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Bosonic Halperin (441) Fractional Quantum Hall Effect at Filling Factor $\nu = 2/5$

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Quantum Hall effects with multicomponent internal degrees of freedom facilitate the playground of novel emergent topological orders. Here, we explore the correlated topological phases of two-component hardcore bosons at a total filling factor $\nu = 2/5$ in topological lattice models under the interplay of intracomponent and intercomponent repulsions. We give the numerical demonstration of the emergence of Halperin (441) fractional quantum Hall effect based on exact diagonalization and density-matrix renormalization group methods. We elucidate its topological features including the degeneracy of the ground state, fractionally quantized topological Chern number matrix and chiral edge modes.

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Multicomponent systems provide an avenue for realizing a tremendous amount of physics that have no analogue in one-component systems, which also results in a question of fundamental interest: what kind of topological order can emerge in a microscopic interacting model. Two-component fractional quantum Hall (FQH) effect is such an example, which is conjectured to produce new incompressible ground states beyond Laughlin states. The earlier theoretical studies of an electron gas in spin-unpolarized and bilayer (or double quantum wells) systems such as AlGaAs, exemplified by integer quantum Hall state at $\nu = 1$[1–3] and fractional quantum Hall state at $\nu = 1/2$[4–7] (note that, whereas for single layer, the other competing candidates could be either a composite fermion liquid in the lowest Landau level or a Moore–Read state in the second Landau level), can be described as Halperin's two-component (mmn) wave functions with the $\boldsymbol{K} = \begin{pmatrix} m & n \\ n & m \end{pmatrix}$ matrix.[8] Recently, experimental observations in graphene have greatly expanded our understanding of a widespread zoology of multicomponent FQH effects[9–12] with an $SU(4)$ theoretical generalization of Halperin's wave function to these FQH effects in graphene,[13] and at the same time inspired the current study of multicomponent FQH effect in bosonic systems as the counterparts.

Compared to the fermionic systems, the ongoing study of two-component bosonic quantum Hall effects supplements our knowledge of multicomponent topological phases and yet demands further attention. Besides FQH states of spinless bosons at sequential filling factors in early days,[14] it was realized that there exists an intimate theoretical generalization of Halperin's wave functions to clustered spin-singlet quantum Hall states[15–21] for bosons in the lowest Landau level with an internal degree of freedom like pseudospin, and even a classification scheme for bosonic symmetry-protected topological (SPT) phases with no intrinsic topological order for multicomponent bosons,[22] such as two-component bosonic integer quantum Hall liquid with the associated $\boldsymbol{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ matrix in the presence of mutual flux attachment,[23,24] followed by the numerical hunt for two-component bosonic fractional quantum Hall states at various filling factors where several proposals of new quantum Hall structures mentioned above have been examined, i.e., spin-singlet FQH states at $\nu = 2k/3$ and integer quantum Hall state at $\nu = 2$[25–32] based on either strong synthetic gauge fields or topological bands with high Chern number[33,34] in cold atomic neutral systems, and the possible emergence of tunneling-induced Moore–Read quantum Hall state at $\nu = 1$ from two copies of Laughlin $\nu = 1/2$ FQH state[35,36] that are not available in

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electronic FQH effect. Nevertheless, the vague outline of multicomponent bosonic systems discloses a large uncharted theoretical area of two-dimensional bosonic topological order[37] to be explored, aside from the experimental interest.

With these motivations, in this work, we are concerned with the internal correlation structures of two-component bosonic FQH effects at fixed filling factor $\nu = 2/5$, where a convincing theoretical evidence is still lacking in topological flat band models. Early pioneering theoretical studies[38−42] suggest that Laughlin-like FQH state of neutral atoms can be simulated in lattice models in the presence of artificial gauge field, which plays the role of magnetic field acting on charged electrons. While Landau level problem directly relates to the problems in the high magnetic field, topological flat band models dubbed "Chern insulators" are becoming an appealing candidate platform for studying the quantum Hall effect at zero magnetic field,[43] along with lots of experimental interests in topological Hofstadter–Harper[44−47] and Haldane-honeycomb[48] bands for cold atom realization, topological Moiré minibands for twisted multilayer graphene[49−51] and other flat bands with nontrivial topology in geometrically frustrated lattice.[52] The emergence of FQH effect in topological flat bands (namely "fractional Chern insulators"[53−58]) requires a demanding understanding of the internal topological structure of interacting fractionalized phases, where an integer valued symmetric $\boldsymbol{K}$ matrix was proposed to characterize different topological orders for Abelian multicomponent systems according to the Chern–Simons theory.[59−63] Indeed, at partial fillings $\nu = 1/(kC+1)$ (odd k for hardcore bosons and even k for spinless fermions) in topological flat bands with higher Chern numbers C, there fractionalized Abelian C-color-entangled states host a close relationship to C-component FQH states,[64−71] where the general one-to-one correspondence is built up based on the classification of their $\boldsymbol{K}$ matrices from the inverse of Chern number matrix for these multicomponent topological phases[72−74] (even for topological phases in Bose–Fermi mixtures[75]), where the quantized intercomponent drag Hall transport is identified as a primary experimental evidence for the emergence of exotic correlated many-body topological states in multicomponent systems.[11,12] Together with synthetic magnetic gauge fields in cold atomic neutral systems, these related progresses thus enable new relevant prospects for the study of two-component bosonic FQH effects in topological lattice models, which is the focus of our work.

The main findings of the present work is that we characterize a bosonic FQH state at filling factor $\nu = 2/5$ with the spin-S_z-rotation symmetric interaction, which is Halperin (441) FQH state with $\boldsymbol{K} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$, as presented below. Through the state-of-the-art methods, we determine their topological nature, including ground state degeneracy, topological Chern number matrix on the torus manifold and fractional charge pumping, chiral edge excitations on the cylindrical geometry.

Topological Lattice Models. Here, we utilize both exact diagonalization (ED) and density-matrix renormalization group (DMRG) to study the low-energy properties of the Hamiltonian for two-component hardcore bosons with pseudospin degrees of freedom via intercomponent and intracomponent interactions at a total filling $\nu = 2/5$ in topological flat bands, and elucidate the physical mechanism of the competing Halperin FQH effects. The Hamiltonian built on topological π-flux checkerboard (CB) and Haldane-honeycomb (HC) lattices is given by

$$H_{\rm CB} = \sum_{\sigma}\Big[-t\sum_{\langle \boldsymbol{r},\boldsymbol{r}'\rangle} e^{i\phi_{\boldsymbol{r}'\boldsymbol{r}}} b^{\dagger}_{\boldsymbol{r}',\sigma} b_{\boldsymbol{r},\sigma} - \sum_{\langle\langle \boldsymbol{r},\boldsymbol{r}'\rangle\rangle} t'_{\boldsymbol{r},\boldsymbol{r}'} b^{\dagger}_{\boldsymbol{r}',\sigma} b_{\boldsymbol{r},\sigma} - t''\sum_{\langle\langle\langle \boldsymbol{r},\boldsymbol{r}'\rangle\rangle\rangle} b^{\dagger}_{\boldsymbol{r}',\sigma} b_{\boldsymbol{r},\sigma} + \text{H.c.}\Big] + V_{\rm int}, \tag{1}$$

$$H_{\rm HC} = \sum_{\sigma}\Big[-t\sum_{\langle \boldsymbol{r},\boldsymbol{r}'\rangle} b^{\dagger}_{\boldsymbol{r}',\sigma} b_{\boldsymbol{r},\sigma} - t'\sum_{\langle\langle \boldsymbol{r},\boldsymbol{r}'\rangle\rangle} e^{i\phi_{\boldsymbol{r}'\boldsymbol{r}}} b^{\dagger}_{\boldsymbol{r}',\sigma} b_{\boldsymbol{r},\sigma} - t''\sum_{\langle\langle\langle \boldsymbol{r},\boldsymbol{r}'\rangle\rangle\rangle} b^{\dagger}_{\boldsymbol{r}',\sigma} b_{\boldsymbol{r},\sigma} + \text{H.c.}\Big] + V_{\rm int}, \tag{2}$$

where $b^{\dagger}_{\boldsymbol{r},\sigma}$ is the bosonic creation operator of pseudospin $\sigma = \uparrow,\downarrow$ at site $\boldsymbol{r}$, $\langle \ldots \rangle$, $\langle\langle \ldots \rangle\rangle$ and $\langle\langle\langle \ldots \rangle\rangle\rangle$ denote the nearest-neighbor, the next-nearest-neighbor, and the next-next-nearest-neighbor pairs of sites, respectively. We take the on-site and nearest-neighbor interactions

$$V_{\rm int} = U\sum_{\boldsymbol{r}} n_{\boldsymbol{r},\uparrow} n_{\boldsymbol{r},\downarrow} + V_{\uparrow\uparrow}\sum_{\sigma=\uparrow,\downarrow}\sum_{\langle \boldsymbol{r},\boldsymbol{r}'\rangle} n_{\boldsymbol{r}',\sigma} n_{\boldsymbol{r},\sigma}, \tag{3}$$

where $n_{\boldsymbol{r},\sigma}$ is the particle number operator of pseudospin σ at site $\boldsymbol{r}$. In the hardcore bosonic limit where $n_{\boldsymbol{r},\sigma} = 0, 1$, the onsite intracomponent Bose–Hubbard term $U n_{\boldsymbol{r},\sigma}(n_{\boldsymbol{r},\sigma} - 1)$ vanishes. Here the last term $\sum_{\sigma=\uparrow,\downarrow}\sum_{\langle \boldsymbol{r},\boldsymbol{r}'\rangle} n_{\boldsymbol{r}',\sigma} n_{\boldsymbol{r},\sigma} = \sum_{\langle \boldsymbol{r},\boldsymbol{r}'\rangle}[n_{\boldsymbol{r}'} n_{\boldsymbol{r}} + 4S_z(\boldsymbol{r}')S_z(\boldsymbol{r})]/2$ only preserves the spin-S_z-rotation symmetry with local spin and density operators $S_z(\boldsymbol{r}) = (n_{\boldsymbol{r},\uparrow} - n_{\boldsymbol{r},\downarrow})/2$, $n_{\boldsymbol{r}} = n_{\boldsymbol{r},\uparrow} + n_{\boldsymbol{r},\downarrow}$. In what follows, we impose the tunnel couplings $t' = 0.3t$, $t'' = -0.2t$, $\phi = \pi/4$ for checkerboard lattice, while $t' = 0.6t$, $t'' = -0.58t$, $\phi = 2\pi/5$ for honeycomb lattice,[57,76] and choose the interaction strengths $U = \infty, V_{\uparrow\uparrow}/t \sim 100$. We emphasize that the two component layers will never be decoupled since the large on-site Hubbard repulsion $U/t \gg 1$ though our numerical calculations.

For finite system sizes with translational symmetry, $\nu = \nu_\uparrow + \nu_\downarrow$, $\nu_\uparrow = 2N_\uparrow/N_\mathrm{s} = \nu_\downarrow = 2N_\downarrow/N_\mathrm{s} = 1/5$, where $N_\mathrm{s} = 2 \times N_x \times N_y$ is the total number of sites, and N_σ is the particle number of pseudospin σ. The energy states are labeled by the total momentum $K = (K_x, K_y)$ in units of $(2\pi/N_x, 2\pi/N_y)$ in the Brillouin zone. For cylindrical system sizes, we exploit infinite DMRG on the cylindrical geometry with open boundary condition in the x direction and periodic boundary condition in the y direction, and the bond dimension of DMRG is kept up to $M = 5000$ here.

In the following, we first delve into the topological ground state degeneracies for the interaction with spin-S_z-rotation symmetry, and discuss its stability. Then, we give a detailed demonstration of the $\boldsymbol{K}$ matrix classification from the inverse of the Chern number matrix $\boldsymbol{C}$ based on ED calculation of topologically invariant Chern number and DMRG simulation of drag charge pumping in the periodic parameter plane.

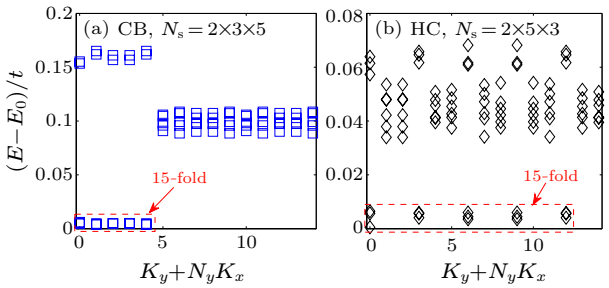

Fig. 1. Numerical ED results for the low energy spectrum of two-component bosonic systems $\nu = 2/5$ with $U(1)\times U(1)$-symmetry for spin-S_z-rotation symmetric interaction in different topological lattice models for (a) checkerboard lattice and (b) honeycomb lattice. The red dashed box indicates the ground state degeneracy.

Ground State Degeneracy. The key property of the existence of topological fractionalized ordered phases lies in their ground state degeneracies. For Halperin (mmn) quantum Hall state, the ground state degeneracy is given by the determinant of the $\boldsymbol{K}$ matrix. Thus, first we demonstrate the ground state degeneracy on periodic lattice in different interacting regimes. As shown in Figs. 1(a) and 1(b) for different topological systems, there exists a low-energy manifold with fifteen-fold degenerate ground states, which are separated from higher energy levels by a robust gap.

Further, in order to demonstrate the topological equivalence of these ground states, we calculate the low energy spectra flux under the insertion of flux quanta, by utilizing twisted boundary conditions $\psi(\boldsymbol{r}_\sigma + N_\alpha) = \psi(\boldsymbol{r}_\sigma)\exp(i\theta_\sigma^\alpha)$, where θ_σ^α is the twisted angle for pseudospin σ particles in the α direction. For spin-S_z-rotation symmetric interaction, as shown in Fig. 2(a), these fifteen-fold ground states evolve into each other without mixing with the higher levels, and the system returns back to itself upon the insertion of five flux quanta for both $\theta_\uparrow^\alpha = \theta_\downarrow^\alpha = \theta$ and $\theta_\uparrow^\alpha = \theta, \theta_\downarrow^\alpha = 0$. The robustness of degenerate ground states reveals the existence of universal internal structures with the behavior of fractional quantization.

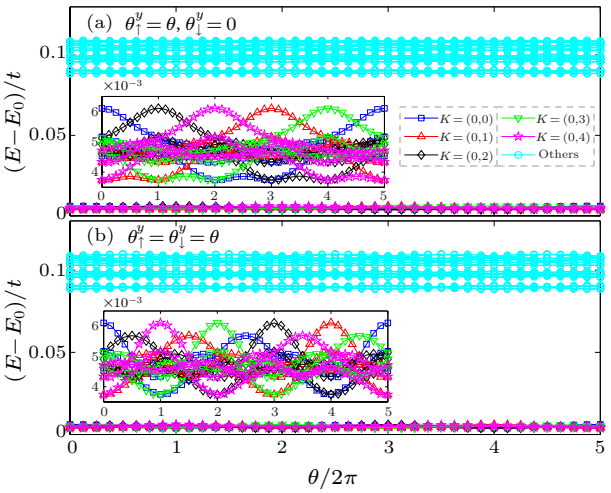

Fig. 2. Numerical ED results for the low energy spectral flow of two-component bosonic systems $\nu = 2/5$, $N_\mathrm{s} = 2\times3\times5$ with $U(1)\times U(1)$-symmetry in topological checkerboard lattice under the insertion of two types of flux quanta: (a) $\theta_\uparrow^y = \theta$, $\theta_\downarrow^y = 0$, (b) $\theta_\uparrow^y = \theta_\downarrow^y = \theta$.

Chern Number Matrix. Following the above discussion, we continue to analyze the fractionally topological quantization, characterized by the Chern number matrix $\boldsymbol{C} = \begin{pmatrix} C_{\uparrow\uparrow} & C_{\uparrow\downarrow} \\ C_{\downarrow\uparrow} & C_{\downarrow\downarrow} \end{pmatrix}$ of the many-body ground state wavefunction ψ for interacting systems.[77,78] In the parameter plane $(\theta_\sigma^x, \theta_{\sigma'}^y)$, the matrix elements are defined by $C_{\sigma\sigma'} = \int d\theta_\sigma^x d\theta_{\sigma'}^y F_{\sigma\sigma'}(\theta_\sigma^x, \theta_{\sigma'}^y)/2\pi$ with the Berry curvature

$$F_{\sigma\sigma'}(\theta_\sigma^x, \theta_{\sigma'}^y) = \mathrm{Im}\left(\left\langle \frac{\partial\psi}{\partial\theta_\sigma^x}\middle|\frac{\partial\psi}{\partial\theta_{\sigma'}^y}\right\rangle - \left\langle \frac{\partial\psi}{\partial\theta_{\sigma'}^y}\middle|\frac{\partial\psi}{\partial\theta_\sigma^x}\right\rangle\right).$$

On a coarsely discretized parameter plane $(\theta_\sigma^x, \theta_{\sigma'}^y) = (2k\pi/m, 2l\pi/m)$, where $0 \le k, l \le m$, we first define the Berry connection of the wavefunction on a Wilson loop plaquette as $A_{k,l}^{\pm x} = \langle\psi(k,l)|\psi(k\pm1,l)\rangle$, $A_{k,l}^{\pm y} = \langle\psi(k,l)|\psi(k,l\pm1)\rangle$. Then the Berry curvature on this Wilson loop plaquette $(k,l)\to(k+1,l)\to(k+1,l+1)\to(k,l+1)\to(k,l)$ is given by the gauge-invariant expression $F_{\sigma\sigma'}(\theta_\sigma^x, \theta_{\sigma'}^y)\Delta\theta_\sigma^x\Delta\theta_{\sigma'}^y = \mathrm{Im}\ln\left[A_{k,l}^x A_{k+1,l}^y A_{k+1,l+1}^{-x} A_{k,l+1}^{-y}\right]$ with each equal difference step $\Delta\theta_\sigma^x = \Delta\theta_{\sigma'}^y = 2\pi/m$. By numerically calculating the Berry curvatures using $m\times m$ mesh Wilson loop plaquette in the boundary phase space with $m \ge 10$, one can obtain the quantized topological invariant $C_{\sigma\sigma'}$ of the gapped ground states at momentum K, and $C_{\uparrow\uparrow} = C_{\downarrow\downarrow}, C_{\uparrow\downarrow} = C_{\downarrow\uparrow}$ under the prescribed symmetry $b_{\boldsymbol{r},\uparrow} \leftrightarrow b_{\boldsymbol{r},\downarrow}$. In the ED study of finite system size, we find that for spin-S_z-rotation symmetric interaction, the three ground

states at momentum $K = (0, 0)$ couple with each other under the flux thread, and host the average many-body Chern numbers $\frac{1}{3}\sum_{i=1}^{3} C^i_{\uparrow\uparrow} = 0.2621 \simeq 4/15$ and $\frac{1}{3}\sum_{i=1}^{3} C^i_{\uparrow\downarrow} = -0.0641 \simeq -1/15$ for 10×10 mesh points with the absolute error around 0.002, as indicated in Figs. 3(a) and 3(b), respectively.

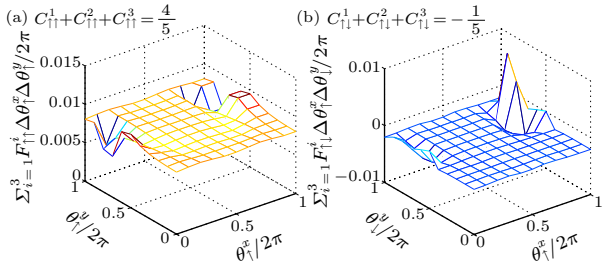

Fig. 3. Numerical ED results for Berry curvatures $F^{xy}_{\sigma\sigma'}\Delta\theta^x_\sigma\Delta\theta^y_{\sigma'}/2\pi$ of the three ground states at $K = (0, 0)$ of two-component bosonic systems $N_\uparrow = N_\downarrow = 5$, $N_s = 2 \times 3 \times 5$ in topological checkerboard lattice in different parameter planes: (a) $(\theta^x_\uparrow, \theta^y_\uparrow)$, (b) $(\theta^x_\uparrow, \theta^y_\downarrow)$.

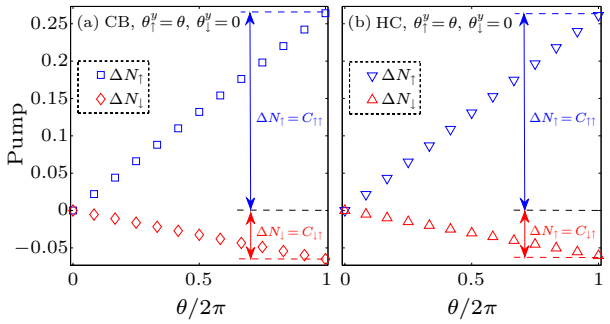

Fig. 4. Quantized charge transfers for two-component bosonic systems $\nu = 2/5$ with $U(1) \times U(1)$-symmetry on the infinite $N_y = 5$ cylinder for spin-S_z-rotation symmetric interaction under the insertion of flux quantum $\theta^y_\uparrow = \theta, \theta^y_\downarrow = 0$ in one component for (a) checkerboard lattice and (b) honeycomb lattice.

As topological Chern number is related to the Hall transport response,[79] we further calculate the charge pumping induced by the Berry curvature once the flux quantum is adiabatically inserted on infinite cylinder systems using DMRG.[80] Numerically we cut the infinite cylinder into left-half and right-half parts along the x direction, and obtain the net charge transfer of the pseudospin σ from the right side to the left side by calculating the the evolution of $N_\sigma(\theta^y_{\sigma'}) = tr[\widehat{\rho}_L(\theta^y_{\sigma'})\widehat{N}_\sigma]$ as a function of $\theta^y_{\sigma'}$ (here $\widehat{\rho}_L$ the reduced density matrix of the left part, classified by the quantum numbers $\Delta Q_\uparrow, \Delta Q_\downarrow$. $N_\sigma(\theta^y_{\sigma'})$ is the particle number of pseudospin σ in the left part. In DMRG partition of the wavefunction with the total particle number N^t_σ into two halves, $\Delta Q_\sigma = N_\sigma - N^t_\sigma/2$ denotes the deviation of the particle number of pseudospin σ in the left part relative to the equal partition). For spin-S_z-rotation symmetric interaction, as shown in Figs. 4(a) and 4(b), we get fractionally quantized charge transfer in different topological lattice models with pumping values $\Delta N_\uparrow = N_\uparrow(2\pi) - N_\uparrow(0) \simeq C_{\uparrow\uparrow} = 4/15$, $\Delta N_\downarrow = N_\downarrow(2\pi) - N_\downarrow(0) \simeq C_{\downarrow\uparrow} = -1/15$ upon threading one flux quantum $\theta^y_\uparrow$ of pseudospin $\uparrow$ with $\theta^y_\downarrow = 0$ for two-component bosons with the absolute error around 0.001.

In view of quantized topological invariants, our ED and DMRG studies establish the essential diagnosis of distinct topological orders, benefiting from the merit of the well-defined Chern number matrix of the gapped ground state, namely

$$\boldsymbol{C} = \frac{1}{15}\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}, \tag{4}$$

which is just the inverse of the $\boldsymbol{K} = \boldsymbol{C}^{-1}$ matrix of Halperin's (441) FQH state.

Entanglement Spectroscopy of Edge Physics. Moreover, we analyze the edge physics of these topological ordered phases according to the bulk-edge correspondence, especially the chiral Luttinger liquid character which can be used to characterize the topological orders in the different FQH states.[81,82] The edge chirality is determined by the signs of the eigenvalues $m \pm n$ of the $\boldsymbol{K} = \begin{pmatrix} m & n \\ n & m \end{pmatrix}$ matrix.

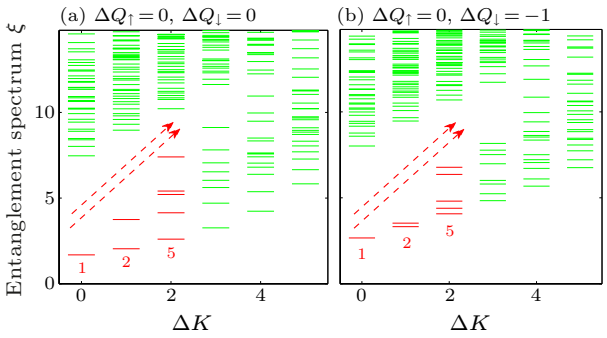

Fig. 5. Chiral edge mode identified from the momentum-resolved entanglement spectrum for two-component bosonic systems $\nu = 2/5$ with $U(1) \times U(1)$-symmetry for spin-S_z-rotation symmetric interaction on the infinite $N_y = 6$ cylinder in the typical charge sectors: (a) $\Delta Q_\uparrow = \Delta Q_\downarrow = 0$ and (b) $\Delta Q_\uparrow = 0, \Delta Q_\downarrow = -1$. The horizontal axis shows the relative momentum $\Delta K = K_y - K^0_y$ (in units of $2\pi/N_y$). The numbers below the red dashed line label the level counting: $1, 2, 5, \cdots$.

Here we harness the low-lying momentum-resolved entanglement spectrum to identify the topological nature on the infinite cylinder based on the general relationship between the entanglement spectrum and the spectrum of a physical edge state:[83,84] (i) for $m > n$, two propagating chiral modes in the same direction are obtained,[72] (ii) for $m < n$, two chiral modes propagate in the opposite directions, i.e., bosonic integer quantum Hall phase,[85] (iii) for $m = n$, only one chiral mode is left, i.e., Halperin (111) exciton phase.[86] For spin-S_z-rotation symmetric interaction,

Figs. 5(a) and 5(b) depict two parallel forward-moving branches of low-lying excitations, matching with the level counting $1, 2, 5, \cdots$ of WZW conformal field description for two gapless free bosonic edge theories of Halperin (441) FQH state: for given sectors $\Delta Q_\uparrow$ and $\Delta Q_\downarrow$, apart from certain constant, these edge modes are approximately described by the charge and spin Hamiltonian $H_{\rm edge} = 2\pi/N_y \times v_{\rm s} \sum_{m=1}^{\infty} m(n_m^c + n_m^s)$, where $n_m^{c(s)}$ denotes the set of non-negative integers of edge charge (spin) excitation mode,[85] and we can derive the degenerate level pattern of $n_m^{c(s)}$ for any momentum $\sum_{m=1}^{\infty} m(n_m^c + n_m^s) = \Delta K$ with charge and spin branches mixed.

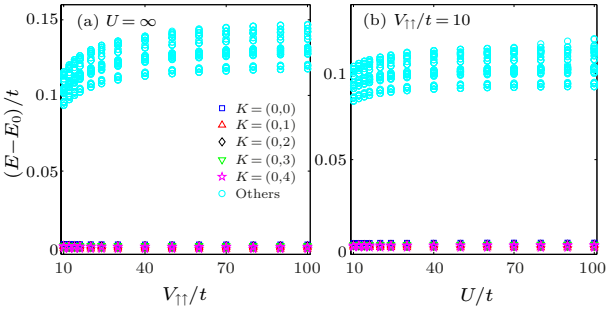

Fig. 6. Numerical ED results for the low energy spectrum evolution of two-component bosonic systems $\nu = 2/5$, $N_{\rm s} = 2 \times 3 \times 5$ with $U(1) \times U(1)$-symmetry in topological checkerboard lattice as a function of (a) $V_{\uparrow\uparrow}/t$ at $U = \infty$ and (b) U/t at $V_{\uparrow\uparrow}/t = 10$.

Finite Interaction Effects. Finally, we examine the effect of finite onsite intercomponent repulsion U and nearest-neighbor intracomponent repulsion $V_{\uparrow\uparrow}$ on the stability of topological ground state degeneracy at $\nu = 2/5$, qualitatively from the ED study of the low energy spectrum in topological lattice models. As shown in Fig. 6(a) for fixed large repulsion U, once nearest-neighbor intracomponent repulsion $V_{\uparrow\uparrow}$ decreases from $V_{\uparrow\uparrow}/t = 100$ to $V_{\uparrow\uparrow}/t = 10$, a robust ground degenerate manifold of Halperin (441) states is still preserved with a slight softening of spectrum gap. Meanwhile, in the collision of cold atom with short-range interactions, both intracomponent and intercomponent repulsions are usually controllable through the essential Feshbach resonance. Hence we further plot the low energy spectrum evolution as a function of intercomponent repulsion U, as shown in Fig. 6(b). For fixed large repulsion $V_{\uparrow\uparrow}/t = 10$, as U decreases from $U/t = 100$ to $U/t = 10$, the fifteen-fold ground state degeneracy remains unchanged. Based on these observations, we judge that hardcore bosonic Halperin (441) state is closely promising, within the comparable interaction energy scale of the order 10. In this work we only consider the hardcore boson, where the onsite particle repulsion is infinite large. The effect of finite onsite intracomponent repulsion, which is possible in softcore bosons in realistic cold atom experiment, has been neglected. The fact is that the discussion of softcore boson is beyond our computational ability due to the huge Hilbert space of softcore bosons. Nevertheless we anticipate the proposed (441) state is robust if the onsite repulsion is strong enough for softcore bosons.

Conclusion and Outlook. So far we have introduced two microscopic topological lattice models that realize two-component bosonic FQH effects at the filling $\nu = 2/5$, including Halperin (441) fractional quantum Hall states in the extremely spin-S_z-rotation symmetric interacting limit. Using ED and DMRG simulations, we find that the ground state shows several characteristic topological properties in connection to the $\boldsymbol{K}$ matrix: (i) the ground state degeneracy, (ii) fractional topological Chern number in relation to Hall conductance, and (iii) chiral edge modes. Nevertheless, our study may furthermore serve as a promising paradigm for engineering quantum Hall physics in the clean system with multiple components just by tuning intercomponent and intracomponent interactions using artificial topological bands in future experiments on cold atomic gases.[87]

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