Chinese Physics Letters **39**, 120501 (2022) **Express Letter**

Theory of Critical Phenomena with Memory

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(Received 21 October 2022; accepted manuscript online 15 November 2022)

Memory is a ubiquitous characteristic of complex systems, and critical phenomena are one of the most intriguing phenomena in nature. Here, we propose an Ising model with memory, develop a corresponding theory of critical phenomena with memory for complex systems, and discover a series of surprising novel results. We show that a naive theory of a usual Hamiltonian with a direct inclusion of a power-law decaying long-range temporal interaction violates radically a hyperscaling law for all spatial dimensions even at and below the upper critical dimension. This entails both indispensable consideration of the Hamiltonian for dynamics, rather than the usual practice of just focusing on the corresponding dynamic Lagrangian alone, and transformations that result in a correct theory in which space and time are inextricably interwoven, leading to an effective spatial dimension that repairs the hyperscaling law. The theory gives rise to a set of novel mean-field critical exponents, which are different from the usual Landau ones, as well as new universality classes. These exponents are verified by numerical simulations of the Ising model with memory in two and three spatial dimensions.

DOI: 10.1088/0256-307X/39/12/120501

Equilibrium critical phenomena are intriguing and have been well understood.[1–5] Classical critical dynamics is usually imposed by a Langevin equation with a Gaussian noise on the basis of instantaneous interactions and has also been well established.[4–8] Quantum critical dynamics of closed systems is determined by their Hamiltonians and can be obtained from analytical continuation through a quantum-classical correspondence.[9] However, open quantum systems may contain temporally nonlocal interactions when their coupled environmental degrees of freedom are integrated away.[10–22] In fact, such a long-range temporal correlation or memory is ubiquitous in complex systems such as reaction-diffusion systems,[23,24] disordered systems,[25–37] active matters,[38,39] epidemic spreading,[40–43] and many others,[44–49] not to mention brain science and biology[50–55] in general. It can result in effects such as anomalous diffusions that are qualitatively different from those exhibited by a memoryless system. What then will one expect memory or non-Markovian dynamics to bring to critical behavior?

To this end, we study a model with a time-dependent Hamiltonian,

$$\mathcal{H}_I(t) = -J \sum_{t_1<t} \sum_{\langle ij \rangle} \frac{s_i(t)s_j(t_1)}{(t-t_1)^{1+\theta}}, \tag{1}$$

which extends the standard Ising model with a nearest-neighbor coupling constant J to account for a long-range temporal interaction parameterizing by a constant $\theta > 0$, which ensures convergence at long times[56] but demands $t_1 < t$ to avoid apparent divergence at $t_1 = t$. In Eq. (1), the Ising spin $s_i = \pm 1$ at site i may model, besides spins,[11] the phase of a Josephson junction in an array,[10] a density of a chemical species in intracellular processes[54] or complex reaction-diffusion systems,[23,34] positions of active particles,[38,39] or even a memorized pattern in a Hopfield neuron network,[55] and can be generalized to more realistic forms. We have chosen a power-law decay as a simplification of various possible memory types.[10–55] The reason is that it is an analogy of a spatial counterpart[57] and thus the single parameter θ anticipatorily enables us to classify different universality classes, which can then provide clues to the interactions in real systems. Such an interaction may be realized at least in artificial systems noticing that the decay exponent of the spatial counterpart can even be continuously tuned in trapped ions.[58–63] Model (1) is to be simulated using (random-)update Monte Carlo with Metropolis algorithm,[64,65] which is interpreted as dynamics[65,66] with the time unit of the usual Monte Carlo step per spin. Due to the inherent memory in the model, time average is most likely different from ensemble average over samples. Consequently, we sample configurations only at time t but not the earlier configurations at any $t_1 < t$, resulting in t-dependent observable quantities. In fact, such ensemble averages have been successfully exploited to tackle Hamiltonian with a time-dependent term in finite-time scaling (FTS),[67–69] in which scaling laws are found to be satisfied even if the fluctuation-dissipation theorem is not.[69] We will come back to FTS when we test the theory.

Theoretically, the long-wavelength low-frequency behavior of model (1) is described by a naïve effective Hamiltonian[70]

$$\mathcal{H}(t) = \int d^d x \Big\{ \frac{1}{2}\Big[\tau\phi(\boldsymbol{x},t)^2 + [\nabla\phi(\boldsymbol{x},t)]^2\Big] + \frac{1}{4!}u\phi(\boldsymbol{x},t)^4 + \frac{1}{2}v\int_{-\infty}^{t} dt_1 \frac{\phi(\boldsymbol{x},t)\phi(\boldsymbol{x},t_1)}{(t-t_1)^{1+\theta}} - h(\boldsymbol{x},t)\phi(\boldsymbol{x},t)\Big\}, \tag{2}$$

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which is the usual ϕ^4 theory for an order-parameter field $\phi(\boldsymbol{x},t)$ at position $\boldsymbol{x}$ and time t, where τ, v, u, h, and d denote a reduced temperature, the strength of the long-range interaction, a coupling constant, an additional ordering field, and the spatial dimensionality, respectively. In accordance with the Metropolis method, dynamics is governed by the Langevin equation

$$\lambda_1\partial\phi(\boldsymbol{x},t)/\partial t = -\delta\mathscr{H}/\delta\phi(\boldsymbol{x},t) + \zeta(\boldsymbol{x},t) \tag{3}$$

with a Gaussian white noise ζ of zero mean $\langle\zeta(\boldsymbol{x},t)\rangle = 0$ and correlator

$$\langle\zeta(\boldsymbol{x},t)\zeta(\boldsymbol{x}_1,t_1)\rangle = 2\lambda_2\delta(\boldsymbol{x}-\boldsymbol{x}_1)\delta(t-t_1) \tag{4}$$

under the ensemble average. Here, $\mathscr{H} \equiv \int dt\mathcal{H}(t)$, and we have introduced two kinetic coefficients λ_1 and λ_2 in consideration of the memory effect; $\lambda_1 = \lambda_2$ in the usual dynamics and an equilibrium Boltzmann distribution is retrieved. The dynamics has been well-known to be equivalent to a dynamic Lagrangian [4–7,71,72]

$$\mathcal{L} = \int dtd^dx\{\tilde{\phi}[\lambda_1\partial\phi/\partial t + \delta\mathscr{H}/\delta\phi] - \lambda_2\tilde{\phi}^2\}, \tag{5}$$

where $\tilde{\phi}$ is an auxiliary response field. [73]

Hamiltonian (2) is similar to its spatial counterpart, which correctly describes critical behavior of long-range spatial interaction systems. [57,74] Here, surprisedly, we find that the naïve theory of its temporal counterpart is not correct anymore. We show below that such a theory violates a hyperscaling law [1–5,75]

$$\beta/\nu + \beta\delta/\nu = d, \tag{6}$$

for *all* d, once the memory dominates, where β, ν, and δ are critical exponents. This is in sharp contrast to the usual Landau mean-field exponents which break Eq. (6) only above the upper critical dimension d_c. From the singular part of a time-dependent free energy

$$f(\tau,h,t) = b^{-d}f(\tau b^{1/\nu}, hb^{\beta\delta/\nu}, tb^{-z}). \tag{7}$$

Equation (6) ensures a fundamental thermodynamic relation $M = -(\partial f/\partial h)_\tau \sim b^{-d+\beta\delta/\nu} \sim b^{-\beta/\nu}$ between the order parameters M and h, and violation of the hyperscaling law then implies violation of the thermodynamics, where z is the dynamic critical exponent. Moreover, to correctly describe the critical behavior of model (1), we demonstrate that a proper dimensional correction of the theory is indispensable. This leads to a series of novel results in the corrected theory such as the role of $\mathcal{H}$ in dynamics, the intimate relation between space and time, and a set of novel mean-field critical exponents different from the usual Landau ones.

We will utilize the renormalization-group (RG) technique. To this end, we write Eqs. (2) and (5) together as

$$\mathcal{L} = \int dtd^dx\Big\{\tilde{\phi}\Big[\lambda_1\frac{\partial\phi}{\partial t} + \tau\phi - K\nabla^2\phi + \frac{1}{3!}u\phi^3 + v\int dt_1\frac{\phi(t_1)}{(t-t_1)^{1+\theta}} - h - \lambda_2\tilde{\phi}\Big]\Big\}. \tag{8}$$

Here we have inserted a new constant K for renormalization. The RG method is first to integrate out the degrees of freedom within the momentum shell $\Lambda/b \le k \le \Lambda$ and then to rescale k to $k_r = kb$ so that the cutoff Λ of the remaining degrees of freedom recovers its original value, where $b > 1$ is a rescaling factor and r denotes renormalized quantity. To keep the form of $\mathcal{L}$ after renormalization, we also rescale

$$x_r = xb^{-1},\ t_r = tb^{\gamma_t},\ \phi_r = \phi b^{\gamma_\phi},\ \tilde{\phi}_r = \tilde{\phi}b^{\gamma_{\tilde{\phi}}},\ h_r = hb^{\gamma_h}, \tag{9}$$

where the constant γ_o is referred to hereafter as the dimension of o defined as $o_r = ob^{\gamma_o}$. Critical exponents can be obtained through

$$\beta/\nu = \gamma_\phi, \qquad \beta\delta/\nu = \gamma_h, \qquad z = -\gamma_t, \tag{10}$$

via scaling hypothesis [1–5] similar to Eq. (7). For an infinitesimal transformation, $b = 1+dl$ with $dl \ll 1$, keeping only terms to one-loop order, [1–5] we obtain

$$\partial_l v = (d - \gamma_t - \gamma_\phi - \gamma_{\tilde{\phi}} + \theta\gamma_t)v, \tag{11a}$$
$$\partial_l u = (d - \gamma_t - 3\gamma_\phi - \gamma_{\tilde{\phi}})u - 3u^2I_2/dl, \tag{11b}$$
$$\partial_l \ln K = d - 2 - \gamma_t - \gamma_\phi - \gamma_{\tilde{\phi}}, \tag{11c}$$
$$\partial_l \ln\lambda_2 = d - \gamma_t - 2\gamma_{\tilde{\phi}}, \tag{11d}$$
$$\partial_l \ln\lambda_1 = d - \gamma_\phi - \gamma_{\tilde{\phi}} = \gamma_{\lambda_1}, \tag{11e}$$
$$\partial_l \ln h = d - \gamma_t - \gamma_{\tilde{\phi}} - \gamma_h, \tag{11f}$$
$$\partial_l \ln\tau = d - \gamma_t - \gamma_\phi - \gamma_{\tilde{\phi}} + uI_1/2\tau dl, \tag{11g}$$

where $\partial_l o \equiv \partial o/\partial l = (o_r - o)/dl$ and the two integrals are defined as

$$I_1 = \frac{1}{(2\pi)^{d+1}}\int_{\Lambda/b}^{\Lambda} dk\int d\omega C_0(k,\omega) \equiv \int_k^b\int_\omega C_0(k,\omega),$$
$$I_2 = \int_k^b\int_\omega C_0(k,\omega)G_0(-k,-\omega), \tag{12}$$

with the response $\langle\phi'(\boldsymbol{k},\omega)\tilde{\phi}'(-\boldsymbol{k},-\omega)\rangle$ and correlation $\langle\phi'(\boldsymbol{k},\omega)\phi'(-\boldsymbol{k},-\omega)\rangle$ functions [4–7,71,72,76] given by

$$G_0(k,\omega)^{-1} = \tau + Kk^2 + v\Gamma(-\theta)(-i\omega)^\theta - i\lambda_1\omega \tag{13}$$

(Γ is the gamma function) and $C(\boldsymbol{k},\omega) = 2\lambda_2G_0(k,\omega)G_0(k,-\omega) \equiv \lambda_2C_0(k,\omega)$, respectively. Since even the mean-field results of the naïve theory breach the fundamental law, we need essentially a Gaussian analysis without the results of the two integrals.

We now analyze the fixed points of the RG equations and their properties. This is obtained by equating all equations in Eq. (11a) with zero. [77] There are two different cases. The first case is $v^* = 0$ from Eq. (11), where the star denotes the fixed-point value. The memory is irrelevant and we recover the usual short-range interaction ϕ^4 theory. Indeed, solving the fixed-point equations from Eqs. (11c)–(11f) yields

$$\gamma_t^* = -2, \quad \gamma_\phi^* = (d-2)/2, \quad \gamma_{\tilde{\phi}}^* = \gamma_h^* = (d+2)/2, \tag{14}$$

and hence $u^*(4 - d - 3I_2u^*/dl) = 0$ from Eq. (11b). Standard textbook analysis [1–5] shows that the Gaussian fixed point $u^* = 0$ is stable above $d_{c0} = 4$ with the Gaussian dimensions, Eq. (14), and hence, from Eq. (10), the correct Gaussian critical exponents β/ν, $\beta\delta/\nu$, and z satisfy the

hyperscaling law (6). For $d < d_{c0}$, since $I_2 > 0$, there exists a new fixed point $u^* = dl(4-d)/3I_2$ that controls a nonclassical regime.[1–5]

For the long-range fixed point, $v^* \neq 0$, instead one finds

$$u^*(6 - 2/\theta - d - 3I_2 u^*/dl) = 0, \tag{15}$$

using the fixed-point equations from Eqs. (11a)–(11d). The same analysis shows that the Gaussian fixed point $u^* = 0$ is now stable above $d_c = 6 - 2/\theta$ with the Gaussian dimensions now changing to

$$\gamma_t^* = -\frac{2}{\theta}, \quad \gamma_\phi^* = \frac{d}{2} + \frac{1}{\theta} - 2, \quad \gamma_{\tilde{\phi}}^* = \gamma_h^* = \frac{d}{2} + \frac{1}{\theta}. \tag{16}$$

As a result, we have now $z = 2/\theta$ and

$$\beta/\nu + \beta\delta/\nu = d - 2 + 2/\theta. \tag{17}$$

Equation (17) indicates that the hyperscaling law is radically violated for all d except $\theta = 1$ and therefore the naïve theory cannot be right.

We note that both sets of the exponents, Eqs. (14) and (16), give rise to $\gamma_\tau^* = d - \gamma_t^* - \gamma_\phi^* - \gamma_{\tilde{\phi}}^* = 2$ in Eq. (11g). This indicates that τ increases with b and thus is relevant. Moreover, the Gaussian exponent $\nu = 1/\gamma_\tau^* = 1/2$ for both the short-rang and the long-range fixed points.[1–5] Similarly, $\gamma_{\lambda_1}^* = 2 - 2/\theta$ for the long-range fixed point from Eq. (11e). Accordingly, it is negative for $\theta < 1$ and therefore is irrelevant.

We now focus on the theory with memory and show that the violation of the hyperscaling law originates from the dimension of $\mathcal{H}$. Indeed, using Eq. (9) and the short-range results, Eq. (14), one sees that all terms in $\mathcal{H}$ have a zero dimension except inconsistently the memory term. However, with the long-range results, Eq. (16), all terms then share an identical dimension

$$\gamma_{\mathcal{H}} = 2 - 2/\theta = \gamma_a, \tag{18}$$

which is minus the extra term in Eq. (17). This prompts us to eliminate $\gamma_{\mathcal{H}}$ in order to save the hyperscaling law and the theory.

To this end, we multiply $\mathscr{H}$ by a constant a such that $\gamma_{\mathcal{H}'} = -\gamma_a + \gamma_{\mathcal{H}} = 0$. However, introducing an additional scaling field to the theory breaks the correspondence between Eqs. (1) and (2). This means that a just contributes a factor $b^{-\gamma_a}$ (note the special minus sign to emphasize its speciality) instead of changing to a_r upon rescaling. In addition, ζ ought to be multiplied by a accordingly. Its correlator in Eq. (4) is then proportional to a^2. Consequently, $\mathcal{L}$, Eq. (5), becomes

$$\mathcal{L}' = \int dt d^d x [\tilde{\phi}(\boldsymbol{x}, t) a \delta \mathscr{H}/\delta\phi - a^2 \tilde{\phi}(\boldsymbol{x}, t)^2], \tag{19}$$

where we have omitted λ and the irrelevant time derivative term for $\theta < 1$. One sees from Eq. (19) that if one defines $\tilde{\phi}_1 = a\tilde{\phi}$, $\mathcal{L}$ recovers its original form. This is reasonable as $\tilde{\phi}$ originates from integration out of ζ.[73] The replacement implies that one ought to replace $\gamma_{\tilde{\phi}}$ with $\gamma_{\tilde{\phi}} + \gamma_a$ in Eq. (11a) and thus the new hyperscaling law (17) does hold.

However, this is not enough. To see why, we make a transformation,

$$\phi = \phi' a^{-n}, \qquad \tilde{\phi} = \tilde{\phi}' a^{-\tilde{n}}, \tag{20}$$

with two constants n and $\tilde{n}$ and obtain

$$\begin{aligned} \mathcal{H}' &= a^{1-2n} \int_k \Big[\frac{1}{2} G_0^{-1} |\phi'|^2 + \frac{1}{4!} a^{-2n} u \mathcal{H}_u - a^n h \phi'\Big], \\ \mathcal{L}' &= \int_{\omega,k} \tilde{\phi}' \Big[a^{1-n-\tilde{n}} G_0^{-1} \phi' + \frac{1}{3!} a^{1-3n-\tilde{n}} u \frac{\delta \mathcal{H}_u}{\delta \phi'} \\ &\quad - a^{1-\tilde{n}} h - a^{2-2\tilde{n}} \tilde{\phi}'\Big], \end{aligned} \tag{21}$$

after Fourier transform, where $\mathcal{H}_u$ denotes the Fourier transform of $\int dt d^d x \phi^4$ with one frequency and wavenumber integration extracted, and we have used identical symbols for both direct spacetime and its reciprocal and suppressed the arguments for clarity.

In Eq. (21), we have drawn a^{1-2n} out of the integration for $\mathcal{H}'$. In this way, we find that the perturbation contribution to the renormalized vertex, $\tilde{u}$, becomes

$$\tilde{u} = u\Big\{1 - \frac{3}{4} a^{1-4n} u \int_k G_0(k, 0)^2\Big\}, \tag{22}$$

to one-loop order for all t, t_1, and $(t - t_1)$ in Eq. (2) approaching infinity. It can be shown that all higher orders contain powers of $a^{1-4n}u$ combinatorially.[70] Therefore, if we choose $n = 1/4$, a does not contribute to the renormalization. This is also true for other parameters such as τ and substantially disentangles the involvement of a in the theory.

For $\mathcal{L}'$ in Eq. (21), the transformation Eq. (20) represents only a redistribution of γ_a between the fields and must not introduce an overall factor. Accordingly, from Eq. (21), we have $n = 1 - \tilde{n}$, or $\tilde{n} = 3/4$ to ensure identical h transformations ($h' = ha^{1/4}$), propagators, and u for $\mathcal{H}'$ and its $\mathcal{L}'$. Although the last term in $\mathcal{L}'$ leads to an a-dependent correlation function $C(\boldsymbol{k}, \omega) = a^{2n} C_0(k, \omega)$, one can again show that all of a are exactly canceled in the two integrals in Eq. (12) for the renormalized τ_r and u_r to one-loop order and to all orders combinatorially.[70] Conversely, without the transformation (20), a could not serve as a simple constant only, since it would be differently involved in $\mathcal{H}$ and $\mathcal{L}$, which are then inconsistent with each other.

We note that for $\mathcal{H}'$ and $\mathcal{L}'$ to be really consistent, each integration over wavenumber for $\mathcal{L}'$ and hence in Eq. (12) must also contain the same $a^{1-2n} = a^{1/2}$ factor. However, it must be exactly canceled by $a^{-1/2}$ for the frequency integration. The same happens also to the integrations in $\mathcal{H}_u$. Accordingly, the correct theory in place of Eqs. (2) and (5) is

$$\begin{aligned} \mathcal{L}' = \int \big(dt a^{-\frac{1}{2}}\big)\big(d^d x a^{\frac{1}{2}}\big)\tilde{\phi}'\Big\{&\tau\phi' - \nabla^2\phi' + \frac{1}{3!} a^{-\frac{1}{2}} u \phi'^3 \\ &+ v \int_{-\infty}^{t} dt_1 \frac{\phi'(\boldsymbol{x}, t_1)}{(t - t_1)^{1+\theta}} - h' - a^{\frac{1}{2}} \tilde{\phi}'\Big\}. \end{aligned} \tag{23}$$

Equation (23) and its associated $\mathcal{H}'$ constitute our corrected theory of critical phenomena with memory. The RG transformations of the theory are similar to Eq. (11a) except that the transformations for u and λ_2 need add and

subtract $\gamma_a/2$, respectively, because of the extra factors. The fixed-point solution for u is again Eq. (15) while

$$\gamma_t^* = -2/\theta, \quad \gamma_{\phi'}^* = d/2 + 1/2\theta - 3/2,$$
$$\gamma_{\tilde{\phi}}^* = d/2 + 3/2\theta - 1/2, \quad \gamma_{h'}^* = d/2 + 1/2\theta + 1/2, \quad (24)$$

which leads to

$$\beta/\nu + \beta\delta/\nu = d - \gamma_a/2 \equiv d_{\rm eff}, \quad (25)$$

using Eq. (10). Although Eq. (25) appears to violate both Eqs. (6) and (17), $d_{\rm eff}$ is just the dimension of $d^d x a^{1/2}$ and hence an effective spatial dimension. Accordingly, the hyperscaling law (6) holds again in it! This is corroborated by the fact that the coupling term $a^{-1/2}u$ now scales as $b^{d_{\rm c}-\gamma_a/2-d} \equiv b^{d_{\rm c,eff}-d}$, viz., the effective upper critical dimension $d_{\rm c,eff}$ changes consistently to $d_{\rm c} - \gamma_a/2 = 5 - 1/\theta$. Similarly, the usual hyperscaling law ought to be modified to $d_{\rm eff}\nu = (d-\gamma_a/2)\nu = 2-\alpha$ accordingly, while the others such as $\gamma = (2-\eta)\nu$ [1–5] remain unchanged, where α and γ are the critical exponents for specific heat and susceptibility χ, respectively. From Eq. (18), $d_{\rm eff}$ is always larger than d in the memory-dominant regime, a fact which is qualitatively reasonable as the temporal correlation inevitably suppresses spatial fluctuations and contributes an effective spatial dimension.

Moreover, in the memory-dominated regime, the correct mean-field critical exponents, Eq. (24) at $d_{\rm c}$, become

$$\beta/\nu = 3/2 - 1/2\theta, \qquad \beta\delta/\nu = 7/2 - 1/2\theta, \quad (26)$$

though $\nu = 1/2$ since $\gamma_\tau^* = 2$ too and $\gamma = 1$ as usual. For $\theta = 1/2$ and $2/3$, for example, $d_{\rm c} = 2$ and 3 and $\beta/\nu = 1/2$ and $3/4$, respectively, different from the Landau result $\beta/\nu = 1$ obtained from both Eq. (14) and Eq. (16).

In addition, since the fixed-point Eq. (15) remains intact, $d_{\rm c}$ again separates the classical regime above it from and nonclassical regimes below it. Also, from Eq. (24), $\gamma_{\lambda_1}^* = 2 - 2/\theta$ too, the division of the short-range and long-range regimes remains intact. In fact, to $\epsilon^2 = (d_{\rm c} - d)^2$, we find that this division shifts to $\theta = 1 - \kappa$ with $\kappa = \ln(4/3)\epsilon^2/18$. [78] These divisions of different regimes are similar to the spatial counterpart. [57,74] However, the central difference is that $\gamma_{\mathcal{H}} = 0$ in the latter case and hence no corresponding behavior as presented here emerges. Detailed comparison of the two systems are given elsewhere. [78]

Turning to $a^{-1/2}$ in the temporal integration in Eq. (23), we need to transform t to $t' = a^{-1/2}t$, which just accounts for the last extra $a^{1/2}$ in Eq. (23) because $a^{1/2}\delta(t) = \delta(t')$ in Eq. (4). Indeed, an RG analysis with t' instead of t results again in Eq. (24) with a new $z' = -\gamma_{t'}^* = z + \gamma_a/2$, [79] which indicates that part of the temporal dimension has been transferred to the space and which leads correctly to $G_0 = b^2/[\tau_{\rm r} + K_R k_{\rm r}^2 + v_{\rm r} a^{-\theta/2}\Gamma(-\theta)(-i\omega_{\rm r}')^\theta]$ upon neglecting the irrelevant λ_1 term in Eq. (13), where $\omega_{\rm r}' = a^{1/2}\omega b^{-\gamma_{t'}^*}$. A similar scaling obviously holds for G_0 in terms of t as well.

We have seen that $\mathcal{H}$ plays an essential role due to Eq. (18). This draws attention to the eluded role played by $\mathcal{H}$ in dynamics, as nowadays one usually solely focuses on $\mathcal{L}$ for critical dynamics though $\mathcal{H}$ controls the equilibrium properties. [4–7,71,72] Since a arises from the memory, the scaling laws thus connect both static and dynamic properties, demonstrating the intimate relation between space and time similar to quantum phase transitions but adjustable through θ in the memory-dominated regime.

To verify the exponents, we utilize FTS. [67–69] FTS is to change a controlling parameter in Hamiltonian linearly with time at a rate R through a critical point and is perfect for dealing with the time-dependent Hamiltonian (1). The driving imposes on the system a controllable external finite timescale that plays the role of system size in finite-size scaling and leads to FTS. [67] FTS has been successfully applied to many systems to efficiently study their equilibrium and nonequilibrium critical properties. [13,67–69,80–98] Here, we change the temperature T through $\tau = T - T_{\rm c} = Rt'$ [70] and choose R as a variable in place of t'. Accordingly, Eq. (7) becomes [67–69]

$$f(\tau, h, R) = b^{-d_{\rm eff}} f(\tau b^{1/\nu}, h' b^{\beta\delta/\nu}, R b^r), \quad (27)$$

where r is a rate exponent. [99,100] Since t' scales as $t' b^{-z'}$, one finds $r = z' + 1/\nu = 3 + 1/\theta$, which enables us to verify z'. Consequently, one arrives at the FTS forms

$$M = R^{\beta/r\nu} f_1(\tau R^{-1/r\nu}), \ \chi = R^{-\gamma/r\nu} f_2(\tau R^{-1/r\nu}), \quad (28)$$

for M and χ at $h' = 0$, where f_1 and f_2 are scaling functions. At the peak of χ, $T_{\rm p} = T_{\rm c} + cR^{1/r\nu}$ with a constant c satisfying $df_2(x)/dx|_{x=c} = 0$ from Eq. (28). [67,68] This yields $1/r\nu$ and $T_{\rm c}$, at which $M_{\rm c} \propto R^{\beta/r\nu}$ and $\chi_{\rm c} \propto R^{-\gamma/r\nu}$, and thereby $\beta/r\nu$ and $\gamma/r\nu$. However, $T_{\rm c}$ so estimated has a relatively large error and hence affects others. Thus, we apply the theoretical $1/r\nu$ to fit $T_{\rm c}$ and then use it to determine other exponents and check the consistency by curve collapsing.

We simulate model (1) with $J = 0.2$ and a cutoff $\mathcal{T}$ of the long-range interaction. Periodic boundary conditions are applied throughout and initial ordered states are carefully chosen to be far away from $T_{\rm c}$ and equilibrated before heating and thus do not affect results. In Fig. 1, we display the results for the 2D model. We estimate, with the above-mentioned method, $T_{\rm c} = 1.402(18)$, which leads to Figs. 1(a)–1(c), where the straight lines have slopes of $1/r\nu = 0.405(22)$, $\beta/r\nu = 0.0941(4)$, and $-\gamma/r\nu = -0.372(9)$, close to the theoretical values of $2/5$, $1/10$, and $-2/5$, respectively. Note that if t were not transformed to t', the three exponents would be $1/3$, $1/12$, and $-1/3$, respectively, and further, if β were $1/2$, $\beta/r\nu = 1/6$, all far away from the numerical values. Moreover, these estimated exponents result in the rather good curve collapses in Figs. 1(d) and 1(e) around $T_{\rm c}$, though far away from it, the collapses are not quite good near the peaks where fluctuations are large as expected. [93,95] Similar results are obtained for $\mathcal{T} = 15$. [78] This is reasonable once the driven time is shorter than the cutoff memory time so that the latter becomes sub-leading. Similar condition applies to the lattice sizes as well. [68,69,82] Although these sub-leading factors may still contribute to the small difference of the numerical results, any logarithmic correction needed in the spatial counterpart [101] is found to deteriorate the collapses.

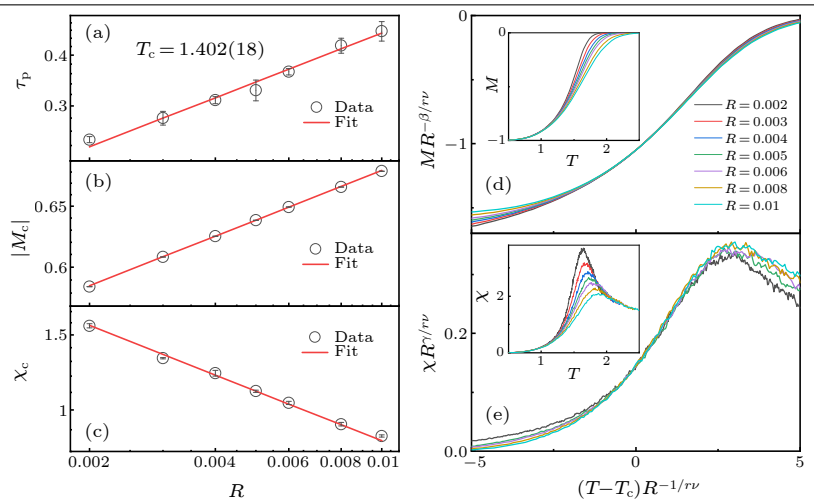

Fig. 1. Results of (a) $\tau_p = T_p - T_c$, (b) $|M_c|$, (c) χ_c, (d) and (e) rescaled M and χ, respectively, of the Ising model with memory with $\theta = 0.5$ and $\mathcal{T} = 20$ and subject to heating at a series of R given in the legend for 40000 samples on 100×100 square lattices. The insets in (d) and (e) are the respective original curves. Each column shares identical abscissa axes. Double logarithmic scales are used in (a)–(c).

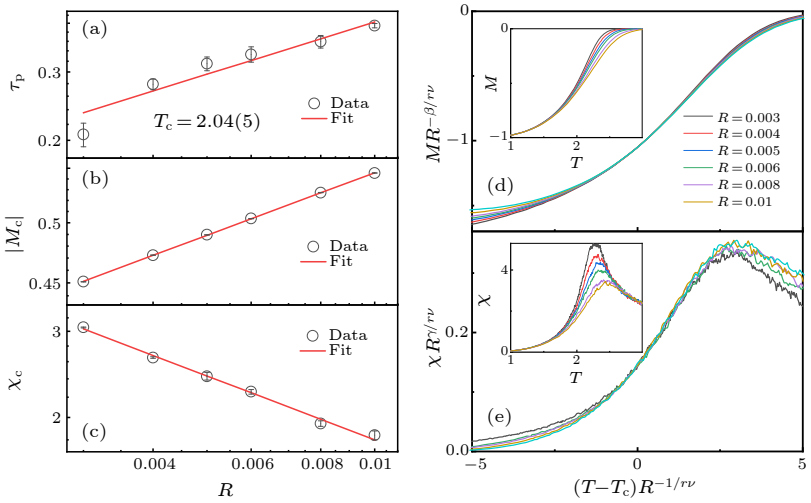

Fig. 2. Results of the 3D Ising model with $\theta = 2/3$ and $\mathcal{T} = 18$ for 40000 samples on $25 \times 25 \times 25$ simple cubic lattices. All style formats are the same as Fig. 1.

The 3D exponents are estimated to be $1/r\nu = 0.45(6)$, $\beta/r\nu = 0.1580(7)$, and $-\gamma/r\nu = -0.435(14)$ shown in Figs. 2(a)–2(c), again close to the theoretical values of 4/9, 1/6, and $-4/9$, respectively. They again yield rather good curve collapses in Figs. 2(d) and 2(e) around T_c.

In summary, we have proposed a model and derived the corresponding theory for critical phenomena with memory arising from a temporal power-law decaying interaction. Due to the dimension of the Hamiltonian, the theory contains a unique dimensional constant that serves (1) to inextricably interweave space and time into an effective dimension, (2) to repair the radically violated hyperscaling law, (3) to transform the time, (4) to change the Gaussian and mean-field critical exponents, (5) to draw attention to the eluded role played by the Hamiltonian in dynamics, (6) to dramatically distinguish the theory from its spatial counterpart, and yet, (7) not to be a scaling field that alters the fixed points. New classical and nonclassical universality classes divided by d_c emerge for $\theta < 1 - \kappa$.

Acknowledgment. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11575297 and 12175316).

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