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Soliton Rectangular Pulses and Bound States in a Dissipative System Modeled by the Variable-Coefficients Complex Cubic-Quintic Ginzburg–Landau Equation

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The complex cubic-quintic Ginzburg–Landau equation (CQGLE) is a universal model for describing a dissipative system, especially fiber laser. The analytic one-soliton solution of the variable-coefficients CQGLE is calculated by a modified Hirota method. Then, phenomena of soliton pulses splitting and stable bound states of two solitons are investigated. Moreover, rectangular dissipative soliton pulses of the variable-coefficients CQGLE are realized and controlled effectively in the theoretical research for the first time, which breaks through energy limitation of soliton pulses and is expected to provide theoretical basis for preparation of high-energy soliton pulses in fiber lasers.

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Solitons have been applied in fields such as optics, fluid mechanics, plasma physics and the Bose–Einstein condensates.[1−7] As one of the momentous parts of solitons, optical solitons have valuable applications and prospects in optical communication, all-optical network, signal processing and other systems because of their long-distance stable transmission and low loss.[8−12] In an optical fiber, which is an energy balanced system (Hamiltonian system), the formation and transmission mechanism of optical solitons can be simulated by the non-linear Schrödinger equation (NLSE).[13−17] Those solitons appear as hyperbolic secant pulses, which are called traditional solitons, and there are abundant results in the experiment and theory about them.[18−23] However, fiber lasers and most systems in nature are dissipative systems with energy changing at any time.[24−26] Thus, the Ginzburg–Landau equation (GLE), which is frequently used to simulate a dissipative system, has been a wide concern for scholars from some different fields.[27−32] In recent years, the complex cubic-quintic Ginzburg–Landau equation (CQGLE) has been applied in much nonlinear optical research, such as mode-locked lasers, multimode lasers, short pulse propagation, parametric oscillators, and long-haul optical communication.[33−36] In addition, in order to better reflect the complexity and variability of a dissipative system, it is necessary to choose variable-coefficient CQGLE as the theoretical model.

Nozaki and Bekki took effort firstly to defining and using the modified bilinear method to study the exact solutions of the generalized GLE in 1984.[37] So far, some kinds of soliton solutions to the CQGLE have been obtained in the theoretical research.[38−41] However, the discussions on bound states of solitons, especially rectangular soliton pulses, are rare,[42,43] even though they have been studied in the experiment.[44,45] When the energy of the soliton pulse is too high, the single soliton pulse with traditional hyperbolic secant shape usually splits to form the bound state of solitons, which is helpful for improving multi-channel communication and expanding channel capacity.[43] Also, the pulse splitting hinders the formation of soliton pulses with higher energy. The new rectangular solitons just make up for the deficiency, because they break through the bottleneck of soliton area theory and complete the stable output of high energy pulses.[46,47] Therefore, in this Letter, we mainly analyze the dissipative soliton bound states and new rectangular pulses of the variable-coefficient CQGLE, which not only enriches the theoretical system of solitons but also is expected to be helpful for improving the upper limit of the energy output of nonlinear optical devices such as fiber lasers.

The variable-coefficient CQGLE can be written as[48]

$$\begin{aligned} &i\frac{\partial u}{\partial z} - \frac{B(z)}{2}\frac{\partial^2 u}{\partial \tau^2} + \psi(z)|u|^2 u - \rho(z)|u|^4 u \\ =&\frac{i}{2}[g(z) - \beta(z)]u + \frac{i}{2}d(z)\frac{\partial^2 u}{\partial \tau^2} - iK(z)|u|^2 u \\ &- i\nu(z)|u|^4 u. \end{aligned} \tag{1}$$

Here, $u(z,\tau)$ is the normalized amplitude of optical field, z is the normalized propagation distance, and τ is the retarded time. The variable coefficient $B(z)$

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is related to group velocity dispersion (GVD), $\psi(z)$ and $\rho(z)$ represent the cubic nonlinearities and the quintic nonlinearities, respectively. Accordingly, the first term on the right side of Eq. (1) determines the net gain $\Delta g_0(z) = g(z) - \beta(z)$, with $g(z)$ being the peak gain and $\beta(z)$ the linear loss. Then, the coefficient $d(z)$ represents gain dispersion, $\nu(z)$ and $K(z)$ contribute to three-photon absorption and two-photon absorption effects, respectively. In the following, the analytic one-soliton solution to Eq. (1) will be deduced using the modified Hirota method for the first time.

Analytic One-Soliton Solution to Eq. (1). On the basis of the modified Hirota method and the general dependent variable transformation, $u(z,\tau)$ can be set as[49]

$$u(z,\tau) = G(z,\tau)/F(z,\tau)^{1/2+i\alpha}, \tag{2}$$

here $F(z,\tau)$ is assumed to be a real function, and $G(z,\tau)$ is a complex differentiable function. Then the modified bilinear forms of Eq. (1) can be expressed as

$$\begin{aligned}
&iD^1_{z,\alpha}G\cdot F - \Big[\frac{B(z)}{2} + \frac{i}{2}d(z)\Big]D^2_{\tau,\alpha}G\cdot F\\
=&\frac{i}{2}[g(z)-\beta(z)]G\cdot F,\\
&\Big(\frac{1}{2}+i\alpha\Big)\Big(\frac{3}{2}+i\alpha\Big)\Big[\frac{B(z)}{2}+\frac{i}{2}d(z)\Big]D^2_\tau F\cdot F\\
=&2[\rho(z)-i\nu(z)]|G|^4 - 2[\psi(z)+iK(z)]|G|^2\cdot F,
\end{aligned} \tag{3}$$

where $D^2_{\tau,\alpha}G\cdot F$ and $D^1_{z,\alpha}G\cdot F$ are modified Hirota's bilinear operators, and $D^2_\tau F\cdot F$ is a Hirota bilinear operator. They are defined as follows:[37,50]

$$\begin{aligned}
&D^n_{z,\alpha}D^m_{\tau,\alpha}G(z,\tau)\cdot F(z,\tau)\\
=&\Big(\frac{\partial}{\partial z} - \Big(\frac{1}{2}+i\alpha\Big)\frac{\partial}{\partial z'}\Big)^n \times \Big(\frac{\partial}{\partial \tau} - \Big(\frac{1}{2}+i\alpha\Big)\frac{\partial}{\partial \tau'}\Big)^m\\
&\times G(z,\tau)\cdot F(z',\tau')|_{z'=z,\tau'=\tau},
\end{aligned} \tag{4}$$

$$\begin{aligned}
&D^n_z D^m_\tau G(z,\tau)\cdot F(z,\tau) = \Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\Big)^n\Big(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'}\Big)^m\\
&\times G(z,\tau)\cdot F(z',\tau')|_{z'=z,\tau'=\tau}.
\end{aligned} \tag{5}$$

Further, we can set the form of bright one-soliton solution as

$$\begin{aligned}
G(z,\tau) &= \varepsilon G_1(z,\tau) + \varepsilon^3 G_3(z,\tau) + \varepsilon^5 G_5(z,\tau) + \cdots,\\
G_1(z,\tau) &= \exp(b_1(z) + ib_2(z) + (a_1+ia_2)\tau\\
&\quad + k_1 + ik_2),\\
F(z,\tau) &= 1 + \varepsilon^2 F_2(z,\tau) + \varepsilon^4 F_4(z,\tau) + \varepsilon^6 F_6(z,\tau)\\
&\quad + \cdots,\\
F_2(z,\tau) &= \sigma(z)\exp(2b_1(z) + 2a_1\tau + 2k_1).
\end{aligned} \tag{6}$$

Here, $b_1(z)$, $b_2(z)$ and $\sigma(z)$ are differentiable functions of z to be determined; ε is a formal expansion parameter, real constants a_1 and a_2 are complex frequencies, real constants k_1 and k_2 are initial phases.

By substituting Eq. (6) into Eq. (3), the coefficients of different powers of ε can be extracted, and each coefficient can be divided into two independent equations according to the imaginary part and the real part, and then the expressions of a series of parameters can be solved in turn as follows:

$$\begin{aligned}
b_1(z) &= \int \frac{1}{2}[g(z) + 2a_1a_2B(z) + (a_1^2 - a_2^2)d(z)\\
&\quad - \beta(z)]dz,\\
b_2(z) &= \int [d(z)a_1a_2 + \frac{1}{2}B[z](a_2^2 - a_1^2)]\mathrm{d}z,\\
\Psi[z] &= \frac{1}{2}a_1^2[(4\alpha^2-3)B[z] + 8\alpha d(z)]\sigma(z),\\
K(z) &= \frac{1}{2}a_1^2[(4\alpha^2-3)d(z) - 8\alpha B(z)]\sigma(z),\\
\sigma(z) &= m\exp\Big[\int [d(z)(a_2 - 2\alpha a_1)^2 + \beta(z) - g(z)]\mathrm{d}z\Big],\\
&\quad \{m : m\in\Re\cap m\neq 0\},\\
B(z) &= 2d(z)(a_2/a_1 - \alpha),\\
\rho(z) &= d(z)a_1[(7\alpha - 4\alpha^3)a_1 + (4\alpha^2-3)a_2]\sigma(z)^2,\\
\nu(z) &= \frac{1}{2}d(z)a_1[(3-20\alpha^2)a_1 + 16\alpha a_2]\sigma(z)^2,\\
G_x(z,\tau) &= 0,\ x = (3,5,7,\ldots);\\
F_x(z,\tau) &= 0,\ x = (4,6,8,\ldots).
\end{aligned} \tag{7}$$

Without loss of generality, we set $\varepsilon = 1$. Finally, on the basis of all the above constraints, the one-soliton solution to Eq. (1) can be written as

$$\begin{aligned}
u(z,\tau) = &\Big\{\exp\Big[y(z) + k_1 + ik_2 + \tau(a_1+ia_2)\\
&+ i\frac{(\alpha a_1^3 - \alpha a_1a_2^2 + a_2^3)}{a_1}\int d(z)dz\Big]\Big\}\\
&\cdot\Big\{\Big[1 + m\exp\Big[\int d(z)[(2\alpha a_1 - 2a_2)^2 + a_1^2]dz\\
&+ 2\tau a_1 + 2k_1\Big]\Big]^{\frac{1}{2}+i\alpha}\Big\}^{-1},
\end{aligned} \tag{8}$$

and

$$y(z) = \frac{1}{2}\int [g(z) + d(z)(a_1^2 - 4\alpha a_1a_2 + 3a_2^2) - \beta(z)]dz. \tag{9}$$

The free parameters are peak gain $g(z)$, linear loss $\beta(z)$, gain dispersion $d(z)$, modified parameter α, frequency parameter a_1 and a_2, real constant m, and phase parameter k_1 and k_2.

Results and Discussions—Bound States of Two Solitons. In fiber lasers, optical soliton pulses are formed under the balance of dispersion, nonlinearity and dissipation effect. However, when the effect of dissipation is weak and the balance of dispersion effect and nonlinear effect plays a major role, the dissipative system will be similar to a Hamiltonian system, so the traditional hyperbolic secant solitons can be observed. For the traditional soliton pulse, a single

pulse will split to form a soliton bound state when its energy is too high, as shown in Fig. 1. In Fig. 1(a), we set all the coefficients as follows: $a_1 = 0.0001$, $a_2 = 3$, $\alpha = 1.46$, $k_1 = 8$, $k_2 = -7$, $g(z) = 0.3$, $d(z) = 0.2z$, $m = 60$, $\beta(z) = -0.079 + 2/(0.32 - 0.6z)$, and we will keep α, k_1, k_2 unchanged in all of the following discussions. In this figure, a single soliton pulse with high energy is split into two soliton molecules due to the energy area theory, resulting in the formation of the soliton bound state. Furthermore, we can see that the energy distributions of the two soliton molecules are not equal to each other.

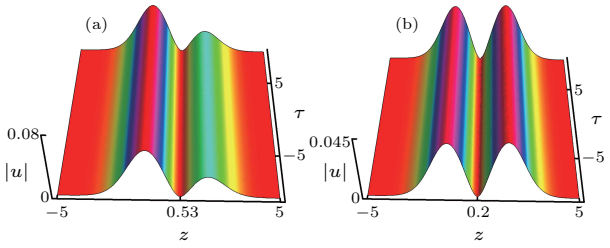

Fig. 1. Evolution of soliton bound states. The corresponding parameters are selected to be $a_1 = 0.0001$, $a_2 = 3$, $\alpha = 1.46$, $k_1 = 8$, $k_2 = -7$, $g(z) = 0.3$, $d(z) = 0.2z$, $m = 60$ with (a) $\beta(z) = -0.079 + 2/(0.32 - 0.6z)$ and (b) $\beta(z) = -0.079 + 2/(0.12 - 0.6z)$.

Then we can achieve the soliton bound state with uniform energy distribution by adjusting $\beta(z) = -0.079 + 2/(0.12 - 0.6z)$ in Fig. 1(b). Moreover, we observe that the splitting point of the soliton pulse is determined by the point of discontinuity of $\beta(z)$, so we can change the splitting position by controlling the point of discontinuity of $\beta(z)$. The research on the bound states of optical solitons can help to carry out more complex information modulation and coding in optical communication, and may help to advance the multi-channel sharing technology.

Soliton Rectangular Pulses. When the dispersion effect is not dominant in fiber lasers, meanwhile the interaction between dissipative and nonlinear effect is more prominent, the shapes of optical soliton pulses will evolve into rectangles as shown in Fig. 2. This is the first time that the rectangular solitons are presented in the variable-coefficient CQGLE. In Fig. 2(a), all coefficients are set as $\alpha = 1.46$, $k_1 = 8$, $k_2 = -7$, $a_1 = 0.6025$, $a_2 = 3$, $g(z) = 5.138$, $d(z) = 2$, $\beta(z) = 2.06$, $m = 60$, and in this case, the frequency propagation width of the rectangular soliton pulse is about 42, and its amplitude is about 0.13. For the dissipative solitons with a rectangular shape, they are absolutely stable, and can be enhanced with the increase of the net gain in the system without splitting, so as to ensure the output of a high-energy single pulse.

In order to further increase the total energy of the soliton pulse, we can make efforts to broaden the propagation width and to increase the amplitude of the soliton. On the one hand, in Fig. 2(b), the net gain $\Delta g_0(z) = g(z) - \beta(z)$ of the system is increased from 3.078 in Fig. 2(a) to 10.078, while the values of a_1, a_2 and $d(z)$ are replaced with -1.3, 2 and 0.3, respectively, to keep the balance among dissipation effect, dispersion effect and nonlinear effect, then the rectangular pulse with nearly doubled frequency range is shown. On the other hand, in Fig. 2(c), the value of m is reduced from 60 to 10, which leads to the enhancement of the nonlinear effect in the dissipative system, and then the amplitude of the rectangular wave increases from 0.14 in Fig. 2(b) to 0.34, that is, the energy of the soliton increases significantly. Our results are beneficial to the preparation of high energy rectangular soliton pulses in the fiber laser. Moreover, due to the advantages of high energy and stability of rectangular pulses, they show the surprising potential in the applications of high-speed and long-haul all-optical networks, optical sensing, optical ranging, all-optical square wave clock, and so on.

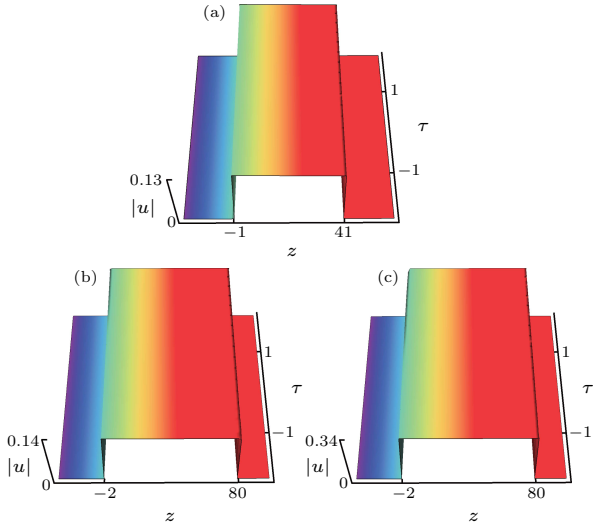

Fig. 2. Evolution of soliton rectangular pulses. The corresponding parameters are selected to be $\alpha = 1.46$, $k_1 = 8$, $k_2 = -7$ with (a) $a_1 = 0.6025$, $a_2 = 3$, $g(z) = 5.138$, $d(z) = 2$, $\beta(z) = 2.06$, $m = 60$; (b) $a_1 = -1.3$, $a_2 = 2$, $g(z) = 10$, $d(z) = 0.3$, $\beta(z) = -0.078$, $m = 60$; (c) $a_1 = -1.3$, $a_2 = 2$, $g(z) = 10$, $d(z) = 0.3$, $\beta(z) = -0.078$, $m = 10$.

In summary, an analytic one-soliton solution to the variable-coefficient CQGLE has been obtained with the modified Hirota method. Then the stable soliton bound states in the weak dissipative system and the rectangular soliton pulses in the strong dissipative system have been observed and studied in detail for the first time. In the weak dissipative system, solitons have appeared as hyperbolic secant pulses, and most of the high energy solitons have split into soliton bound states, as well as the linear loss $\beta(z)$ is the key to the splitting point and energy distribution of soliton molecules. In the strong dissipative system, solitons behave as rectangular pulses without splitting, and increasing the net gain $\Delta g_0(z)$ or enhancing the non-

linear effect by reducing m can lead to the increase of the total energy of solitons. Moreover, it is necessary to pay attention to the harmony of dispersion, nonlinear and dissipation effect in the dissipative system by adjusting the relevant parameters. The present results not only are perfect for theoretical systems of solitons but also provide valuable references for applications of optical solitons in optical communication and preparation of ultra-high energy soliton pulses in fiber lasers.

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