Wave-Particle Duality via Quantum Fisher Information

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(Received 9 August 2023; accepted manuscript online 25 September 2023)

Quantum Fisher information (QFI) plays an important role in quantum metrology, placing the ultimate limit to how precise we can estimate some unknown parameter and thus quantifying how much information we can extract. We observe that both the wave and particle properties within a Mach–Zehnder interferometer can naturally be quantified by QFI. Firstly, the particle property can be quantified by how well one can estimate the a priori probability of the path taken by the particle within the interferometer. Secondly, as the interference pattern is always related to some phase difference, the wave property can be quantified by how well one can estimate the phase parameter of the original state. With QFI as the unified figure of merit for both properties, we propose a more general and stronger wave-particle duality relation than the original one derived by Englert.

DOI: 10.1088/0256-307X/40/11/110301

Wave-particle duality exemplifies Bohr's complementarity principle,[1] stating that there are mutually exclusive physical properties. As a particle traverses, for example, a Mach–Zehnder interferometer for the extraction of which-way information which represents the particle property, and the observation of interference pattern which represents the wave property, are incompatible. The consistent efforts[2–6] of trying to simultaneously measure these two incompatible properties culminate in a duality inequality,[7] showing the trade-off between the which-way information, quantified by the path distinguishability, and the interference pattern, quantified by the fringe visibility. As it turns out, the quantitative description of wave-particle duality is intimately related to the error tradeoffs exhibited in the joint measurement of incompatible observables.[8–10]

Other quantifications have also been employed such as min and max entropies,[4,9,11] coherence for wave property,[12] unambiguous discrimination for path information,[13,14] guessing games,[15] and also from classical metrology.[16] In all these cases, unlike the preparation uncertainty relation in which the relevant quantities have the same kind of figures of merit, e.g., the variance or the entropy, the wave and particle properties appearing in previous duality tradeoffs were quantified by different figures of merit. One question arises naturally: Can duality tradeoffs be manifested in terms of a unified measure for both properties? Quantum Fisher information provides such a unified measure.

Eventually, a measurement aims to acquire information about some relevant parameters that characterize the physical properties of a quantum system via suitable observables. Some physically relevant parameters do not correspond to observables, such as time and phase.[17] The precision of estimating an unknown parameter is naturally quantified by the distinguishability of the signal parameter that is encoded in quantum states. This distinguishability, according to the quantum estimation theory,[18–20] is quantified by the quantum Fisher information (QFI), which sets the ultimate precision to the estimation of an unknown parameter by the celebrated quantum Cramér–Rao bound.[18] Moreover, the QFI is a suitable extension of variance to the case of mixed states.[21] Intuitively, from the complementarity principle, the ultimate precisions of estimating two parameters via joint measurement of two complementary observables should exhibit some kind of tradeoff, providing a quantitative complementarity.

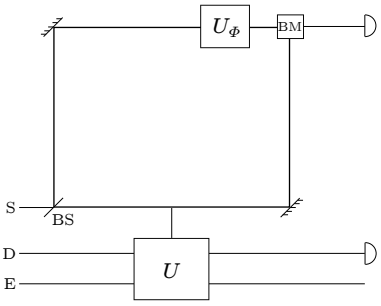

Fig. 1. A typical Mach–Zehnder interferometer with a specific which-way detector that is implemented by a controlled unitary operation between the system plus the environment and the detector.

Let us consider a typical Mach–Zehnder interferometer equipped with a which-way detector as illustrated in Fig. 1. A particle transversing the interferometer can be treated as an effective qubit system with two paths spanning the Hilbert space of the system. We suppose that the effective qubit system can be prepared into an arbitrary state, say,

$$\rho = e^{-i\frac{\Phi}{2}\sigma_z}\frac{1+x\sigma_x+z\sigma_z}{2}e^{i\frac{\Phi}{2}\sigma_z}, \tag{1}$$

for some real x, z satisfying $x^2+z^2 \le 1$. We note that $|x| = V$ and $|z| = P$ are respectively the *a priori* visibility

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and distinguishability in the context of the wave-particle duality. Since $P = 1$ corresponds to a definite path and $V = 0$ corresponds to a classical mixture of two paths, in what follows we shall assume $P \neq 1$ and $V \neq 0$.

The wave and particle properties can be characterized, respectively, by two parameters, namely, the phase Φ and the *a priori* probability of which path is taken, or equivalently the parameter $z = \mathrm{Tr}(\rho\sigma_z)$. The simultaneous extraction of which-way information by the which-way detector and the observation of interference pattern is viewed as a joint estimation of these two parameters via a joint measurement of some two incompatible observables. As will be shown below, with the QFIs quantifying the ultimate precisions of estimation taken as the figures of merit, the wave-particle duality can be demonstrated in the form of an uncertainty relation with the same kind of uncertainty measure.

For a joint estimation of two parameters, we adopt a slightly more general measurement model than what is used in deriving the original duality inequality [7] by introducing an environment in addition to a detector system. Initially, the detector plus environment can be assumed without loss of generality to be in some pure state $|e_0\rangle_{\mathrm{DE}}$. To extract which-way information, a unitary interaction U is applied to the whole system such that

$$U|a\rangle_{\mathrm{S}}|e_0\rangle_{\mathrm{DE}} = |a\rangle_{\mathrm{S}}|e_a\rangle_{\mathrm{DE}}, \quad (a = 0, 1) \tag{2}$$

where $|e_1\rangle = u|e_0\rangle$ with some unitary u acting on the detector plus environment. Roughly speaking the unitary interaction U can be realized by a controlled u gate with system qubit as source and detector plus environment as target. After the interaction U, the joint state of the whole system evolves into a final state

$$\rho_{\mathrm{f}} = U(\rho \otimes |e_0\rangle\langle e_0|_{\mathrm{DE}})U^{\dagger}. \tag{3}$$

Since the which-way information has been imprinted in the detector system while the phase information is preserved in the system qubit, the joint estimation of two parameters z and Φ can be achieved by separate single-parameter estimation on two independent systems. That is, we can estimate the parameter z by performing a suitable measurement on the detector system, which is in the state $\rho_{\mathrm{D}} = \mathrm{Tr}_{\mathrm{SE}}\rho_{\mathrm{f}}$, while we can estimate the phase φ by performing some suitable measurement on the system qubit, which is in the state $\rho_{\mathrm{S}} = \mathrm{Tr}_{\mathrm{DE}}\rho_{\mathrm{f}}$. The variances of estimating both parameters z and Φ, after n runs of measurements, are lower bounded by the corresponding quantum Cramér–Rao bounds,

$$\delta_z^2 \geq \frac{1}{nF_z[\rho_{\mathrm{D}}]}, \quad \delta_\Phi^2 \geq \frac{1}{nF_\Phi[\rho_{\mathrm{S}}]}, \tag{4}$$

with the details of two QFIs $F_z[\rho_{\mathrm{D}}]$ and $F_\Phi[\rho_{\mathrm{S}}]$ given in the first part of the Supplementary Materials. Being the ultimate precision of estimating these two parameters, namely, z and Φ, which capture the particle and wave properties, respectively, these two QFIs can be used as quantifications of information regarding particle and wave properties. Therefore the following trade-off between these two ultimate precisions (proofs given in the second part of the Supplementary Materials)

$$(1 - P^2)F_z[\rho_{\mathrm{D}}] + \frac{F_\Phi[\rho_{\mathrm{S}}]}{V^2} \leq 1, \tag{5}$$

together with its envelop

$$F_\Phi[\rho_{\mathrm{S}}]F_z[\rho_{\mathrm{D}}] \leq \frac{1}{4} \tag{6}$$

provides a quantitative characterization of wave-particle duality, which is our main result as illustrated in Fig. 2.

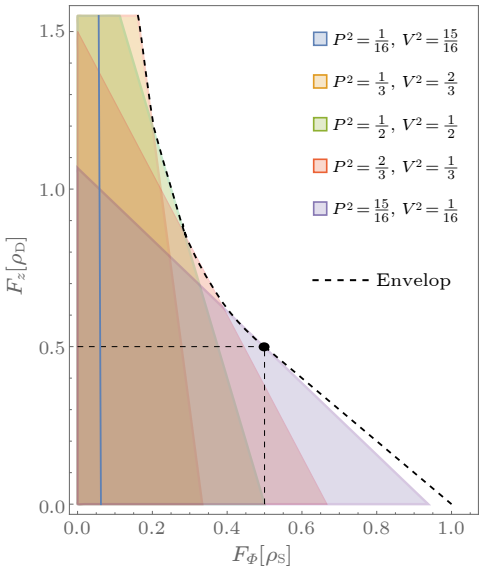

Fig. 2. Wave-particle duality reflected by the trade-off between two QFIs $F_z[\rho_{\mathrm{D}}]$ and $F_\Phi[\rho_{\mathrm{S}}]$. Regions enclosed by straight lines correspond to the linear trade-off Eq. (5) with different values of V, P satisfying $P^2 + V^2 = 1$. The dashed curve represents the envelop that is composed of a hyperbola, corresponding to the trade-off Eq. (6), and a straight line, corresponding to the linear trade-off Eq. (5) with $V^2 = 1$, the largest possible value of V.

Some remarks are in order. First, several interpretations can be attached to the above certainty relation. Above all it can be regarded as a kind of duality inequality, quantifying the trade-off between the path information and the interaction pattern. In fact, the QFI $F_z[\rho_{\mathrm{D}}]$ quantifies the path information via the ultimate precision of estimating the *a priori* path information while the QFI $F_\Phi[\rho_{\mathrm{S}}]$ quantifies the coherence in the system state ρ_{S} after the interaction or it can be used to quantify the ability to interfere, i.e., the interference pattern.

Second, our certainty relation (5) turns out to be stronger than the original duality inequality. In fact, after the interaction, the detector system will evolve into the following final state:

$$\rho_{\mathrm{D}} = \sum_{a=0}^{1} p_a \varrho_a, \quad p_a = \frac{1 + z(-1)^a}{2}, \tag{7}$$

with $\varrho_a = \mathrm{Tr}_{\mathrm{E}}|e_a\rangle\langle e_a|_{\mathrm{DE}}$ being essentially two arbitrary states of the detector. The optimal probability of guessing the path right with the help of information from the which-way detector reads $P_{\mathrm{opt}} = \frac{1+\mathcal{D}}{2}$ with

$$\mathcal{D} = \mathrm{Tr}|p_0\varrho_0 - p_1\varrho_1| \tag{8}$$

being the distinguishability. In terms of QFI, as proved in the third part of the Supplementary Materials, the distinguishability has the following upper bound:

$$\mathcal{D}^2 \leq P^2 + (1-P^2)^2 F_z[\rho_{\rm D}], \tag{9}$$

from which, together with the fact that $\mathcal{V} = \sqrt{F_\Phi[\rho_{\rm S}]}$ is exactly the fringe visibility and our trade-off Eq. (5), it follows immediately Englert's duality inequality [7]

$$\mathcal{D}^2 + \frac{1-P^2}{V^2}\mathcal{V}^2 \leq 1.$$

Third, our certainty relation (5) is tight, i.e., the equality is attainable. In particular, we consider the system of a qubit and the detector (another qubit) without involving any environment as shown in Fig. 1. Initially we prepare the system into a pure state

$$|\psi\rangle_{\rm S} = \alpha_+|0\rangle + \alpha_-|1\rangle, \quad \alpha_\pm = e^{\mp i\frac{\Phi}{2}}\sqrt{\frac{1\pm z}{2}}, \tag{10}$$

and the detector into a pure state $|e_0\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$ for some given θ. To extract the which-way information we apply a CNOT with the system as the source and the detector as a target, leading to the final state

$$|\Psi\rangle = U|\psi\rangle_{\rm S}|e_0\rangle = \alpha_+|0\rangle|e_0\rangle + \alpha_-|1\rangle|e_1\rangle \tag{11}$$

with $|e_1\rangle = \cos\theta|1\rangle + \sin\theta|0\rangle$. The Bloch vectors of two reduced density matrices $\rho_{\rm D} = {\rm Tr}_{\rm S}|\Psi\rangle\langle\Psi|$ and $\rho_{\rm S} = {\rm Tr}_{\rm D}|\Psi\rangle\langle\Psi|$ are, respectively,

$$\boldsymbol{r}_{\rm D} = {\rm Tr}(\boldsymbol{\sigma}\rho_{\rm D}) = [\sin(2\theta), 0, z\cos(2\theta)],$$
$$\boldsymbol{r}_{\rm S} = {\rm Tr}(\boldsymbol{\sigma}\rho_{\rm S}) = [x\sin(2\theta)\cos\Phi, x\sin(2\theta)\sin\Phi, z],$$

where we have used the fact $x^2 + z^2 = 1$ in this case. The corresponding QFIs are given by

$$F_z[\rho_{\rm D}] = \frac{\cos^2 2\theta}{1-z^2}, \quad F_\Phi[\rho_{\rm S}] = \sin^2 2\theta(1-z^2), \tag{12}$$

from which we see that Eq. (5) is attained. Also the envelop of these straight lines is given by $F_z[\rho_{\rm D}]F_\Phi[\rho_{\rm S}] = 1/4$ via choosing $\theta = \pi/8$.

To conclude, we have proposed a quantification of the wave-particle duality within a typical two-path interferometer from a novel perspective, i.e., in terms of quantum Fisher information. In particular, we employ the ultimate precision, as quantified by QFI, of estimating two parameters as a unified measure of the path information, i.e., particle property, and the phase information, i.e., the wave property. The trade-off between these QFIs quantifies naturally the wave-particle duality. In comparison to Englert's duality inequality, ours not only deal with a slightly more general model but also provide a stronger quantification in the sense that the former can be derived from our result. In future works, we shall consider even more general which-way detectors, e.g., not of the form of a controlled unitary operation, and the wave-particle duality in a multi-path interferometer.

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