + b^2)$ and $\rho = b^2\cosh(r)$ are also candidates for constructing RBHs. The geometric difference between these two candidates is their asymptotic behavior at infinity. The former is asymptotically flat, while the latter is asymptotic AdS because of $R \to -6$ (negative) as $r \to \infty$. Generally, if $\rho \sim r^n$ with $n > 0$ as r approaches infinity, the corresponding RBHs are asymptotically flat; if $\rho \sim \exp(r)$ as $r \to \infty$, the RBHs are asymptotic AdS and $R \to -6$. The entropy of RBHs with this algebraic structure also meets the area law, $S = \pi\rho_{\rm H}^2$, if the source is the combination of a nonlinear electromagnetic field and a phantom field.

Incidentally, another well-known example of RBHs is a solution of conformal gravity; however, it cannot be interpreted in Einstein's gravity. Its action reads [42]

$$I_{\rm conf} = -\frac{1}{2}\int d^4x\sqrt{-g}\phi\Big(\frac{1}{6}R\phi - \Box\phi\Big), \tag{38}$$

in which the algebraic structures of gravitational and matter (scalar field) sectors are inconsistent with each other. The former has the third class of algebras in Eq. (9), [1, 1(11)], while the latter has the second class of algebras in Eq. (9), [1(11, 1)]. One remedy is to use the Lagrangian, $\mathcal{L}_{\rm cof} \propto W^{\mu\nu\alpha\beta}W_{\mu\nu\alpha\beta}$ instead of Eq. (38), because the corresponding Bach tensor vanishes; the other remedy is to introduce a nonlinear electric field into Eq. (38). Although the entropy in these two explanations differs by a factor, it obeys the area law.

Regular Black Holes with Algebra [1, (111)]. The RBHs with the fourth class of algebras in Eq. (9), [1, (111)], have no field interpretations (linear/nonlinear scalar, vector, spinor fields or their combinations), but they can be generated by the particular type of anisotropic fluids mentioned above. The thermodynamics is self-contained if this class of RBHs is regarded as a grand canonical ensemble.

To give an example, we apply a linear equation of state, $p = w\epsilon$, where w is constant. Further, by choosing $\rho = e^r$, we determine the other metric function $f(r) = 1 - 2M\sigma/r$ with the following $\sigma(r)$,

$$\sigma(r) = \frac{e^{-r}r(w\sinh r - \cosh r)}{M(w-1)} + c_3 r e^{r(\frac{2}{w+1}-3)}, \tag{39}$$

where c_3 is constant and its different values correspond to different types of spacetime, such as flat, AdS or dS, in the center of RBHs. The Kretschmann scalar is finite for $r \in [0, \infty)$ as long as $w \neq \pm 1$, and such RBHs are Ricci flat if $w > 0$. Another integrable example is $\rho = \cosh r$, which is well-defined even as $r \to -\infty$. The area law of this class of RBHs holds, which can be verified by following the procedure given above.

In summary, we address an issue encountered in the field of RBHs, namely, how to deal with the conflict between the geometric and quantum natures of RBHs.

Usually there are two approaches to calculate entropy, where one is associated with the geometric characteristics of black holes and the other with quantum ones. The former begins with the metric of black holes and then establishes the connections between ADM mass and energy and between surface gravity and temperature, and ultimately employs the thermodynamic formula $dM = TdS$. Because this approach is closely related to the metric of BHs, it shows their geometric characteristics. The latter approach starts with the action of gravitational theories and then utilizes either the Euclidean path-integral method or Wald's formalism. Because an action shows a quantum phase (see the explanation in the introduction), the latter approach presents the quantum characteristics of BHs. The former approach violates the entropy-area law in Einstein's gravity, whereas the latter, as demonstrated in our current work, does not. It is commonly accepted, see Ref. [6], that the entropy-area law should hold in Einstein's gravity, which leads to the conflict between the geometric and quantum natures of RBHs.

Therefore, it is necessary to verify the entropy-area law for RBHs because the law correlates with the thermodynamics of RBHs. Meanwhile, this verification is also nontrivial because our result shows that the efforts reported

in literature[19–23] are unnecessary in the modifications of the first law of thermodynamics and of temperature for the RBHs in the framework of Einstein's gravity. In our current work, we provide solid evidence that the entropy-area law of the Segré classes of RBHs still holds by using the path-integral method. Moreover, the path-integral method coincides[41] with the introduction of Gliner pressure at horizons. That is, we verify that the Gliner pressure equals the thermal pressure if the thermal volume of RBHs is applied, which is derived from the partition function of the spacetime with no matter.

Further, it is known that the one-loop correction of entropy for SBHs has a logarithmic divergence in the UV limit at the quantum level. Thus, it is reasonable to check whether the UV divergence still exists in RBHs. Intuitively, the UV divergence may be improved in one-loop corrections since the essential singularity is removed in RBHs. We shall present our work in the near future.

Acknowledgment. This work was supported by the National Natural Science Foundation of China (Grant No. 12175108).

References

[1] Bekenstein J D 1972 *Lett. Nuovo Cimento* **4** 737
[2] Bekenstein J D 1973 *Phys. Rev. D* **7** 2333
[3] Dyer E and Hinterbichler K 2009 *Phys. Rev. D* **79** 024028
[4] Sotiriou T P and Faraoni V 2010 *Rev. Mod. Phys.* **82** 451
[5] Wald R M 1993 *Phys. Rev. D* **48** R3427
[6] Jacobson T, Kang G, and Myers R C 1994 *Phys. Rev. D* **49** 6587
[7] Brustein R, Gorbonos D, and Hadad M 2009 *Phys. Rev. D* **79** 044025
[8] Querella L 1998 arXiv:gr-qc/9806005
[9] Gover A R and Nurowski P 2006 *J. Geometry Phys.* **56** 450
[10] Myers R C and Simon J Z 1998 *Phys. Rev. D* **38** 2434
[11] Lan C, Yang H, Guo Y, and Miao Y G 2023 *Int. J. Theor. Phys.* **62** 202
[12] Bronnikov K A, Melnikov V N, and Dehnen H 2007 *Gen. Rel. Grav.* **39** 973
[13] Ansoldi S 2008 arXiv:0802.0330 [gr-qc]
[14] Zaslavskii O B 2010 *Phys. Lett. B* **688** 278
[15] Markov M A 1982 *JETP Lett.* **36** 266
[16] Zakhary E and Mcintosh C B G 1997 *Gen. Relat. Gravit.* **29** 539
[17] Zhang Y and Gao S 2018 *Class. Quantum Grav.* **35** 145007
[18] Gibbons G W and Hawking S W 1977 *Phys. Rev. D* **15** 2752
[19] Myung Y S, Kim Y W, and Park Y-J 2007 *Phys. Lett. B* **656** 221
[20] Myung Y S and Yoon M 2009 *Eur. Phys. J. C* **62** 405
[21] Spallucci E, Smailagic A, and Nicolini P 2009 *Phys. Lett. B* **670** 449
[22] Nam C H 2018 *Eur. Phys. J. C* **78** 581
[23] Naveena Kumara A, Rizwan C L A, Hegde K, and Ajith K M, 2020 *Phys. Lett. B* **807** 135556
[24] Petrov A Z 1961 *Einstein Spacetime* (Moscow: Fizmatlit) (in Russian)
[25] Fan Z Y and Wang X 2016 *Phys. Rev. D* **94** 124027
[26] Bronnikov K A and Fabris J C 2006 *Phys. Rev. Lett.* **96**251101
[27] Bronnikov K A, Rybakov Y P, and Saha B 2020 *Eur. Phys. J. Plus* **135** 124
[28] Saha B 2006 *Phys. Rev. D* **74** 124030
[29] Saha B 2018 *Eur. Phys. J. Plus* **133** 461
[30] Simpson A and Visser M 2019 *J. Cosmol. Astropart. Phys.* **2019(02)** 42
[31] Bronnikov K A and Walia R K 2022 *Phys. Rev. D* **105** 044039
[32] Hu H W, Lan C, and Miao Y G 2023 *Eur. Phys. J. C* **83** 1047
[33] Kiefer C 2012 *Quantum Gravity* 3nd edn (Oxford: Oxford University Press)
[34] Poisson E 2009 *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge: Cambridge University Press) p. 12
[35] Ayon-Beato E and Garcia A 1998 *Phys. Rev. Lett.* **80** 5056
[36] Ayon-Beato E and Garcia A 2000 *Phys. Lett. B* **493** 149
[37] Mann R B and Nicolini P 2011 *Phys. Rev. D* **84** 064014
[38] Lan C, Miao Y G, and Yang H 2021 *Nucl. Phys. B* **971** 115539
[39] Gliner E B 1966 *Sov. Phys. JETP* **22** 378
[40] Christodoulou D 1970 *Phys. Rev. Lett.* **25** 1596
[41] Lan C and Miao Y G 2022 *Eur. Phys. J. C* **82** 1152
[42] Toshmatov B, Bambi C, Ahmedov B, Abdujabbarov A, and Stuchlík Z 2017 *Eur. Phys. J. C* **77** 542