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Quantum Hydrodynamics of Fractonic Superfluids with Lineon Condensate: From Navier–Stokes-Like Equations to Landau-Like Criterion

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Fractonic superfluids are exotic states of matter with spontaneously broken higher-rank $U(1)$ symmetry. The broken symmetry is associated with conserved quantities, including not only particle number (i.e., charge) but also higher moments, such as dipoles, quadrupoles, and angular moments. Owing to the presence of such conserved quantities, the mobility of particles is restricted either completely or partially. Here, we systematically study the hydrodynamical properties of fractonic superfluids, especially focusing on the fractonic superfluids with conserved angular moments. The constituent bosons are called "lineons" with d components in d-dimensional space. From the Euler–Lagrange equation, we derive the continuity equation and Navier–Stokes-like equations, in which the angular moment conservation introduces extra terms. Further, we discuss the current configurations related to the defects. Like the conventional superfluid, we study the critical values of velocity fields and density currents, which gives rise to a Landau-like criterion. Finally, several future directions are discussed.

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Superfluidity is a well-known macroscopic quantum phenomenon in which interacting bosons are organized into a macroscopic coherent state. Theoretically, we can realize the superfluid phase in a weakly interacting boson system. In the presence of positive chemical potential and repulsive interaction, the $U(1)$ symmetry is spontaneously broken and bosons are condensed. An unconventional superfluid, namely, fractonic superfluid,[1−4] has been proposed recently, motivated by the advances in the field of "fracton topological order".[5,6] The basic idea of fractonic superfluids is to consider the quantum many-body physics of fracton-like particles. Instead of regarding fractons, lineon, and other subdimensional particles as topological excitations of fracton topological order states,[1−28] we regard them as constituent particles of real many-body systems. In fractonic superfluids, these particles are bosonic, and the conserved quantities include not only particle number but also higher moments, such as dipoles, quadrupoles, and angular moments.[24−26,29] The associated $U(1)$ symmetry is said to be of higher rank, whereas the usual $U(1)$ symmetry with particle number conservation has rank-0. Thus, fractonic superfluids are symmetry-breaking phases with spontaneously broken higher-rank symmetry. Recently, this line of thinking with general considerations of higher moment conservation and symmetry breaking has been focused on by both condensed matter and high energy community.[28,30−35]

In fractonic superfluids, the constituent bosons are subject to mobility restriction. If the constituent bosons do not have quadratic kinetic energy (hopping term), the mobility of such bosons (fractons) is completely restricted, which can be realized by implementing dipole conservation. Likewise, if the constituent bosons have d components and the ith component has quadratic kinetic energy only along the spatial direction $\hat{x}_i$, the mobility of such bosons (lineons) is partially restricted, which can be realized by implementing conservation of angular moment. Following the notation introduced above, we may use $d\mathcal{SF}^0$ to represent fractonic superfluid phase with fracton condensation in d-dimensional space, where the superscript 0 means that the constituent particles do not have kinetic energy at all. Likewise, $d\mathcal{SF}^1$ represents fractonic superfluid phase with lineon condensation in d-dimensional space, where the superscript 1 encodes the above definition of lineons. In this notation, the conventional superfluid is represented by $d\mathcal{SF}^d$ in which bosons have normal kinetic energy along all d spatial directions.

As a quantum fluid, fractonic superfluids are expected to exhibit exotic quantum hydrodynamical behaviors entirely different from those of the conventional superfluids denoted by $d\mathcal{SF}^d$. In this study, we explore highly unconventional hydrodynamics of

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a fractonic superfluid with lineon condensation, i.e., $d\mathcal{SF}^1$ originally introduced in Ref. [2]. Following the Euler–Lagrange equation, we subsequently derive the continuity equation as well as the Navier–Stokes-like hydrodynamic equation. Particularly, for $d = 2$, we discuss the velocity fields and the vorticity of superfluid defects. We find a Landau-like criterion for $d\mathcal{SF}^1$ to survive against the current occurrence. Finally, we conclude and highlight research future directions.

Model and Symmetry. When a many-body system obeys multimoment conservation laws, the constituent particles are subject to mobility constraints. One example is "lineon", which only moves in one certain direction in d spatial dimensions, in contrast to a conventional particle that can propagate freely in the entire space. First, we consider a system of many d-component bosons with Hamiltonian,[2]

$$\begin{aligned}\mathcal{H} =& \sum_{a=1}^{d} \partial_a \hat{\Phi}_a^\dagger \partial_a \hat{\Phi}_a + \sum_{a,b\neq a}^{d} \frac{1}{2} K_{ab} \left(\hat{\Phi}_a^\dagger \partial_a \hat{\Phi}_b^\dagger + \hat{\Phi}_b^\dagger \partial_b \hat{\Phi}_a^\dagger\right) \\ &\times \left(\hat{\Phi}_a \partial_a \hat{\Phi}_b + \hat{\Phi}_b \partial_b \hat{\Phi}_a\right) + V\left(\hat{\Phi}^\dagger, \hat{\Phi}\right), \end{aligned} \tag{1}$$

where $\hat{\Phi}_a^\dagger$ and $\hat{\Phi}_a$ are, respectively, creation and annihilation operators with a bosonic communicative relation, $\left[\hat{\Phi}_a(\boldsymbol{x}), \hat{\Phi}_b^\dagger(\boldsymbol{y})\right] = \delta_{ab}\delta^d(\boldsymbol{x}-\boldsymbol{y})$ with the coordinates $\boldsymbol{x} = (x^1, \ldots, x^d)$. The first term in Eq. (1) denotes the kinetic energy of the ath component freely propagating toward the ath direction. The second term describes the cooperative motion involving the two components with the coefficients $K_{ab} = K_{ba} > 0$ and $K_{aa} = 0$. In other words, the specified form of kinetic terms impose a constraint on mobility, which is manifested by the invariance under the transformations, $(\hat{\Phi}_a, \hat{\Phi}_b) \to (\hat{\Phi}_a e^{i\lambda_{ab} x_b}, \hat{\Phi}_b e^{-i\lambda_{ab} x^a})$, aside from the particle number conservation of each component, $\hat{\Phi}_a \to e^{i\lambda_a}\hat{\Phi}_a$. The according charges are angular charge moments $\hat{Q}_{ab} = \int d^d x \left(\hat{\rho}_a x_b - \hat{\rho}_b x_a\right)$ and the particle number $\hat{Q}_a = \int d^d x \hat{\rho}_a$ with $\hat{\rho}_a = \hat{\Phi}_a^\dagger \hat{\Phi}_a$ being a density operator. Owing to the conservation of angular moment $\hat{Q}_{ab}$, a single a-component becomes a lineon that can freely move only in the ath direction if other components have zero particle density. The potential $V = \sum_{a=1}^{d}\left(-\mu\hat{\rho}_a + \frac{g}{2}\hat{\rho}_a^2\right)$ contains chemical potential μ and short-range repulsive interaction $g > 0$.

Once the chemical potential μ is tuned positive, the lineons get condensed to form a fractonic superfluid phase $d\mathcal{SF}^1$. The phase $d\mathcal{SF}^1$ is described by the order parameter, i.e., the classical configuration as a plane wave, $\langle\hat{\Phi}_a(\boldsymbol{x})\rangle = \sqrt{\rho_0} e^{i\theta_a^{\rm cl}(\boldsymbol{x})}$ with $\rho_0 = \mu/g$ and $\theta_a^{\rm cl}(\boldsymbol{x}) = \theta_a + \sum_b \beta_{ab} x^b$ with a finite momentum $\boldsymbol{k}_a = (\beta_{a1}, \ldots, \beta_{\rm ad})$. The real parameters θ_a and β_{ab} describe the degenerate ground state manifold with $\beta_{ab} = -\beta_{ba}$. The low-energy theory can be obtained by an expansion around the classical configuration, $\hat{\Phi}_a(\boldsymbol{x}) = \sqrt{\rho_0 + \delta\rho_a(\boldsymbol{x})} e^{i[\theta_a^{\rm cl}(\boldsymbol{x}) + \delta\theta_a(\boldsymbol{x})]}$, where $\delta\rho_a(\boldsymbol{x})$ and $\delta\theta_a(\boldsymbol{x})$ are density and phase fluctuation fields, respectively. In $d = 2$ spatial dimensions, the system supports thermal defect excitations. Owing to the particle number conservation and conservation of angular moment, one may construct three types of defects in $2\mathcal{SF}^1$, $\Theta_{0x} = \ell_{0x}(\varphi, 0), \Theta_{0y} = \ell_{0y}(0, \varphi)$, $\Theta_1 = \ell_1\left(\frac{y}{a}\varphi - \frac{x}{a}\ln\frac{r}{a}, -\frac{x}{a}\varphi - \frac{y}{a}\ln\frac{r}{a}\right)$, where $\varphi = \mathrm{Arctan}\,(y/x)$, and a is the defect core size. We use the notation $\Theta \equiv (\theta_1, \theta_2)$. The defects $\Theta_{0x,0y}$ resemble vortices with topological charges $\ell_{0x,0y}$ in a conventional superfluid phase $2\mathcal{SF}^2$. The defects Θ_1 arise as a consequence of the conservation of angular moment, and the quantization of the topological charge ℓ_1 needs a lattice for regularization.[2]

Hydrodynamic Equations. In Ref. [2] we presented the Euler–Lagrange equation of many lineon system as given in Eq. (1). Following the Euler–Lagrange equation, here we derive a Navier–Stokes-like equation from the perspective of hydrodynamics.

In the framework of the coherent-state path integral, we can express the Lagrangian $\mathcal{L} = i\sum_a \phi_a^* \partial_t \phi_a - H(\phi_a)$. The Euler–Lagrange equation can be obtained straightforwardly as follows:

$$\begin{aligned} i\partial_t\phi_a =& -\partial_a^2\phi_a - \mu\phi_a + g\phi_a^2\phi_a^* \\ &+ \sum_{b(\neq a)} K_{ab}\left(\partial_a\phi_b^*\right)\left(\phi_b\partial_b\phi_a + \phi_a\partial_a\phi_b\right) \\ &- \sum_{b(\neq a)} K_{ab}\partial_b\left[\phi_b^*\left(\phi_b\partial_b\phi_a + \phi_a\partial_a\phi_b\right)\right]. \end{aligned} \tag{2}$$

Further, in terms of the relation $\phi_a = \sqrt{\rho_a} e^{i\theta_a}$, we can reformulate the Euler–Lagrange Eq. (2) with respective to the density field $\rho_a(\boldsymbol{x})$ and phase field $\theta_a(\boldsymbol{x})$,

$$\begin{aligned} \partial_t\rho_a =& -\partial_a\left(2\rho_a\partial_a\theta_a\right) \\ &- \sum_{b(\neq a)} \partial_b\left[2K_{ab}\rho_a\rho_b\left(\partial_a\theta_b + \partial_b\theta_a\right)\right], \end{aligned} \tag{3}$$

$$\begin{aligned} \partial_t\theta_a =& -\left(\partial_a\theta_a\right)^2 + p_a(\rho)/2 \\ &- \sum_{b(\neq a)} K_{ab}\rho_b\left(\partial_a\theta_b + \partial_b\theta_a\right)^2, \end{aligned} \tag{4}$$

where $p_a(\rho)$ is independent of the θ_a fields,

$$\begin{aligned} p_a(\rho) = \sum_{b(\neq a)} \frac{K_{ab}}{2}\Big[&-\rho_b\frac{(\partial_b\rho_a)^2}{\rho_a^2} - \frac{(\partial_a\rho_b)^2}{\rho_b} + \frac{2\partial_b(\rho_b\partial_b\rho_a)}{\rho_a} \\ &+ 2\partial_a\partial_b\rho_b\Big] + \frac{2\partial_a^2\sqrt{\rho_a}}{\sqrt{\rho_a}} + 2(\mu - g\rho_a). \end{aligned} \tag{5}$$

In fact, Eq. (3) can be reformulated in the standard form of the continuity equation:

$$\partial_t\rho_a = -\sum_i \partial_i\left(\rho_a v_{ai}\right), \tag{6}$$

where v_{ai} is the ith component of the velocity of the ath component of lineon field ϕ_a:

$$v_{ai} = 2\partial_a\theta_a\delta_{ai} + 2K_{ai}\rho_i\left(\partial_i\theta_a + \partial_a\theta_i\right)\left(1 - \delta_{ia}\right). \tag{7}$$

This expression indicates a strong anisotropic velocity field v_{ai} of lineon field ϕ_a. More explicitly,

$$v_{ai} = 2\partial_a\theta_a \quad (\text{if } a = i), \tag{8}$$

$$v_{ai} = 2K_{ai}\rho_i\left(\partial_i\theta_a + \partial_a\theta_i\right) \quad (\text{if } a \neq i). \tag{9}$$

Therefore, for the ath component of the lineon field, i.e., ϕ_a, there is always a nonvanishing velocity along the ath spatial direction if the phase field θ_a has a nonvanishing gradient along the ath spatial direction. This is similar to the usual velocity field of conventional superfluids. However, when $i \neq a$, to have a nonzero v_{ai}, i.e., a nonzero velocity along the ith spatial direction, we at least require that ρ_i is nonzero. Consequently, when $i \neq a$, ϕ_a and ϕ_i must be combined such that ϕ_a can gain velocity along the ith direction. In other words, a single lineon ϕ_a can only move along the ath direction; ϕ_a may move along other directions only in the form of collective motion. An alternative explanation from the symmetry argument was presented in Appendix A of Ref. [3]. Keeping this scenario in mind, for ϕ_a, we denote the ath direction [cf. $i = a$ in Eq. (7)] as a movable direction and others [cf. $i \neq a$ in Eq. (7)] as immovable directions. Moreover, from Eq. (4), by means of velocity fields in Eq. (7), we can obtain the following Navier–Stokes-like equation:

$$\begin{aligned}\partial_t v_{ai} =& \delta_{ai}\Big(-\partial_i\frac{v_{aa}^2}{2} - \sum_{b\neq a}\frac{\partial_i v_{ab}^2}{2K_{ab}} + \partial_a p_a\Big)\\ &+ (1-\delta_{ai})\Big[-\sum_{b\neq a}K_{ai}\rho_i\Big(\frac{\partial_i v_{ab}^2}{2K_{ab}} + \frac{\partial_a v_{ib}^2}{2K_{ib}}\Big)\\ &- K_{ai}\rho_i\frac{\partial_i v_{aa}^2 + \partial_a v_{ii}^2}{2} - \frac{v_{ai}}{\rho_i}\sum_b\partial_b\left(\rho_i v_{ib}\right)\\ &+ K_{ai}\rho_i\left(\partial_i p_a + \partial_a p_i\right)\Big].\end{aligned} \tag{10}$$

We introduce the notations T_a of a-component:

$$T_a = \frac{v_{aa}^2}{2} + \sum_{b(\neq a)}\frac{v_{ab}^2}{2K_{ab}\rho_b}. \tag{11}$$

Then, the Navier–Stokes-like equations for the diagonal components v_{aa} $(a = 1, \ldots, d)$ in Eq. (10) can be expressed in a more compact form:

$$\partial_t v_{aa} = \partial_a(-T_a + p_a). \tag{12}$$

For the off-diagonal components v_{ai} $(i \neq a)$, we have

$$\begin{aligned}\partial_t v_{ai} =& K_{ai}\rho_i[\partial_i(-T_a + p_a) + \partial_a(-T_i + p_i)]\\ &- \frac{v_{ai}}{\rho_i}\sum_b\partial_b(\rho_i v_{ib}).\end{aligned} \tag{13}$$

The diagonal components in Eq. (12) behave similarly to the Navier–Stokes equation in conventional superfluid $d\mathcal{SF}^d$,[36]

$$\partial_t v_i = \partial_i\Big(-\sum_j\frac{v_j^2}{2} + \frac{1}{2\sqrt{\rho}}\sum_j\partial_j^2\sqrt{\rho} - g\rho\Big), \tag{14}$$

where v_i and ρ are the velocity and density fields in $d\mathcal{SF}^d$, respectively. Here, $\sum_j v_j^2/2$ is the kinetic density, and $\frac{1}{2\sqrt{\rho}}\sum_j\partial_j^2\sqrt{\rho}$ is the quantum pressure term in $d\mathcal{SF}^d$. The comparison with Navier–Stokes equation in $d\mathcal{SF}^d$ gives the meaning of T_a in Eq. (11) as kinetic density and p_a in Eq. (5) as pressure term from lineon effects. The kinetic density of immovable direction x^b containing the bth component density field $\rho_b(\boldsymbol{x})$ may characterize the difference between fractonic and conventional hydrodynamics.

For the off-diagonal components in Eq. (13), the density $\rho_i(\boldsymbol{x})$ outside the partial derivative makes currents in the immovable direction beyond the conventional Navier–Stokes equation. However, for the ground state of $d\mathcal{SF}^1$, i.e., $\rho_0 = \mu/g$ and $\theta_a^{\rm cl}$, the pressure term $p_a \equiv 0$ and the off-diagonal components in Eq. (13) can be further simplified to

$$\partial_t v_{ai} = -K_{ai}\rho_0(\partial_i T_a + \partial_a T_i) - v_{ai}\sum_b\partial_b v_{ib}, \tag{15}$$

where the first term is similar to those in Eq. (14), and the second term is the pressure term from the ith component to the ath component. The physical meaning of the third term in Eq. (15) is still an open issue.

In addition to the above discussion, we can study the velocity fields in the presence of topological defects due to the multivaluedness of phase fluctuation field $\theta(\boldsymbol{x})$, playing an important role in the Kosterlitz–Thouless transition of superfluids. For simplicity, we denote the phase fluctuations as $\theta(\boldsymbol{x})$. The formation of the bound states will eliminate the divergent energy of a single defect. In $2\mathcal{SF}^1$, the bound states of $\Theta_{0x,0y}$ constitute two defects when the bound states of Θ_1 constitute four defects.[4]

In two spatial dimensions, we have four component velocity fields v_{ai} $(a, i = 1, 2)$. In $2\mathcal{SF}^1$, we discuss the vorticity to the topological charges of defects. For a single defect Θ_{0x} with the form with $\theta_2 = 0$, we have the velocity fields $(K_{12} = K, \rho_0 = \mu/g)$ as follows:

$$v_{11} = 2\partial_1\theta_1, \quad v_{12} = 2K\rho_0\partial_2\theta_1, \quad v_{22} = 0. \tag{16}$$

Then, the topological charge can be obtained from the vorticity

$$\ell_{0x} = \oint\Big(\frac{v_{11}}{2}dx + \frac{v_{12}}{2K\rho_0}dy\Big). \tag{17}$$

It resembles a conventional vortex in $2\mathcal{SF}^2$. Similarly, for a single Θ_{0y} with $\theta_1 = 0$, the velocity fields take the form

$$v_{11} = 0, \quad v_{12} = 2K\rho_0\partial_1\theta_2, \quad v_{22} = 2\partial_2\theta_2, \tag{18}$$

and we have the vorticity

$$\ell_{0y} = \oint \left(\frac{v_{12}}{2K\rho_0} dx + \frac{v_{22}}{2} dy \right). \tag{19}$$

In Figs. 1(a)–1(d), we depict the velocity fields (v_{11}, v_{12}) and (v_{12}, v_{22}) for the Θ_{0x} and Θ_{0y}, respectively.

The defect Θ_1 has no correspondence in a conventional superfluid phase. From Ref. [2], a bound state of two Θ_1 with opposite charges reduces to Θ_{0x} and Θ_{0y}, which reflect on their corresponding velocity fields. The velocity fields (v_{21}, v_{22}) of two Θ_1's with opposite charges are similar to the velocity fields (v_{11}, v_{12}) of a single Θ_{0y} [Figs. 1(e) and 1(f)], and velocity fields (v_{11}, v_{12}) of two Θ_1's with opposite charges to (v_{21}, v_{22}) of a single Θ_{0x}.

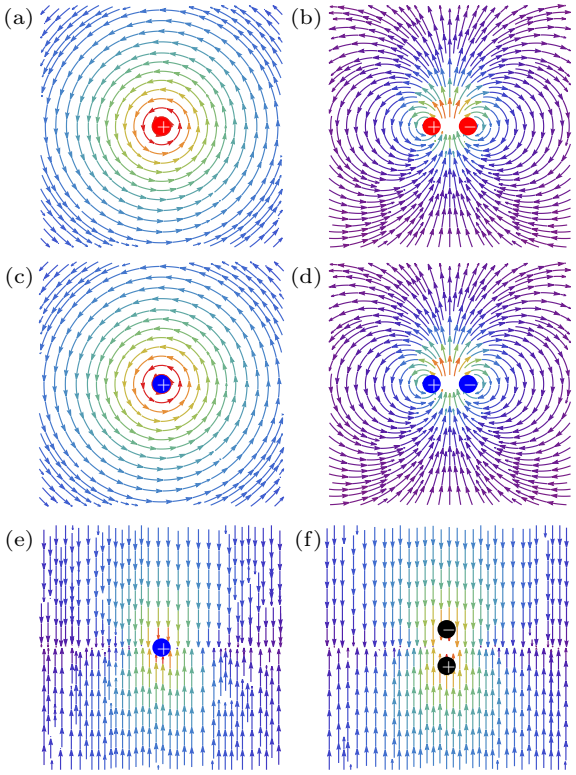

Fig. 1. The velocity fields in Eq. (7) of defects in $2\mathcal{SF}^1$: (v_{11}, v_{12}) in [(a), (b), (e)] and (v_{12}, v_{22}) in [(c), (d), (f)]. The colors and directions of arrows represent the strength (the brighter the color, the greater the strength) and direction of the velocity fields. The red, blue, and black dot mark the cores of the defects Θ_{0x}, Θ_{0y}, and Θ_1 with topological charge ± 1.

Landau-Like Criterion and Critical Currents. In the lineon condensation phase $d\mathcal{SF}^1$, the occurrence of currents harms the superfluidity. Now, we discuss the critical current, which is the critical value for the stability of $d\mathcal{SF}^1$.

For simplicity, we consider the isotropic case $K_{ab} = K$ with uniform velocity fields. In the presence of uniform superfluid density $\rho_a(\boldsymbol{x}) = \rho$, one may define $\nu = \sum_a |\partial_a \theta_a|^2 / d, w = \sum_a \sum_{b(\neq a)} |\partial_a \theta_b + \partial_b \theta_a|^2 / d$, as two average phase fluctuations. In addition, the energy density of $d\mathcal{SF}^1$ should be

$$\mathcal{E} = d\Big[\frac{\rho^2}{2} (g + Kw) + \rho (\nu - \mu) \Big], \tag{20}$$

which reaches its minimum when

$$\rho_0 = \frac{\mu - \nu}{Kw + g}. \tag{21}$$

Because $\rho_0 > 0$ in the superfluid phase, we obtain a condition $\nu < \mu$. It implies the critical value for velocity fields in the movable direction, $v_m^{\max} = 2\sqrt{\mu}$, which plays a role as the Landau criterion in a conventional superfluid phase, $d\mathcal{SF}^d$. Further, $w \equiv 0$ in $d\mathcal{SF}^d$, a conventional superfluid phase, as the condensed bosons can move in the entire space and $\nu \equiv 0$ in $d\mathcal{SF}^0$ as a single fracton is totally immovable. In $d\mathcal{SF}^1$, the current

$$J_{\mathrm{m}} = \sum_a |J_{aa}| / d \tag{22}$$

is along the movable direction and the current

$$J_{\mathrm{im}} = \sqrt{\sum_i \sum_{a(\neq i)} |J_{ai}|^2 / d} \tag{23}$$

is along the immovable direction. These two currents have different critical values. For $\nu = \mu/3$ and $w = 0$, currents only occur in the movable direction and J_{m} becomes maximal with

$$J_{\mathrm{m}}^{\max} = \frac{4\mu^{3/2}}{3\sqrt{3g}}, \tag{24}$$

whereas for $\nu = 0$ and $w = g/3K$, currents J_{im} only occur in the immovable direction with the maximal value $J_{\mathrm{im}}^{\max}$,

$$J_{\mathrm{im}}^{\max} = \frac{3\sqrt{3\,K}\mu^2}{8g^{3/2}}. \tag{25}$$

Once the currents exceed the critical values, the superfluid density vanishes and the system is no longer in the superfluid phase. In addition, thermal fluctuations may destroy the superfluid phase, which has been studied in Refs. [2,4].

Outlook. In this study, we derive the continuity equations and Navier–Stokes-like equations of a many-lineon system. We investigate the critical currents of fractonic superfluid phase $d\mathcal{SF}^1$ after lineons are condensed. The currents in the movable direction behave like currents in a conventional superfluid denoted by $d\mathcal{SF}^d$, whereas the currents in the immovable direction behave like currents in $d\mathcal{SF}^0$. With the equations we established, in the future work, we can investigate the basic hydrodynamic properties of $d\mathcal{SF}^1$, such as compressibility, viscosity, and irrotationality. As the equations are written with general d, it will be interesting to investigate the potential dimension reduction and underlying physical consequences.[37]

By putting $d\mathcal{SF}^1$ on a curved space, one can further study how the background gravitational field enters into the Navier–Stokes-like equations and how hydrodynamical field is entangled with geometric quantities of based manifold.[31,38] Finally, we can also regard bosonic lineon field Φ as the vacuum expectation of a bilinear form of a more fundamental fermion such that the Hamiltonian we wrote is a mean-field theory of a more fundamental fermion system in the language of projective construction using techniques in, e.g., Refs. [39–41]. Along this line of thinking, it is interesting to ask the analytic change of hydrodynamical properties in the presence of gauge fluctuations, which potentially signals exotic non-Fermi liquid theory or spin liquid states. We can also regard hydrodynamical behaviors in open systems, where an exotic non-Hermitian quantum effect may be expected,[42] and nontrivial entanglement properties[43,44] may be hidden in a set of non-Hermitian hydrodynamical equations.

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