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Multi-Type Solitons in Spin-Orbit Coupled Spin-1 Bose–Einstein Condensates

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Recently, research of solitons in Bose–Einstein condensates has become a popular topic. Here, we mainly study exact analytical solutions of Gross–Pitaevskii equations describing spin-orbit coupled spin-1 Bose–Einstein condensates. To begin with, we show the analytical relation between different types of one-dimensional spin-orbit coupling and Zeeman effect. In addition, we find a transformation that can simplify the three-component Gross–Pitaevskii equations with spin-orbit coupling into the nonlinear Schrödinger equation. The abundant stripe phase and dynamic characteristics of the system are investigated.

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When Bose–Einstein condensate (BEC) was first created using ^{87}Rb atomic gas in 1995[1] and a spinor BEC is realized in an optical dipole trap in 1998,[2] the BEC has provided an important platform for investigation of nonlinear phenomena. The soliton of nonlinear problems has been studied extensively in BECs[3−5] and other systems, such as shallow water waves, nonlinear optical and ferromagnetic systems.[6−10] Afterwards, one-dimensional artificial spin-orbit coupling (SOC) with equal strengths of coupling of Bychkov and Rashba[11] and Dresselhaus[12] was realized for the first time in 2011 by Spielman *et al.*[13] The artificial SOC provides possibility for realization of many novel quantum states and promotes research of some important physical phenomena.[14,15] It greatly stimulates the interest of researchers in SOC-BECs. In particular, researchers found a variety of solitons species, such as stripe bright solitons,[16] stripe dark solitons,[17] and vortex solitons,[18,19] which play important roles in quantum information and precision measurement.

For pseudospin-1/2 BECs with SOC, there have been many discussions about soliton solutions with vector properties, such as bright-bright solitons, bright-dark solitons and dark-dark solitons.[20,21] In addition, when spin-1/2 BECs have helicoidal SOC, the system can be transformed into the Manakov system by a gauge transformation.[22] Therefore, a diversity of analytical solutions can be constructed using the integrable methods, such as stripe solitons and breathing solitons.[23,24] However, there are relatively few studies on the exact analytical solutions in spin-1 BECs.[25] Most of them are about approximate solutions obtained by the variational method.[26,27] Especially, the solutions can only be obtained by multi-scale expansion method when considering Raman coupling and Zeeman effect.[28,29] It is undeniable that the exact solutions can explicitly describe the relation between various physical parameters. Consequently, exploring more exact analytical soliton solutions in spin-1 BECs is crucial.

In this Letter, we mainly focus on multi-type soliton solutions of one-dimensional spin-1 BECs with SOC. Firstly, we find that the different types of one-dimensional SOC are essentially equivalent when ignoring the Zeeman effect and Raman coupling. Secondly, we derive the analytical connection between SOC and Zeeman effect using the gauge transformation. Furthermore, we obtain a transformation that can make the original equations become an integrable nonlinear Schrödinger equation (NLSE). By this transformation, we can obtain the soliton solutions and various different types of analytical solutions for the spin-1 SOC-BECs.

The dynamics of BECs can be described by the Gross–Pitaevskii (GP) equation under the mean field approximation.[30] Considering the Zeeman effect, the dimensionless form of the GP equations describing spin-1 BECs[31] is

$$i\frac{\partial\psi_m}{\partial t} = \Big[-\frac{\nabla^2}{2} + V_{\rm trap} - pm + qm^2\Big]\psi_m + c_0\psi_m\sum_{n=-1}^{+1}|\psi_n|^2 + c_2\sum_{n=-1}^{+1}\boldsymbol{F}\cdot\boldsymbol{f}_{mn}\psi_n, \quad (1)$$

where ψ_m are the wave functions of three components ($m = +1, 0, -1$); $V_{\rm trap}$ is the external potential; p and q represent linear and quadratic Zeeman effects, respectively. In addition, the effective constants of mean field c_0 and spin exchange interaction c_2 are connected with the scattering length between atoms, which can be adjusted experimentally by Fes-

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hbach resonance technology.[32] The positive and negative c_0 represent repulsive and attractive interaction, and the positive and negative c_2 indicate that BECs have ferromagnetism and antiferromagnetism, respectively. Here, $\boldsymbol{f}=(f_x,f_y,f_z)$ is the spin-1 matrix; $\boldsymbol{f}_{mn}$ is a vector composed of matrix elements in the mth row and nth column of (f_x,f_y,f_z). The irreducible representations of its three-direction components are

$$f_x=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad f_y=\frac{i}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix},$$
$$f_z=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

$\boldsymbol{F}=(F_x,F_y,F_z)$ is the spin density vector and its three components are defined by $\boldsymbol{\psi}^{\mathrm{T}}f_{x,y,z}\boldsymbol{\psi}$, with $\boldsymbol{\psi}=(\psi_{+1},\psi_0,\psi_{-1})^{\mathrm{T}}$ and T the transposed operation.

First of all, we consider the situation in which the trapping frequencies along the y and z axes are much larger than that along the x axis. In this case, the system can be regarded as a quasi-one-dimensional system. Then we consider the SOC effect, in this case, BECs have rich characteristics and spin textures. As we know, there are three types of one-dimensional SOC $\gamma p_x f_x$, $\gamma p_x f_y$ and $\gamma p_x f_z$, where $p_x=-i\partial/\partial x$ is the momentum operator in the x axis direction, and γ is the SOC strength. Then, we take the SOC type of $\gamma p_x f_x$ as an example. Without considering the Zeeman effect ($q=0$, $p=0$), the quasi-one-dimensional three-component GP equations with SOC are

$$\begin{aligned}
i\frac{\partial\psi_{\pm1}}{\partial t}=&-\frac{1}{2}\frac{\partial^2\psi_{\pm1}}{\partial x^2}+(c_0+c_2)(|\psi_\pm|^2+|\psi_0|^2)\psi_{\pm1}\\
&+(c_0-c_2)(|\psi_\mp|^2)\psi_{\pm1}+c_2\psi_0^2\psi_{\mp1}^*\\
&-i\frac{\sqrt{2}\gamma}{2}\frac{\partial\psi_0}{\partial x},\\
i\frac{\partial\psi_0}{\partial t}=&-\frac{1}{2}\frac{\partial^2\psi_0}{\partial x^2}+(c_0+c_2)(|\psi_{+1}|^2+|\psi_{-1}|^2)\psi_0\\
&+c_0|\psi_0|^2\psi_0+2c_2\psi_0^*\psi_{+1}\psi_{-1}\\
&-i\frac{\sqrt{2}\gamma}{2}\Big(\frac{\partial\psi_{+1}}{\partial x}+\frac{\partial\psi_{-1}}{\partial x}\Big).
\end{aligned}\tag{2}$$

Here, the first-order derivative terms which reflect SOC effect increase the difficulty of solving the equation accurately. For this reason, we try to find a kind of transformation to simplify Eq. (2) and derive its soliton solutions. Firstly, taking linear Hamiltonian on the the right-hand side of Eq. (2) and organizing it into the matrix form

$$L=\begin{pmatrix}-\frac{1}{2}\frac{\partial^2}{\partial x^2}&-\frac{i\gamma}{\sqrt{2}}\frac{\partial}{\partial x}&0\\-\frac{i\gamma}{\sqrt{2}}\frac{\partial}{\partial x}&-\frac{1}{2}\frac{\partial^2}{\partial x^2}&-\frac{i\gamma}{\sqrt{2}}\frac{\partial}{\partial x}\\0&-\frac{i\gamma}{\sqrt{2}}\frac{\partial}{\partial x}&-\frac{1}{2}\frac{\partial^2}{\partial x^2}\end{pmatrix}.$$

We assume that the form of matrix L eigenvector is $e^{ikx}(a,b,c)^{\mathrm{T}}$, where k and $(a,b,c)^{\mathrm{T}}$ are arbitrary constants. Then we can obtain the eigenvalues $\lambda_{1,2,3}$ of L as follows:

$$\lambda_1=\frac{1}{2}k^2-\gamma k,\quad \lambda_2=\frac{1}{2}k^2,\quad \lambda_3=\frac{1}{2}k^2+\gamma k,$$

and the corresponding eigenvectors $\xi_{1,2,3}$

$$\xi_{1,2,3}=e^{ikx}\begin{pmatrix}\frac{1}{2}\\-\frac{\sqrt{2}}{2}\\\frac{1}{2}\end{pmatrix},\ e^{ikx}\begin{pmatrix}\frac{\sqrt{2}}{2}\\0\\-\frac{\sqrt{2}}{2}\end{pmatrix},\ e^{ikx}\begin{pmatrix}\frac{1}{2}\\\frac{\sqrt{2}}{2}\\\frac{1}{2}\end{pmatrix}.$$

Expanding the wave functions with eigenvectors,[22]

$$\boldsymbol{\psi}=e^{-i\lambda_1 x}\xi_1\varphi_{+1}+e^{-i\lambda_2 x}\xi_2\varphi_0+e^{-i\lambda_3 x}\xi_3\varphi_{-1},\tag{3}$$

where $\boldsymbol{\psi}=(\psi_{+1},\psi_0,\psi_{-1})^{\mathrm{T}}$, $\varphi_{\pm1}$ and φ_0 are the complex functions of x and t. In order to balance the exponential terms of equations, we need to take $k=2$. After unitary transformation (3), the new wave functions satisfy the equations

$$\begin{aligned}
i\frac{\partial\varphi_{\pm1}}{\partial t}=&-\frac{1}{2}\frac{\partial^2\varphi_{\pm1}}{\partial x^2}+(c_0+c_2)(|\varphi_\pm|^2+|\varphi_0|^2)\varphi_{\pm1}\\
&+(c_0-c_2)(|\varphi_\mp|^2)\varphi_{\pm1}+c_2\varphi_0^2\varphi_{\mp1}^*\\
&\mp i\gamma\frac{\partial\varphi_{\pm1}}{\partial x},\\
i\frac{\partial\varphi_0}{\partial t}=&-\frac{1}{2}\frac{\partial^2\varphi_0}{\partial x^2}+(c_0+c_2)(|\varphi_{+1}|^2+|\varphi_{-1}|^2)\varphi_0\\
&+c_0|\varphi_0|^2\varphi_0+2c_2\varphi_0^*\varphi_{+1}\varphi_{-1}.
\end{aligned}\tag{4}$$

We can find that Eq. (4) is equivalent to Eq. (2) when considering the SOC of $\gamma p_x f_z$ without Zeeman effect. In addition, Eq. (2) with SOC of $\gamma p_x f_y$ can also be transformed into Eq. (4) by a similar transformation. In other words, the three-component GP equations with different types of SOC can be transformed into each other through unitary transformation. Essentially, this is the rotation of coordinates. Then we can eliminate the first-order derivative terms of the equations by Galilean transformation $\varphi_{\pm1}=u_{\pm1}e^{\mp i\gamma x\mp ipt}$ and $\varphi_0=u_0$, where $u_{\pm1}$ and u_0 are also the complex functions of x and t. The obtained equations are the same as Eq. (1) with the quadratic Zeeman effect $q=-\gamma^2/2$. Therefore, the SOC effect after a specific transformation is equivalent to the quadratic Zeeman effect. Experimentally, q can be changed independently by adjusting a bias field and $q<0$ is accessible.[33] There are some studies on solutions of Eq. (1) when the quadratic Zeeman effect is less than zero.[34]

Moreover, if the Galilean transformation of $\boldsymbol{\varphi}=(\varphi_{+1},\varphi_0,\varphi_{-1})^{\mathrm{T}}$ is

$$\boldsymbol{\varphi}=e^{i\gamma^2 t/2}(u_{+1}e^{-i\gamma x},u_0,u_{-1}e^{i\gamma x})^{\mathrm{T}},\tag{5}$$

then for the case of $u_0=0$, we can obtain the equations

$$\begin{aligned}
i\frac{\partial u_{+1}}{\partial t}&=-\frac{1}{2}\frac{\partial^2u_{+1}}{\partial x^2}+(\alpha|u_{+1}|^2+\beta|u_{-1}|^2)u_{+1},\\
i\frac{\partial u_{-1}}{\partial t}&=-\frac{1}{2}\frac{\partial^2u_{-1}}{\partial x^2}+(\beta|u_{+1}|^2+\alpha|u_{-1}|^2)u_{-1},
\end{aligned}\tag{6}$$

where $\alpha = c_0 + c_2$, $\beta = c_0 - c_2$. Equation (6) is very common in the field of optics[35] and two-component BECs[17] as the non-integrable equation. However, we can select some special relations to make it degenerate into the integrable equation

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma|u|^2 u = 0, \tag{7}$$

where $u = \sqrt{g_0/\sigma}u_{+1}$ is the complex function of x and t, and $\sigma = \pm 1$ determines the types of solitons. The parameter g_0 is related to the selected special relations

$$g_0 = \begin{cases} -(c_0 + c_2), & u_{-1} = 0, \\ -2c_0, & u_{-1} = \pm u_{+1}, \end{cases}$$

when $g_0 > 0$, $\sigma = +1$, otherwise $\sigma = -1$. Equation (7) is the standard NLSE, which has good integrable properties and various local solutions.[36,37] First we consider the case of $u_{-1} = 0$ or $u_{+1} = 0$ and combine with the transformations (3) and (5), the formal solutions of Eq. (2) can be obtained as follows:

$$\begin{pmatrix} \psi_{+1} \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = \frac{e^{\frac{i}{2}\gamma^2 t \pm i\gamma x}}{2\sqrt{g_0/\sigma}} u \begin{pmatrix} 1 \\ \mp\sqrt{2} \\ 1 \end{pmatrix}. \tag{8}$$

Here, $g_0 = -(c_0 + c_2)$. This formula shows that the three-component SOC-GP equations contain all types of solutions of the NLSE, including bright soliton, dark soliton, rouge wave, etc. We can change the phase and speed of solutions by adjusting the SOC intensity γ, and the nonlinear interaction c_0 and c_2 can adjust the amplitude. Nevertheless, when considering $u_{-1} = u_{+1}$ or $u_{-1} = -u_{+1}$, the relation between solutions of Eq. (2) and the NLSE is

$$\begin{pmatrix} \psi_{+1} \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = \frac{e^{\frac{i}{2}\gamma^2 t}}{2\sqrt{g_0/\sigma}} u \begin{pmatrix} e^{i\gamma x} \pm e^{-i\gamma x} \\ -\sqrt{2}(e^{i\gamma x} \mp e^{-i\gamma x}) \\ e^{i\gamma x} \pm e^{-i\gamma x} \end{pmatrix}, \tag{9}$$

where $g_0 = -2c_0$. We can realize that the SOC effect manifests as periodic modulation of the amplitude of solutions, and the system (2) has all NLSE solutions after periodic modulation, such as stripe soliton and rouge wave under periodic background. We can use the Hirota method or Darboux transform to obtain the multi-soliton solutions of Eq. (7), including multiple bright solitons, dark solitons and rogue waves. Then we can reach the stripe-shaped solutions of Eq. (2) by transformation (9). Take single soliton as an example, as shown in Fig. 1. Here, using the Hirota bilinear method,[38] the expression of a bright soliton solution obtained is

$$u = \frac{4\eta k_1^2 e^{(k_1+ik_2)x + i\frac{(k_1+ik_2)^2}{2}t}}{|\eta|^2 e^{-2k_1(k_2 t - x)} + 4k_1^2}, \tag{10}$$

a dark soliton solution is

$$u = \frac{1 - k_3 e^{-2k_1(k_2 t - x)}}{1 + k_3 e^{-2k_1(k_2 t - x)}} k_1 e^{ik_2 x - \frac{i}{2}(k_2^2 + 2k_1^2)t}, \tag{11}$$

and first-order rouge wave[39] is

$$u = \left(1 - \frac{4 + 8it}{4t^2 + 4x^2 + 1}\right)e^{it}, \tag{12}$$

where η is an arbitrary complex parameter, $k_{1,2}$ are real parameters, and k_3 is a real number greater than zero. The expression of second-order rouge wave is complicated and can be found in Ref. [40].

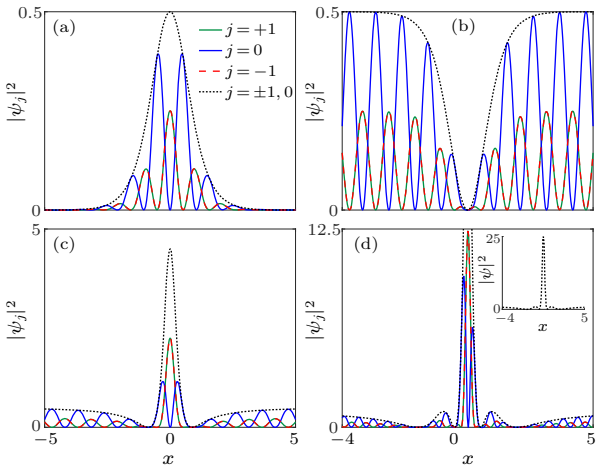

Fig. 1. The three-component density distribution of different types of soliton solutions at time $t = 0$. The green and blue solid lines represent $|\psi_{+1}|^2$ and $|\psi_0|^2$, respectively. The red dotted line indicates $|\psi_{-1}|^2$, and the black line represent the total density $|\psi_{+1}|^2 + |\psi_0|^2 + |\psi_{-1}|^2$. (a) Stripe bright soliton, (b) stripe dark soliton, (c) stripe rouge wave, and (d) second-order stripe rogue wave. The subgraph in (d) represents the total density.

It can be seen from Fig. 1 that the solutions obtained by transformation (9) have some properties. For example, the SOC intensity γ can change the number of stripes in each component solution, but does not affect the total density. In particular, we can find ψ_{+1} and ψ_{-1} to be equal. It is equivalent to the component of spin density vector F_z to be zero.

In the above discussion, there are no constraints on the parameters. The type of soliton solution obtained is relatively single, and three are no mixed soliton solutions. Next we discuss the case of $c_0 = c_2$, and Eq. (6) becomes two independent NLSEs about u_{+1} and u_{-1},

$$i\frac{\partial u_{\pm 1}}{\partial t} + \frac{1}{2}\frac{\partial^2 u_{\pm 1}}{\partial x^2} - 2c_0|u_{\pm 1}|^2 u_{\pm 1} = 0. \tag{13}$$

This means that we can substitute any two types of solutions of the NLSE into the transformations (3) and (5) to obtain the solutions of Eq. (2). However, the interaction types satisfied by the two soliton solutions must be consistent, and both of them are attractive or repulsive. We choose u_{+1} as the second-order rogue wave[41] and u_{-1} as the two-bright soliton solution,[42] as shown in Fig. 2.

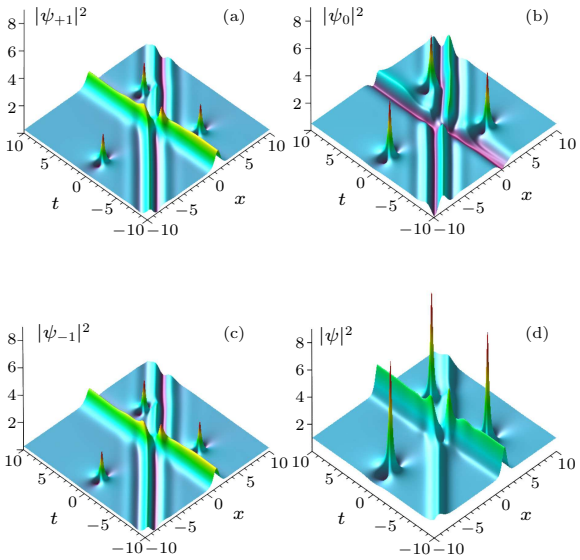

Fig. 2. The exact analytical solutions of rogue waves and bright solitons: (a)–(d) the density distributions of three components and total number of particles, respectively.

We can find the speed of solitons and the position of rogue waves can be changed by adjusting parameters, and the three rogue waves show as a second-order rouge wave when they appear in the same position and time. In addition, the bright solitons have a plane background and appear as striped solitons in three components. However, the velocities of two bright solitons meet the velocity matching condition, and its expression is not found here. Otherwise, the two bright solitons appear as breathing solitons in the three components. Due to the complex forms of solutions, their expression and the choice of parameters are not written here.

In conclusion, we give the connection among three types of one-dimensional SOC by unitary transformation (3), and find that the SOC with intensity γ is equivalent to the Zeeman effect with the quadratic Zeeman intensity $q = -\gamma^2/2$ under gauge transformations. Furthermore, we simplify the three-component SOC-GP equations into the integrable NLSE, and obtain some solutions of multiple solitons, multiple stripe solitons and multiple types of solitons of Eq. (2). Furthermore, we can also find a transformation to simplify the multi-components GP equations with SOC through the method described above.

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