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Unitary Scattering Protected by Pseudo-Hermiticity

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Hermitian systems possess unitary scattering. However, the Hermiticity is unnecessary for a unitary scattering although the scattering under the influence of non-Hermiticity is mostly non-unitary. Here we prove that the unitary scattering is protected by certain type of pseudo-Hermiticity and unaffected by the degree of non-Hermiticity. The energy conservation is violated in the scattering process and recovers after scattering. The subsystem of the pseudo-Hermitian scattering center including only the connection sites is Hermitian. These findings provide fundamental insights on the unitary scattering, pseudo-Hermiticity, and energy conservation, and are promising for light propagation, mesoscopic electron transport, and quantum interference in non-Hermitian systems.

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Pseudo-Hermiticity is important in non-Hermitian physics. The pseudo-Hermiticity ensures the spectrum of a non-Hermitian system to be entirely real or partly complex in conjugate pairs.[1] A non-Hermitian system is pseudo-Hermitian if its Hamiltonian under a unitary transformation equals the Hermitian conjugation of the Hamiltonian.[2] The parity-time ($\mathcal{PT}$) symmetric non-Hermitian systems are the mostly investigated pseudo-Hermitian systems,[3−13] which possess non-unitary dynamics even if their spectra are entirely real. The state involving nonorthogonal eigenmodes exhibits non-unitary intensity oscillation as observed in the coupled optical waveguides.[14−16] Nevertheless, the eigenstates are orthogonal and the time-evolution is unitary under the biorthogonal norm.[17,18] Interestingly, the state only involving real-valued orthogonal eigenmodes in the pseudo-Hermitian systems exhibits an intensity-preserving dynamics.[19] Otherwise, the intensity exponentially increases/decreases in the broken $\mathcal{PT}$-symmetric phase[20,21] or polynomially increases at the exceptional point where the $\mathcal{PT}$-symmetric phase transition occurs.[22−24] The exceptional point in the $\mathcal{PT}$-symmetric systems is experimentally realized in optical/acoustic cavity resonators,[25,26] in the single-photon interferometric quantum simulation,[27] in the single nitrogen-vacancy center,[28] and so on.[29]

Non-Hermitian systems provide unprecedented opportunities in recent decades.[30] The rapid developments in non-Hermitian physics greatly stimulate novel applications in optics, condensed matter physics, quantum physics, and material science.[31−38] For example, the exceptional point enhanced optical sensing,[39−43] robust energy transfer,[44,45] lasing,[46,47] and many other intriguing phenomena including the coherence perfect absorption,[48−53] unidirectional reflectionless/invisibility,[54−60] absorption,[61,62] amplification,[63,64] and lasing were discovered.[65−67] These reveal the non-unitary feature and the asymmetric feature of scattering affected by the non-Hermiticity.[68] In addition, the conservation is an important topic in non-Hermitian physics.[27,69,70] The energy conservation from the unitary scattering was reported in several non-Hermitian scattering centers.[71−75] Thus, the Hermiticity is unnecessary for a unitary scattering. However, the non-unitary scattering more commonly appears in the non-Hermitian systems because of the lack of energy conservation.[76−92] Then, what is essential for a unitary scattering and the energy conservation in non-Hermitian physics? This is a fundamental and important problem. Here we thoroughly solve this problem and unveil that the pseudo-Hermiticity plays a vital role.

In this Letter, we report that the unitary scattering in the non-Hermitian systems is protected by certain type of pseudo-Hermiticity. Under the pseudo-Hermiticity protection, the total probability of wave injection remains unity after scattering; whereas the energy conservation is violated in the scattering process as affected by the non-Hermiticity. We report that the unitary scattering and energy conservation are independent of the degree of non-Hermiticity, but strongly depend on the structure of the scattering center. We provide novel understandings of pseudo-Hermiticity from the perspective of scattering, and present fundamental insights on the unitary and non-unitary scattering. Furthermore, the consequences of symmetry protections on the scattering matrix under the time-reversal symmetry and reciprocity generalized for the non-Hermitian systems are presented.

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A lattice model is schematically illustrated in Fig. 1 and characterizes the discrete systems modeled under the tight-binding approximation in contrast to the continuous models,[93,94] for example, the coupled resonators/waveguides,[56,95] acoustic crystals,[57,96,97] cold atoms in optical lattice,[98] and electronic circuits.[99,100] These experimental platforms are intensively used for studying the non-Hermitian Hamiltonians. In general, the scattering center H_c has N sites; and all the L ports are connected to different sites of the scattering center ($L \leq N$).

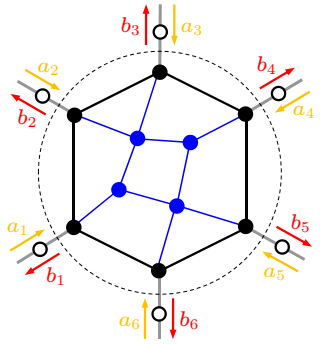

Fig. 1. Schematic of a general scattering system. The scattering center is indicated inside the dashed black circle, where the solid circles are the scattering center sites and the solid lines indicate their couplings. The ports in gray are coupled to the connection sites in black, and the other sites disconnected with the ports are the bulk sites in blue. The first sites of the ports are in white. The arrows indicate the incoming and outgoing waves in the ports.

The properties of the scattering center are fully characterized by the scattering matrix S. Acting the scattering matrix on the incoming wave amplitudes yields the outgoing wave amplitudes

$$B = SA, \tag{1}$$

where $A = [a_1, a_2, \cdots, a_{L-1}, a_L]^{\mathrm{T}}$ represents the incoming wave amplitudes of all the L ports before scattering, and $B = [b_1, b_2, \cdots, b_{L-1}, b_L]^{\mathrm{T}}$ represents the outgoing wave amplitudes of all the L ports after scattering. The scattering matrix element s_{nm} describes the output in the nth port for the wave injection in the mth port.

The wavefunction at the steady state is the superposition of the plane waves with the opposite momenta k propagating in the opposite directions. The incoming wave is e^{-ikj} and the outgoing wave is e^{ikj} for the momentum k, where the integer $j > 0$ indexes the sites of the ports and $j = 0$ represents the connection site of the scattering center. To reflect the properties of the scattering center H_c, all the ports are chosen to be identical and uniform until the scattering center. For the wave injection in the mth port, the wavefunction in the mth port is $\varphi_{p,m}(j) = e^{-ikj} + s_{mm}e^{ikj}$ and the wavefunction in the nth port is $\varphi_{p,n}(j) = s_{nm}e^{ikj}$.

Symmetries are extremely important in physics. The pseudo-Hermitian scattering center satisfies

$$qH_c^* q^{-1} = H_c^{\mathrm{T}}. \tag{2}$$

In the superscripts, * denotes the complex conjugation operation, T represents the transpose operation, and q is the unitary matrix defined in the real space representation of the scattering center with $q^2 = I_N$.[101,102] The progresses on the symmetry classification have greatly advanced our knowledge on the symmetric scattering in non-Hermitian physics.[103] In general case, the pseudo-Hermiticity cannot ensure the unitary scattering. However, the scattering matrix S is unitary if the unitary matrix q satisfies

$$q = \begin{pmatrix} I_L & 0 \\ 0 & q_{\mathrm{b}} \end{pmatrix}, \tag{3}$$

where identical matrix I_L is for the subsystem including only the L connection sites and the unitary matrix q_{b} is for the subsystem including only the $N - L$ bulk sites. For all the bases of the scattering center, the connection sites are exactly mapped to themselves under the unitary transformation q. In Fig. 1, the black part is Hermitian and the blue part is pseudo-Hermitian; otherwise, the unitary scattering is not ensured.

Applying the unitary transformation q to the pseudo-Hermitian scattering system, the bulk sites (the blue circles in Fig. 1) and the couplings between the bulk and connection sites (the blue lines in Fig. 1) are altered, but the subsystem including all the connection sites (the black circles and lines in Fig. 1) is unchanged. Consequently, the wavefunctions of the connection sites and all the ports are invariant under the unitary transformation q. Thus, the scattering matrix for the scattering center $qH_c^*q^{-1}$ is identical with the scattering matrix for the scattering center H_c^*.

From Eq. (1), we obtain $A^* = (S^*)^{-1} B^*$. After applying the complex conjugation operation to the wavefunctions of the scattering system, the vector B^* indicates the incoming wave amplitudes, and the vector A^* indicates the outgoing wave amplitudes. Thus, $(S^*)^{-1}$ stands for the scattering matrix of the scattering center H_c^*.

Then, the scattering matrix $(S^*)^{-1}$ is also the scattering matrix of $qH_c^*q^{-1}$. Notably, the scattering matrix for H_c^{T} is S^{T}.[104] From the pseudo-Hermiticity of H_c, we obtain the relation $(S^*)^{-1} = S^{\mathrm{T}}$. Therefore, the scattering matrix is unitary

$$SS^\dagger = I_L. \tag{4}$$

The unity element in the mth row and mth column of $S^{\mathrm{T}}S^*$ is $\sum_{n=1}^{L} s_{nm}s_{nm}^* = 1$ for the input

in the mth port. The unity diagonal elements of $S^{\mathrm{T}}S^{*}$ yield the unity total probability and energy conservation after scattering for the input in any port. However, the dynamics is non-unitary and the energy conservation is invalid as affected by the non-Hermiticity in the scattering process. Notably, the pseudo-Hermiticity-protected unitary scattering is independent of the degree of non-Hermiticity. Under the pseudo-Hermiticity protection, the unitary scattering is unaffected by the strengths of non-Hermitian couplings and the rates of gains/losses; however, these non-Hermitian elements affect the reflections, the transmissions, and the dynamics in the scattering process.

The pseudo-Hermitian scattering centers holding unitary scattering have featured structures. H_{e} denotes the subsystem that only contains the connection sites; H_{b} denotes the subsystem that only contains the bulk sites; H_{eb} and H_{be} denote the couplings between the connection sites and the bulk sites. The scattering center reads

$$H_{\mathrm{c}} = \begin{pmatrix} H_{\mathrm{e}} & H_{\mathrm{eb}} \\ H_{\mathrm{be}} & H_{\mathrm{b}} \end{pmatrix}. \tag{5}$$

The pseudo-Hermiticity $qH_{\mathrm{c}}^{*}q^{-1} = H_{\mathrm{c}}^{\mathrm{T}}$ and the block diagonal q in Eq. (3) yield

$$\begin{pmatrix} H_{\mathrm{e}}^{*} & H_{\mathrm{be}}^{*}q_{\mathrm{b}}^{-1} \\ q_{\mathrm{b}}H_{\mathrm{eb}}^{*} & q_{\mathrm{b}}H_{\mathrm{b}}^{*}q_{\mathrm{b}}^{-1} \end{pmatrix} = \begin{pmatrix} H_{\mathrm{e}}^{\mathrm{T}} & H_{\mathrm{eb}}^{\mathrm{T}} \\ H_{\mathrm{be}}^{\mathrm{T}} & H_{\mathrm{b}}^{\mathrm{T}} \end{pmatrix}. \tag{6}$$

From $q^2 = I_N$, we have $q = q^{-1}$, $q_{\mathrm{b}} = q_{\mathrm{b}}^{-1}$; and $q_{\mathrm{b}}(q_{\mathrm{b}}^{-1})^{\dagger} = I_{N-L}$. Therefore, $(H_{\mathrm{be}}^{*}q_{\mathrm{b}}^{-1})^{\dagger} = (H_{\mathrm{eb}}^{\mathrm{T}})^{\dagger}$ yields $q_{\mathrm{b}}(q_{\mathrm{b}}^{-1})^{\dagger}H_{\mathrm{be}}^{\mathrm{T}} = q_{\mathrm{b}}H_{\mathrm{eb}}^{*}$.

The unitary scattering requires three constrains

$$\text{(i)}\ H_{\mathrm{e}}^{*} = H_{\mathrm{e}}^{\mathrm{T}},\ \text{(ii)}\ q_{\mathrm{b}}H_{\mathrm{b}}^{*}q_{\mathrm{b}}^{-1} = H_{\mathrm{b}}^{\mathrm{T}},\ \text{(iii)}\ q_{\mathrm{b}}H_{\mathrm{eb}}^{*} = H_{\mathrm{be}}^{\mathrm{T}}. \tag{7}$$

The pseudo-Hermitian scattering center may have all kinds of non-Hermitian elements including the gain/loss, the imaginary/complex coupling, and the asymmetric coupling, etc. These non-Hermitian elements may simultaneously present in the pseudo-Hermitian scattering center that possesses the unitary scattering. The constrain (i) requires that the subsystem H_{e} including only the connection sites is Hermitian. The constrain (ii) requires that the subsystem H_{b} including only the bulk sites is pseudo-Hermitian. The constrain (iii) is the requirement on the couplings H_{eb} and H_{be} between the connection sites and the bulk sites. Thus, the gain and loss cannot appear at the connection sites, but can appear on the bulk sites in the balanced pairs; whereas the non-Hermitian couplings including both the imaginary/complex coupling and the asymmetric coupling cannot appear between the connection sites, but can appear among the bulk sites or as the connection couplings in H_{eb} and H_{be}. Otherwise, the scattering is non-unitary.

The pseudo-Hermiticity-protected two-port scattering centers possess symmetric transmission and reflection for the wave injections from the opposite directions ($|t_{\mathrm{L}}|^2 = |t_{\mathrm{R}}|^2$ and $|r_{\mathrm{L}}|^2 = |r_{\mathrm{R}}|^2$). This is obtained from the unitary scattering $SS^{\dagger} = [1, 0; 0, 1]$ with $S = [r_{\mathrm{L}}, t_{\mathrm{R}}; t_{\mathrm{L}}, r_{\mathrm{R}}]$,[105] where t_{L} and r_{L} (t_{R} and r_{R}) are the transmission and reflection coefficients for the input in the left (right) port. These explain the unitary and symmetric scattering in the two-port non-Hermitian scattering centers.[71–75] However, the symmetric scattering is not promised in the multi-port scattering centers although the scattering is unitary. In a circulator, the wave injected in the port $1, 2, 3$ resonantly outgoes from the port $2, 3, 1$, respectively.[106] The scattering is asymmetric when considering the wave input and output in any two of the three ports.

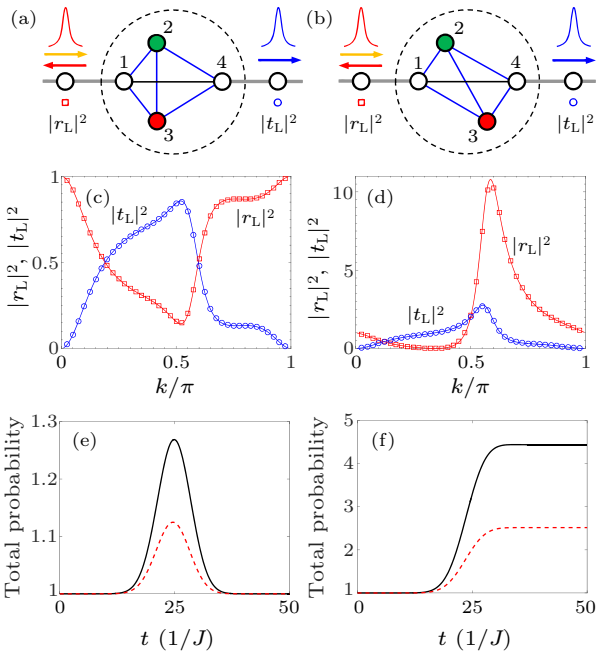

Fig. 2. Schematics of the two-port pseudo-Hermitian systems with (a) unitary and (b) non-unitary scattering. The reflection $|r_{\mathrm{L}}|^2$ and transmission $|t_{\mathrm{L}}|^2$ are plotted in (c) and (d) as indicated by the markers in the ports; and the total probability in the numerical simulations for $k = \pi/2$ are depicted in (e) and (f). In (c) and (d), the curves are analytical results and the markers are numerical simulations. In (e) and (f), the solid black (dashed red) lines are for the left (right) input. The parameters are $\gamma = J_1 = J$, $\kappa_1 = \kappa_2 = J_2 = J/2$.

We have rigorously proved that the unitary scattering and the energy conservation in the non-Hermitian systems are protected by certain pseudo-Hermiticity if the unitary matrix q that has defined the pseudo-Hermiticity satisfies Eq. (3). Furthermore, we elaborate a two-port scattering center to emphasize the importance of the scattering center configuration and a three-port scattering center to emphasize the importance of the port configuration for the uni-

tary scattering. In the schematics, each site stands for a resonator with frequency $\omega_{\rm c}$. The ports until the scattering center are uniform at the coupling $-J$. The light propagation in the coupled resonator optical waveguides is govern by the discrete lattice model,[107] and the dispersion relation supported by the ports is $E=\omega_{\rm c}-2J\cos k$.[65,66]

The two-port pseudo-Hermitian scattering center in Fig. 2(a) includes two resonators with balanced gain and loss (2 and 3) and two connection resonators (1 and 4). If each connection resonator is equally coupled to the gain and loss resonators, the equations of motion for the scattering center are

$$\begin{aligned}
i\dot{\psi}_{{\rm c},1} &= \omega_{\rm c}\psi_{{\rm c},1}-J_1\psi_{{\rm c},2}-J_1\psi_{{\rm c},3}-\kappa_2\psi_{{\rm c},4}-J\psi_{p,1}(1),\\
i\dot{\psi}_{{\rm c},2} &= (\omega_{\rm c}+i\gamma)\,\psi_{{\rm c},2}-J_1\psi_{{\rm c},1}-\kappa_1\psi_{{\rm c},3}-J_2\psi_{{\rm c},4},\\
i\dot{\psi}_{{\rm c},3} &= (\omega_{\rm c}-i\gamma)\,\psi_{{\rm c},3}-J_1\psi_{{\rm c},1}-\kappa_1\psi_{{\rm c},2}-J_2\psi_{{\rm c},4},\\
i\dot{\psi}_{{\rm c},4} &= \omega_{\rm c}\psi_{{\rm c},4}-J_2\psi_{{\rm c},2}-J_2\psi_{{\rm c},3}-\kappa_2\psi_{{\rm c},1}-J\psi_{p,2}(1),
\end{aligned} \tag{8}$$

where $\psi_{{\rm c},j}$ is the wavefunction for the resonator j of the scattering center and $\psi_{p,j}\,(1)$ is the wavefunction for the first resonator of the port j. The transmission and reflection coefficients $t_{\rm L,R}$, $r_{\rm L,R}$ as functions of the input wave vector k are obtained from the steady-state solution; the scattering matrix is unitary $SS^{\dagger}=I_2$.

The four-site scattering center is pseudo-Hermitian with the Hamiltonian $H_{\rm c}$ and the unitary matrix q,

$$\begin{aligned}
H_{\rm c} &= \omega_{\rm c}I_4-\begin{pmatrix} 0 & J_1 & J_1 & \kappa_2\\ J_1 & -i\gamma & \kappa_1 & J_2\\ J_1 & \kappa_1 & i\gamma & J_2\\ \kappa_2 & J_2 & J_2 & 0\end{pmatrix},\\
q &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},
\end{aligned} \tag{9}$$

where $+i\gamma$ and $-i\gamma$ are the gain and loss; all the couplings J_1, J_2, κ_1, κ_2 are real numbers. In the subspaces of connection sites and bulk sites, q satisfies Eq. (3). Thus, the scattering matrix is unitary and symmetric. Notably, the scattering is still unitary if the real couplings J_1 and J_2 are simultaneously imaginary.

The transmission and reflection for the unitary scattering in Fig. 2(a) are depicted in Fig. 2(c). The energy conservation holds after scattering and the total transmitted and reflected wave probability is unity. However, the dynamics is non-unitary and the energy conservation is invalid in the scattering process as numerically simulated in Fig. 2(e) using a Gaussian profile initial excitation $\Omega^{-1/2}\sum_j e^{-(j-n_0)^2\alpha^2/2}e^{-ikj}\,|j\rangle$ of the momentum k centered at the site n_0, where $\alpha=0.1$ controls the width, Ω is the normalization factor, and $|j\rangle$ is the basis of port site j. The scattering process begins when the wave packet reaching the scattering center and the total probability of the excitation starts to change as time because of the influence of non-Hermiticity. The velocity of the Gaussian wave packet obtained from the dispersion relation is $dE/dk=2J\sin k$. The scattering process ends when the wave packet leaving the scattering center and the total probability no longer changes with time. The dynamics after scattering reflects the steady-state solution. The reflected backward going wave packet indicates the reflection and the transmitted forward going wave packet indicates the transmission.[108]

Alternatively, if each connection resonator is unequally coupled to the gain and loss resonators as shown in Fig. 2(b), the four-site scattering center is still pseudo-Hermitian with $H_{\rm c}$ and q:

$$H_{\rm c}=\omega_{\rm c}I_4-\begin{pmatrix} 0 & J_1 & J_2 & \kappa_2\\ J_1 & -i\gamma & \kappa_1 & J_2\\ J_2 & \kappa_1 & i\gamma & J_1\\ \kappa_2 & J_2 & J_1 & 0\end{pmatrix},\quad q=\sigma_x\otimes\sigma_x, \tag{10}$$

where σ_x is the Pauli matrix. Notably, the unitary transformation q is not block diagonalized in the subspaces of connection sites and bulk sites. In this situation, the scattering matrix is non-unitary although the non-Hermitian scattering center is pseudo-Hermitian. The transmission and reflection for the non-unitary scattering in Fig. 2(b) are depicted in Fig. 2(d). The scattering is non-unitary in the entire scattering process and the total probability after scattering is non-unity as demonstrated in Fig. 2(f).

The unitary scattering is still possible if the couplings between the ports and the scattering center are properly redesigned. If both ports in Fig. 2(b) are simultaneously coupled to sites 1 and 4 through the first sites of the ports at the same strength $-J$, the scattering becomes unitary. The first sites of two ports effectively become the connection sites of a six-site scattering center $H_{\rm c}'$ and q' satisfies Eq. (3) with $q_{\rm b}'=\sigma_x\otimes\sigma_x$.

The unitary scattering is also subtle to the port configuration. We consider a multi-port pseudo-Hermitian scattering center with six sites and three ports. The Hamiltonian of the scattering center reads

$$H_{\rm c}=\sum_{j=1}^{6}(\omega_{\rm c}+\Delta_j)\hat{c}_j^{\dagger}\hat{c}_j+i\gamma_j(\hat{c}_j^{\dagger}\hat{c}_{j+1}+\hat{c}_{j+1}^{\dagger}\hat{c}_j), \tag{11}$$

where $\hat{c}_j$ ($\hat{c}_j^{\dagger}$) is the annihilation (creation) operator for the mode of resonator j and satisfies $\hat{c}_{j+6}=\hat{c}_j$ ($\hat{c}_{j+6}^{\dagger}=\hat{c}_j^{\dagger}$). The resonator detuning Δ_j is real. The coupling $i\gamma_j$ is non-Hermitian and can be realized through the gain.[64,109] The scattering center $H_{\rm c}$ is pseudo-Hermitian for the unitary matrix $q=I_3\otimes\sigma_z$. In the couple mode theory, the equation of motion for the light field $\psi_{{\rm c},j}$ of the scattering center resonator j is

$$i\dot{\psi}_{{\rm c},j}=(\omega_{\rm c}+\Delta_j)\,\psi_{{\rm c},j}+i\gamma_{j-1}\psi_{{\rm c},j-1}+i\gamma_j\psi_{{\rm c},j+1}. \tag{12}$$

If the three ports $1,2,3$ are respectively connected to the odd-site $|1\rangle_c, |3\rangle_c, |5\rangle_c$ (or the even-site $|2\rangle_c, |4\rangle_c, |6\rangle_c$) in Fig. 3(a), the additional term $-J\psi_{p,1}, -J\psi_{p,2}, -J\psi_{p,3}$ presents in the right side of the equations of motion for the sites $j = 1,3,5$ (or $j = 2,4,6$). This pseudo-Hermiticity ensures a unitary scattering $SS^\dagger = I_3$. The scattering coefficients are obtained from the steady-state solution. The transmission and reflection in the three ports are depicted in Fig. 3(c). The unity total probability after scattering reflects the energy conservation as demonstrated in Fig. 3(e), where the initial excitation is the one used in the two-port scattering center. Notably, additional Hermitian couplings presented among three connections sites will not affect the unitary scattering.

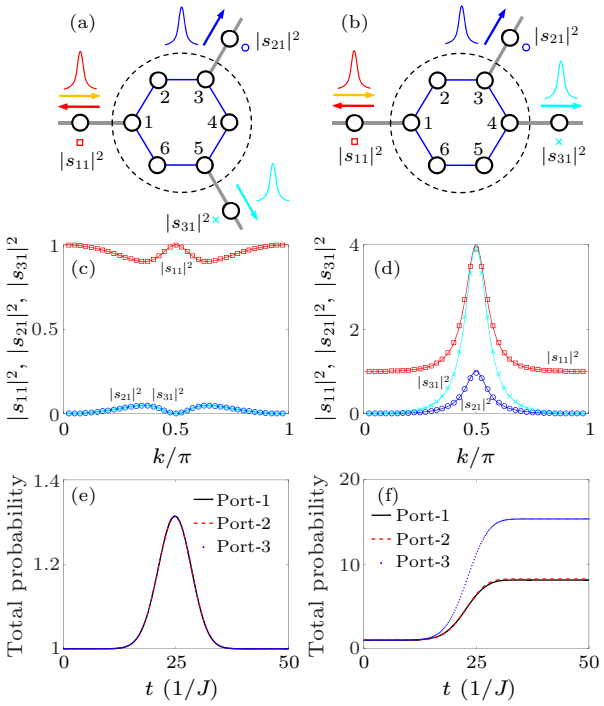

Fig. 3. Schematics of the three-port pseudo-Hermitian systems with (a) unitary and (b) non-unitary scattering. The reflection $|s_{11}|^2$ and transmissions $|s_{21}|^2$, $|s_{31}|^2$ are plotted in (c) and (d) as indicated by the markers in the ports; the curves are analytical results; the markers are numerical simulations. The corresponding total probability in the numerical simulations for $k = \pi/2$ are depicted in (e) and (f). The parameters are $\gamma_j = J$ and $\Delta_j = 0$ for all the sites.

If the three ports are simultaneously connected to the odd-site and even-site of the scattering center, for example, the three ports are connected to the sites $|1\rangle_c, |3\rangle_c, |4\rangle_c$ in Fig. 3(b). The scattering is non-unitary. The transmission and reflection in the three ports are depicted in Fig. 3(d), the non-unity total probability indicates the non-unitary scattering and the absence of energy conservation as observed in Fig. 3(f). The scattering coefficients diverge and lasing occurs at the spectral singularity $2\gamma_j^2 = J^2$ and $\Delta_j = 0$ for all the six sites.[65,67]

All the three constrains in Eq. (7) are satisfied for the unitary scattering, but at least one of the three constrains is not satisfied for the non-unitary scattering in the exemplified pseudo-Hermitian scattering centers. Nevertheless, any pseudo-Hermitian scattering center can exhibit a unitary scattering if the couplings between the scattering center and the ports are properly redesigned according to its structure information, which is completely encoded in the unitary operator q. In this situation, the scattering center is effectively enlarged to include the first sites of the ports. The original pseudo-Hermitian scattering center H_c plays the role as the subsystem H'_b of the enlarged scattering center H'_c. The couplings between the scattering center and the first sites of the ports should be reconstructed to satisfy $q'_b H'^*_{eb} = H'^T_{be}$, where the unitary operator $q'_b = q$ defines the pseudo-Hermiticity of the original scattering center H_c with $qH_c^* q^{-1} = H_c^T$. The unitary operator q' defines the pseudo-Hermiticity of the enlarged scattering center H'_c,

$$q' = \begin{pmatrix} I_L & 0 \\ 0 & q \end{pmatrix}, \quad H'_c = \begin{pmatrix} \omega_c I_L & H'_{eb} \\ H'_{be} & H_c \end{pmatrix}, \\ q'(H'_c)^* q'^{-1} = (H'_c)^T. \tag{13}$$

After the reconstruction, the effective scattering center H'_c has the unitary scattering $S'S'^\dagger = I_L$ and ensures the energy conservation.

In addition, the operations of Hermitian conjugation ($\dagger$), complex conjugation ($*$), transpose (T), and the unit element constitute a V_4 (also called D_2) Abelian group. Three operations define the pseudo-Hermiticity $qH_c^\dagger q^{-1} = H_c$, the time-reversal symmetry $qH_c^* q^{-1} = H_c$, and the reciprocity $qH_c^T q^{-1} = H_c$ for the non-Hermitian systems, respectively. The pseudo-Hermiticity ensures $SS^\dagger = I_L$, the time-reversal symmetry ensures $SS^* = I_L$, and the reciprocity ensures $S = S^T$ if the unitary operator q satisfies Eq. (3).

In conclusion, the unitary scattering and the energy conservation are ensured by certain pseudo-Hermiticity, where the scattering center structure plays an important role. The pseudo-Hermiticity protected unitary scattering is independent of the degree of non-Hermiticity. The energy conservation holds after scattering although it is invalid in the scattering process under the influence of the non-Hermiticity. We also demonstrate how to create a unitary scattering in the pseudo-Hermitian system with a non-unitary scattering through reconstructing the connection couplings. We unveil the physics of unitary scattering, present novel understanding of pseudo-Hermiticity, and provide fundamental insight on the energy conservation in non-Hermitian physics. Our findings are also important guiding principles promising for the non-unitary scattering. These findings shed light on

the fundamental research and potential applications of non-Hermitian scattering including the light propagation, mesoscopic electron transport, and quantum interference.[110]

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[104] For a rigorous proof, please see Eq. (12) in Ref. [103] and compare the scattering coefficients of the scattering centers H and H^{T}. Notice that the symbols t_{L}, t_{R}, r_{L} and r_{R} in Ref. [103] are s_{nm}, s_{mn}, s_{mm} and s_{nn} of the scattering matrix with our current notations for any pair of ports m and n. The fact $(A^T)^{-1} = (A^{-1})^T$ is also used in the proof for any square matrix A.
[105] From the off-diagonal term of $SS^{\dagger}$, we obtain $r_{\mathrm{L}} = -t_{\mathrm{R}} r_{\mathrm{R}}^{*}/t_{\mathrm{L}}^{*}$. From the diagonal terms of $SS^{\dagger}$, we obtain $1 = r_{\mathrm{L}} r_{\mathrm{L}}^{*} + t_{\mathrm{R}} t_{\mathrm{R}}^{*} = (r_{\mathrm{R}}^{*} r_{\mathrm{R}}/t_{\mathrm{L}}^{*} t_{\mathrm{L}} + 1)\, t_{\mathrm{R}} t_{\mathrm{R}}^{*} = t_{\mathrm{R}} t_{\mathrm{R}}^{*}/t_{\mathrm{L}}^{*} t_{\mathrm{L}}$.
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