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Transverse Rutherford Scattering of Electron-Ion Collision in a Uniformly Magnetized Plasma

Chang Jiang(姜畅)[1,2], Chao Dong(董超)[1,2], and Ding Li(李定)[1,3,2*]

[1]*Laboratory of Soft Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China*
[2]*University of Chinese Academy of Sciences, Beijing 100049, China*
[3]*Songshan Lake Materials Laboratory, Dongguan 523808, China*

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Rutherford scattering formula plays an important role in plasma classical transport. It is urgent to investigate influence of magnetic field on the Rutherford scattering since the high magnetic field has been widely used in nowadays magnetic confinement fusion, inertial confinement fusion, and magneto-inertial fusion. In order to elucidate the magnetic field effect in a concise manner, we study the electron-ion collisions transverse to the magnetic field. The scattering angle is defined using the directions of electron velocity before and after collision, which is obtained analytically. It is found that the scattering angle can be influenced by finite magnetic field significantly. The theoretical results agree well with numerical calculation by checking the dependence of scattering angle on the magnetic field.

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Coulomb collision process is the dominant mechanism for plasma classical transport. Collision processes are dominated by binary scattering events for weakly coupled plasmas where multi-particle interaction can be approximated using superposition of a series of independent binary collisions. It is necessary and urgent to understand influence of magnetic field on binary collisional scattering due to increase of magnetic field in fusion devices, such as toroidal magnetic field in magnetic confinement fusion, self-generated magnetic field in inertial confinement fusion and compressed magnetic field in magneto-inertial fusion, which is also important for understanding plasma transport, heating, and confinement.

Rutherford firstly studied the binary scattering process of charged particles in 1911[1] and revealed the structures of an atom. Later, Rosenbluth, MacDonald, and Judd[2] derived the Fokker–Planck[3,4] equation in plasmas using the Rutherford formulas for describing the scattering angle and cross section. Due to the long-distance nature of the Coulomb interaction, a cutoff is made at a small scattering angle. They introduced the Rosenbluth potentials to represent the friction and diffusion coefficients for any form of distribution functions. Different from traditional treatment, Chang and Li used small-momentum-transfer as the cutoff variable to derive the Fokker–Planck equation in 1996.[5] However, the magnetic field effect is absent in these works.

It should be noted that a pioneering study has used the binary collision method to calculate the magnetized Fokker–Planck coefficients. In 1989, Ware[6] calculated the parallel friction and diffusion coefficients with the impact parameter b between the electron Larmor radius ρ_{e} and the Debye length λ_{D}, and treated the binary collision process perturbatively in zero Larmor radius limit. In this way, he derived a magnetized Fokker–Planck collision term. Later, finite Larmor radius effects were considered by Dong *et al.*[7]

Binary collision approach has also been employed to investigate directly the transport processes such as the slowing-down and temperature relaxation processes.[8,9] However, Fokker–Planck equation in the presence of a magnetic field cannot be solved in a closed form using perturbative treatment of binary collision process. The perturbative treatment is valid for weakly coupled plasmas while insufficient for moderately and strongly coupled plasmas.

Some works have dealt the two-body collision process with the scattering angle and the cross section. Siambis solved like and unlike particle collisions numerically where the initial relative velocity was parallel to the magnetic field in 1976.[10] The scattering angle is defined with the difference between the initial and final pitch angles. He found that the scattering angle and cross section are almost the same with the Rutherford formulas. Hu *et al.*[11] have simulated the electron-ion collision case with the electron initial velocity parallel to the magnetic field so that they can use the pitch angle to replace the scattering angle in three-dimensional problem. They found that the scattering angle has a fractal dependence on the impact

*Corresponding author. Email: dli@iphy.ac.cn
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parameter in the chaotic scattering intervals. However, their definitions of scattering angle could not represent the scattering process perpendicular to the magnetic field.

In extremely strong magnetic field limit, Psimopoulos and Li[12] found that the cross-field heat transport occur even through mass transport only along the field lines by using the guiding center approximation and regarding the electron to move only along the magnetic field lines. Later, Koryagin proposed a collision integral for the strong magnetic field.[13] He considered the cases with the impact parameters larger than the Larmor radius. He used the impact parameter of the electron gyro-center to define a differential scattering cross section. A binary collision integral in Boltzmann's form and a cross section of the electron–ion collision in strong magnetic fields were derived. However, the impact parameters smaller than the Larmor radius are still treated as without a magnetic field.

So far, it seems that no one has got a satisfactory Rutherford formula with magnetic field, including scattering angle and cross section. However, it is hard to define a new scattering angle and scattering cross section in three-dimensional cases. In this work, we consider the electron-ion Coulomb collisions transverse to a uniform magnetic field, which could be useful for the two-dimensional plasma transport problems.[14,15]

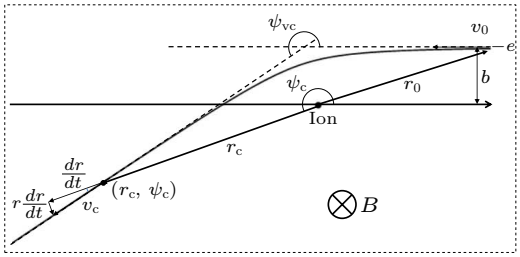

Fig. 1. The electron trajectory in the magnetic fields along the $-Z$ direction. Here (r_c, ψ_c) represents the point at which the magnitudes of the change rates of the electron velocity direction caused by Lorentz force and Coulomb force are equal, v_c is the electron velocity at (r_c, ψ_c).

A new scattering angle in the two-dimensional case is defined with the scattering effects of both Coulomb force and Lorentz force. In Rutherford scattering without magnetic field, the scattering angle is determined by the change of electron velocity before and after the collision. In the presence of a magnetic field, the electron trajectory depends on the superposition of Larmor precession and Coulomb interaction, where the magnetized scattering angle must be defined in a similar way as Rutherford. In order to mitigate the technical complication of the electron trajectory in magnetized collision process, a two-dimensional electron-ion collision is considered perpendicular to the magnetic field B.

The process of electron colliding with a fixed ion is shown in Fig. 1. The initial electron velocity is set as v_0 and the impact parameter is b; r and ψ are the electron radial and angular coordinates in the polar coordinate system; ψ_v is the electron angular velocity; (r_c, ψ_c) represents the point where the magnitudes of the change rates of the electron velocity direction caused by Lorentz force and Coulomb force are equal. The magnitude and the polar angle of electron velocity at this point are v_c and ψ_{vc}. If the changes of the velocity direction caused by the two forces are opposite, the total velocity direction change will be zero. The initial electron-ion distance is chosen as r_0, where the Lorentz force and the Coulomb force are roughly equal for certain magnetic field, and we assume $r_0 \gg b$. It is found that the direction of the electron velocity only changes rapidly when the electron is close to the ion in a relatively weak magnetic field while the direction change becomes slower when the electron is far away from the ion.

When the deflection caused by cyclotron motion partially counteracts the Rutherford scattering, the point (r_c, ψ_c) can be found as shown in Fig. 1. The scattering angle θ is defined as $\psi_{vc} - \pi$, which will be analyzed in the following.

The electron motion equation is

$$-e\boldsymbol{E} - e\boldsymbol{v} \times \boldsymbol{B} = m\frac{d^2\boldsymbol{r}}{dt^2}, \tag{1}$$

where $\boldsymbol{r}$ and $\boldsymbol{v}$ are the electron position vector and velocity, $\boldsymbol{E} = (e/4\pi\varepsilon_0 r^2)\boldsymbol{e}_r$ and $\boldsymbol{B} = B\boldsymbol{e}_z$ are the Coulomb electric field and magnetic field, respectively, e and m represent the elementary charge and mass, ε_0 is the vacuum permittivity, and $\boldsymbol{e}_r$ and $\boldsymbol{e}_z$ are the unit vector in the radial and z directions. Equation (1) can be separated in the radial and angular directions, which gives

$$-\frac{e^2}{4\pi\varepsilon_0 r^2} - er\frac{d\psi}{dt}B = m\Big[\frac{d^2r}{dt^2} - r\Big(\frac{d\psi}{dt}\Big)^2\Big], \tag{2}$$

$$e\frac{dr}{dt}B = m\Big(2\frac{dr}{dt}\frac{d\psi}{dt} + r\frac{d^2\psi}{dt^2}\Big). \tag{3}$$

Integrating Eq. (3) over time yields

$$r^2\Big(\frac{d\psi}{dt} - \frac{eB}{2\,m}\Big) = C_1, \tag{4}$$

where C_1 is a constant. From the conservation of angular momentum, we have $r_0^2 d\psi_0/dt = v_0 b$ so that Eq. (4) becomes

$$\frac{d\psi}{dt} = \frac{\omega}{2} + \frac{1}{r^2}\Big(v_0 b - \frac{\omega r_0^2}{2}\Big), \tag{5}$$

where $\omega = eB/m$ is the electron cyclotron frequency.

The energy conservation equation in polar coordinates is

$$v_0^2 + \frac{2b_0 v_0^2}{r_0} - \frac{2b_0 v_0^2}{r} = \Big(\frac{dr}{dt}\Big)^2 + r^2\Big(\frac{d\psi}{dt}\Big)^2. \tag{6}$$

Here, $b_0 = -e^2/4\pi\varepsilon_0 m v_0^2$ is the impact parameter for 90° scattering. Negative b_0 represents an attractive Coulomb interaction. Equation (6) can be rewritten as

$$\frac{dr}{dt} = \sqrt{v_0^2 + \frac{2b_0 v_0^2}{r_0} - \frac{2b_0 v_0^2}{r} - r^2\Big(\frac{d\psi}{dt}\Big)^2}. \tag{7}$$

From Fig. 1, it is easy to find the expression for ψ_v:

$$\psi_v = \psi + \arctan\Big(r\frac{d\psi}{dt}/\frac{dr}{dt}\Big). \tag{8}$$

From Eqs. (5), (7), and (8), The differential of ψ_v can be expressed as

$$\frac{d\psi_v}{dt} = \omega - \frac{\frac{b_0}{r}\Big[\frac{\omega}{2} + \frac{1}{r^2}\Big(v_0 b - \frac{\omega r_0^2}{2}\Big)\Big]}{1 + \frac{2b_0}{r_0} - \frac{2b_0}{r}}, \tag{9}$$

where the first term on the right hand side (RHS) represents the cyclical motion caused by Lorentz force, the second term on the RHS represents the reflection caused by Coulomb force; r_c is defined as the point where the magnitudes of the first and second terms on the RHS of Eq. (9) are equal to each other, which means

$$|\omega| = -\frac{\frac{b_0}{r_c}\Big[\frac{\omega}{2} + \frac{1}{r_c^2}\Big(v_0 b - \frac{\omega r_0^2}{2}\Big)\Big]}{1 + \frac{2b_0}{r_0} - \frac{2b_0}{r_c}}. \tag{10}$$

Under the assumption of $r_0^2 \gg r_c^2$ and $r_c \gg b_0$, Eq. (10) is simplified to

$$|\omega| \approx -\frac{b_0}{r_c^3}\Big(v_0 b - \frac{\omega r_0^2}{2}\Big). \tag{11}$$

The approximate root of r_c can be expressed as

$$r_c \approx \sqrt[3]{-\frac{b_0(v_0 b - \omega r_0^2/2)}{|\omega|}}. \tag{12}$$

The differential form of the trajectory equation can be written from Eqs. (5) and (7) as

$$\frac{d\psi}{dr} = \frac{\frac{\omega}{2v_0} + \frac{b}{r^2} - \frac{\omega r_0^2}{2v_0 r^2}}{\sqrt{1 + \frac{2b_0}{r_0} - \frac{2b_0}{r} - r^2\Big(\frac{\omega}{2v_0} + \frac{b}{r^2} - \frac{\omega r_0^2}{2v_0 r^2}\Big)^2}}, \tag{13}$$

where $\omega r_0^2/v_0 b \sim b_0/b \ll 1$ for the small angle scattering of our interest. The minimum radial distance $r_{\min}$ has been obtained by setting the denominator of Eq. (13) equal to zero, which is not shown here considering its complexity.

From Fig. 1 and Eq. (8), the expression of the scattering angle θ can be written as

$$\theta = \psi_{vc} - \pi = \arcsin\frac{b}{r_0} + \int_{r_{\min}}^{r_0}\psi' dr + \int_{r_{\min}}^{r_c}\psi' dr + \arctan(r_c\psi_c') - \pi. \tag{14}$$

Here the primes denote the derivatives over r. The integrals in Eq. (14) can be carried out by substituting Eq. (13) and keeping the main contribution from $r \sim O(b)$. We finally obtain

$$\theta = 2\arcsin\Big(\frac{b_0}{\sqrt{b^{*2} + b_0^2}}\Big), \tag{15}$$

where $b^* = b - \omega r_0^2/2v_0$. It is clear that the influence of magnetic field on the scattering angle is manifested through the modification to the impact parameter b. If $B = 0$, $b^* = b$, then θ becomes

$$\theta = 2\arcsin\Big(\frac{b_0}{\sqrt{b^2 + b_0^2}}\Big), \tag{16}$$

which reproduces the Rutherford scattering angle. When $B > 0$, $b^* < b$, the scattering angle θ increases with B. When $B < 0$, the scattering angle θ decreases with B.

To verify Eq. (15), the electron trajectories are calculated in different magnetic fields numerically by Mathematica. Two cases are considered, in which the deflection caused by cyclotron motion reinforces or counteracts the Rutherford scattering.

The temperature is chosen as 40 eV close to that of tokamak boundary where ρ_e is smaller than λ_D; v_0 and b_0 can be correspondingly determined as $b_0 = -2.40 \times 10^{-11}$ m, $v_0 = 3.24 \times 10^6\,\mathrm{m\cdot s^{-1}}$. For each case, we set the magnetic fields as 1.87 T, which is the average magnetic field strength of six operating tokamaks used by the ITER (International Thermonuclear Experimental Reactor) confinement scaling law, 3.5 T (EAST, Experimental Advanced Superconducting Tokamak), 5.3 T (ITER), 6.5 T (CFETR, China Fusion Engineering Test Reactor), 9 T (Alcator C-Mod tokamak) and 12 T (SPARC at MIT). Here, r_0 is taken as 1.54×10^{-8} m.

Equations (2) and (3) are solved numerically by using Mathematica. The electron trajectories are shown in Fig. 2 for different magnetic fields. It is found that the direction of velocity only changes rapidly when the electron is close to the ion. We use minus numbers to represent the cases with $B < 0$.

The scattering angle θ is calculated based on the numerical trajectory. It is shown that θ decreases with the magnetic field in Fig. 3. With the selected parameters, the scattering angles are 22.6°, 20.5°, 18.9°, 17.4°, 16.5°, 14.9° and 13.2°, respectively, when the magnetic fields are 0, −1.87 T, −3.5 T, −5.3 T, −6.5 T, −9 T and −12 T. The differences between the scattering angles with and without magnetic field are about 9.3%, 16.3%, 23.0%, 26.9%, 34.2% and 41.5%, respectively. Obviously, the influence of the magnetic field on scattering angle cannot be ignored in finite magnetic field regime. The comparison between the numerically obtained θ and the analytic form of Eq. (15)

is shown in Fig. 3. The maximal difference is no more than 5%. This indicates that Eq. (15) is a good approximation for θ.

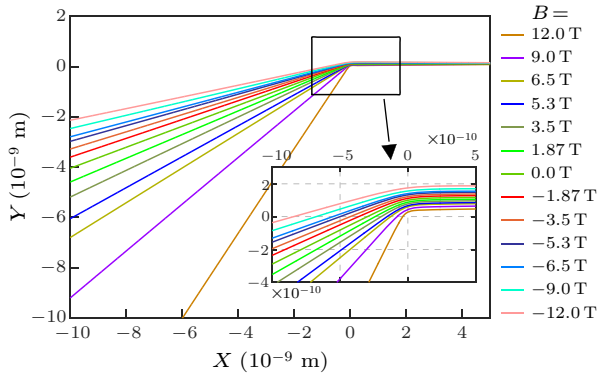

Fig. 2. The electron trajectories in different magnetic fields. Initial conditions are $r_0 = 1.54\times10^{-8}$ m, $T = 40$ eV (corresponding to $b_0 = -2.40 \times 10^{-11}$ m, $v_0 = 3.24 \times 10^6$ m·s^{-1}), $b = 5b_0$.

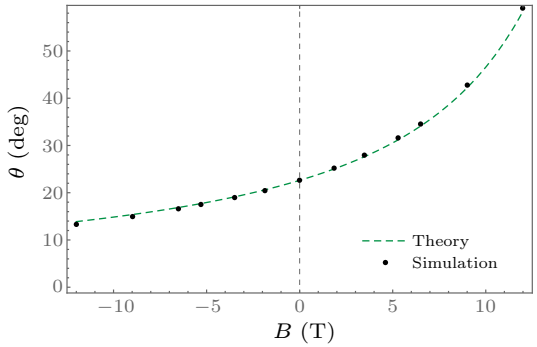

Fig. 3. The scattering angle in different magnetic fields along the Z direction. Initial conditions are the same as those in Fig. 2.

For the case with $B > 0$, under the same initial conditions as before, similar numerical procedure gives the scattering angles 22.6°, 25.2°, 27.9°, 31.5°, 34.5°, 42.7° and 59.1° for $B = 0$, 1.87 T, 3.5 T, 5.3 T, 6.5 T, 9 T and 12 T, respectively. The differences between the scattering angles with and without magnetic field are 11.4%, 23.2%, 39.4%, 52.5%, 88.9% and 161.4%, respectively. Obviously, the influence of magnetic field on scattering angle is even stronger than those in the cases with $B < 0$.

In conclusion, we have investigated the influence of a magnetic field on the scattering angle of electron-ion collision. For simplicity, two-dimensional electron-ion Coulomb collision perpendicular to the magnetic field is studied. The scattering angle is defined by the scattering effects of both Coulomb force and Lorentz force.

For the small angle scattering, in the presence of a magnetic field, the expression of the scattering angle is analytically derived. The influence of magnetic field on the scattering angle is manifested by the modification to the impact parameter b. The scattering angle depends on both strength and direction of the magnetic field.

The electron scattered by a fixed ion in the presence of magnetic field is simulated and the scattering angles are calculated. The numerical results agree well with the theoretical prediction. The present results may be applicable to two-dimensional plasma transport problems. This will be our future work.

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