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Symmetric and Anti-Symmetric Solitons of the Fractional Second- and Third-Order Nonlinear Schrödinger Equation

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The fractional second- and third-order nonlinear Schrödinger equation is studied, symmetric and antisymmetric soliton solutions are derived, and the influence of the Lévy index on different solitons is analyzed. The stability and stability interval of solitons are discussed. The anti-interference ability of stable solitons to the small disturbance shows a good robustness.

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Optical solitons exhibit considerable realizability in experiments and possess enormous potential applications, so research on optical solitons generates important practical significance.[1] As one of the important and typical nonlinear equations, the nonlinear Schrödinger equation (NLSE)[2] is mainly used to describe nonlinear waves in deep water, intense lasers, superconductors and Bose Einstein condensates (BECs), etc. In BECs, the matter wave function of the NLSE and the corresponding energy help researchers to understand properties of microscopic systems.[3,4] In the field of optics, the NLSE can describe propagation of optical pulses in dispersive and nonlinear media.[5–7] Wang *et al.* explained the basis for existence of modulated elliptic waves in the central region using the nonlinear fastest descent method.[8] Dai *et al.* controlled the evolution of spatial solitons in photovoltaic photorefractive crystals using a specific coupled NLSE.[9] Liu *et al.* discussed the similarities and differences of different plane solitons controlled by a (3+1)-dimensional coupled variable coefficient system.[10]

At present, the research on the second- and third-order NLSE is still the case of integer-order equation.[11,12] In real life, many physical phenomena can only be accurately expressed by considering the knowledge of fractional calculus.[13,14] Therefore, it is necessary to expand the research of the NLSE from integer order to fractional order, which can better explain the phenomena in some practical problems. Recently, single cubic and single quintic fractional NLSEs were introduced to describe propagation of optical pulses.[15,16] However, research on the fractional second- and third-order NLSE is hardly reported.

Competitive nonlinearity is a kind of nonlinear response which is caused by different physical processes. Typical examples include BECs with both local and long-range boson interactions,[17] and nematic liquid crystals with both thermal and directional nonlinearities.[18] Many soliton structures, including high topological charge vortex solitons and one-dimensional multimodal solitons, are unstable under the single nonlinearity, whereas the competitive nonlinearity can stabilize these solitons.[19] The competitive nonlinear effect can also support the ground state of bright and dark solitons.[20] In this Letter, a quadratic nonlinear term is added to the fractional cubic NLSE, and the competitive second- and third-order nonlinearities (self-defocusing quadratic and self-focusing cubic nonlinearities) to influence the existence and stability of solitons are discussed.

Model. The fractional second- and third-order NLSE can be rewritten as

$$i\frac{\partial\Psi}{\partial\zeta}-\frac{1}{2}\Big(-\frac{\partial^2}{\partial\xi^2}\Big)^{\alpha/2}\Psi+V(\xi)\Psi+\sigma|\Psi|^2\Psi+\lambda|\Psi|\Psi=0, \tag{1}$$

where the complex envelope Ψ is a function of ξ and ζ. The first term represents the transmission, the second term describes the diffraction effect, the third term denotes the refractive index modulation in optics or the external potential in a BEC, the fourth and last terms represent Kerr and quadratic nonlinear terms, respectively. When the value of α equals 2, Eq. (1) will be the second- and third-order NLSE with the integer order,[11] which describes the nonlinear dynamics of binary BEC.

Here we choose the potential function as[16]

$$V(\xi)=V_0\Big[\mathrm{sech}\Big(\frac{\xi-\xi_0}{\omega_0}\Big)\Big]^2+V_0\Big[\mathrm{sech}\Big(\frac{\xi+\xi_0}{\omega_0}\Big)\Big]^2. \tag{2}$$

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Taking $\Psi(\zeta,\xi)=\psi(\xi)e^{i\beta\zeta}$ into Eq. (1), we obtain

$$-\frac{1}{2}\Big(-\frac{\partial^2}{\partial\xi^2}\Big)^{\alpha/2}\psi+V\psi+\sigma|\psi|^2\psi+\lambda|\psi|\psi-\beta\psi=0. \quad (3)$$

Symmetric and Antisymmetric Solitons and Their Stabilities. Firstly, we use the square operator method to study the existence of soliton solutions for Eq. (1). In this study, we only consider the case of self-defocusing quadratic and self-focusing cubic nonlinearities, i.e., $\sigma=1$ and $\lambda=-1$. The modulation intensity parameter of potential function V_0 is 1.3, the width ω_0 of a single peak of the potential function is 1.4, the distance ξ_0 between the two peaks is 1.5, and the value of Lévy index α is between 1 and 2. For the given soliton propagation constant, we can numerically obtain the symmetric and antisymmetric soliton solutions to Eq. (1), as shown in Fig. 1.

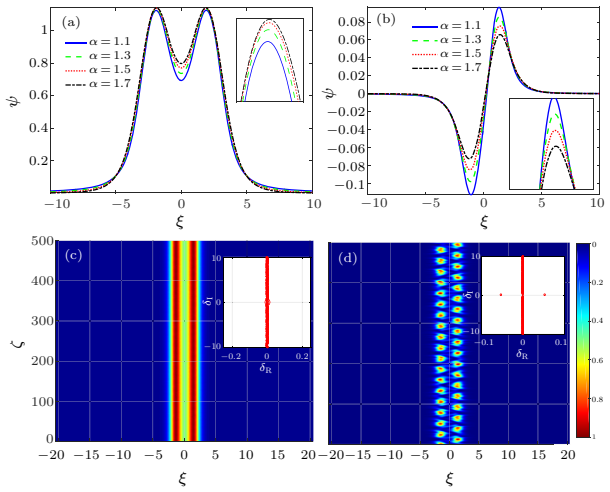

Fig. 1. The influence of different Lévy index on (a) symmetric and (b) antisymmetric soliton solutions for the given propagation constant $\beta=0.8$. The evolution diagrams of (c) symmetric and (d) antisymmetric soliton solutions. The insets in (a) and (b) show enlarged parts at the peak, and in (c) and (d) are linear stability analysis diagrams. Parameters are $V_0=1.3$, $\omega_0=1.2$, $\xi_0=1.5$, $\alpha=1.1$–1.7, $\lambda=-1$, $\sigma=1$.

Referring to Ref. [16], we study the influence of the Lévy index on the wave shape and amplitude of the soliton solution for the fractional NLSE (1). From Fig. 1(a), the amplitude of symmetric soliton increases with the increase of the Lévy index, which is obvious between two peaks. From Fig. 1(b), the amplitude of the antisymmetric soliton decreases with the increase of the Lévy index. This shows that the influences of the Lévy index on the wave shape and amplitude of symmetric and antisymmetric solitons are quite different. For the symmetric soliton, the Lévy index is positively correlated with the amplitude of soliton, while for the antisymmetric soliton, the Lévy index is inversely correlated with the amplitude of soliton. It is worth noting that the breaking bifurcation phenomenon described in Ref. [14] does not occur in the case of self-defocusing quadratic and self-focusing cubic nonlinearities.

Then we analyze the stability of symmetric and antisymmetric solitons. By adding small perturbation $u(\xi)$ and $v(\xi)$ to the stable solution $\psi(\xi)$, we have

$$\Psi(\zeta,\xi)=e^{i\beta\zeta}[\psi(\xi)+u(\xi)e^{\delta\zeta}+v^*(\xi)e^{\delta^*\zeta}], \quad (4)$$

where $u(\xi)$ and $v(\xi)$ are infinitesimal perturbations, $*$ denotes a complex conjugate operation, and δ is an unstable complex growth rate. Taking the linear part of the disturbance, we obtain the following eigenvalue problem:

$$i\begin{pmatrix}L_{11}&L_{12}\\L_{21}&L_{22}\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=\delta\begin{pmatrix}u\\v\end{pmatrix}, \quad (5)$$

with

$$L_{11}=-\frac{1}{2}\Big(\frac{\partial^2}{\partial\xi^2}\Big)^{\alpha/2}-V-u-2\sigma|\psi|^2-\frac{3}{2}\lambda|\psi|,$$

$$L_{12}=\psi^2\Big(-\sigma-\lambda\frac{1}{2|\psi|}\Big),$$

$$L_{21}=(\psi^2)^*\Big(\sigma+\lambda\frac{1}{2|\psi|}\Big),$$

$$L_{22}=\frac{1}{2}\Big(\frac{\partial^2}{\partial\xi^2}\Big)^{\alpha/2}+V+u+2\sigma|\psi|^2+\frac{3}{2}\lambda|\psi|.$$

We use the full spectrum Fourier collocation method to solve the above eigenvalue problem and to obtain the corresponding linear stable spectrum. Generally speaking, the real part of the linear eigenvalue δ represents the unstable growth rate of the soliton solution in the subsequent evolution. During the evolution process, the larger the absolute value of the real part is, the more unstable the soliton solution is. On the contrary, when δ is a pure imaginary number, the soliton has a good linear stability.

In addition, we also use the split step Fourier algorithm to directly simulate the soliton evolution with a 5% perturbation. In the inset of Fig. 1(c), the real parts of the linear eigenvalues are all 0, which shows that the symmetric soliton is stable, which is verified by the direct simulation in Fig. 1(c). Here δ_R and δ_I represent the real and imaginary parts of the linear eigenvalue, respectively. The overall shape symmetric soliton remains unchanged and it only oscillates slightly in the subsequent transmission, which is caused by 5% white noise disturbance. In Fig. 1(d), the real part of the linear eigenvalue is not zero, which indicates that the symmetric soliton is unstable. Figure 1(d) shows the evolution of symmetric soliton. Under the influence of a 5% white noise, the symmetric soliton is unstable and its shape becomes spines, which is consistent with the result of linear stability analysis.

We also find that the soliton with a good linear stability has the characteristic of anti-interference in the process of soliton transmission. In the case of a 5% white noise perturbation, the amplitude and shape of the stable soliton remain unchanged although there is

some oscillation in the subsequent transmission process, the unstable soliton becomes unstable in a very short transmission distance, and its wave shape and amplitude changes greatly along the transmission distance. This shows that the stable soliton has excellent anti-interference performance for small interference, which makes the soliton transmission research have greater applications in real life.

The Stability Interval of Soliton Solution. We study the linear stability interval of symmetric and antisymmetric soliton solutions, that is, we discuss the relationship between the maximum instability growth rate δ_R and the propagation constant β, as shown in Fig. 2.

In Fig. 2(a), when the propagation constant is chosen between [0.4, 1.2], the maximum unstable growth rate of the symmetric soliton, that is, the real part of the eigenvalue, is greater than 0, which indicates that the symmetric soliton is unstable in this interval. When the propagation constant is chosen between [1.2, 1.8], the unstable growth rate of the symmetric soliton solution is 0, which indicates that the symmetric soliton in this interval is stable. In order to further verify the accuracy of the maximum unstable growth rate, we select the points with the maximum $\beta = 0.7$ and the minimum $\beta = 1.25$ in Fig. 2(a) for the stability analysis to numerically simulate the transmission of symmetric soliton in Fig. 3. From Fig. 2(a), for the given the propagation constant $\beta = 0.7$, the symmetric soliton falls into the unstable region. In Fig. 3(a), the real part of the linear eigenvalue in this case is not zero, and the soliton solution is unstable. The numerical simulation in Fig. 3(c) also shows the instability of symmetric soliton. From Fig. 2(a), for the given propagation constant $\beta = 1.25$, the symmetric soliton is in the stable range. In Fig. 3(b), the real part of the linear eigenvalue in this case is 0, which indicates that the symmetric soliton is stable. The numerical simulation in Fig. 3(d) also illustrates this result.

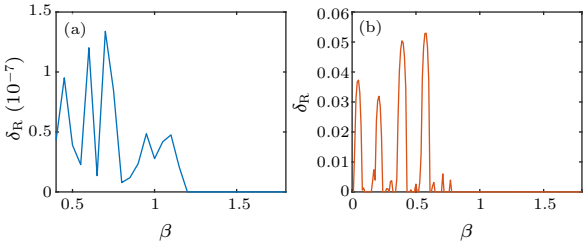

Fig. 2. The relationship between the propagation constant and the maximum instability growth rate of (a) symmetric and (b) antisymmetric solitons. The propagation constant changes from 0.4 to 1.8 with steps of 0.05 in (a), and from 0.01 to 1.8 with steps of 0.01 in (b). Parameters are $V_0 = 1.3$, $\omega_0 = 1.2$, $\xi_0 = 1.5$, $\alpha = 1.1$, $\lambda = -1$, $\sigma = 1$.

In Fig. 2(b), when the propagation constant β is chosen between [0.01, 0.65], the maximum unstable growth rate of the antisymmetric soliton is almost greater than 0, and it is close to 0 only at a few points, which indicates that the antisymmetric soliton is unstable in this interval. When the propagation constant β is chosen between [0.65, 1.8], the unstable growth rate of antisymmetric soliton is almost kept in the range of 10^{-8}, which indicates that the antisymmetric soliton is stable in this range. We select two propagation constants in these two intervals to analyze the stability and to simulate the propagation of antisymmetric solitons. These results are shown in Fig. 4.

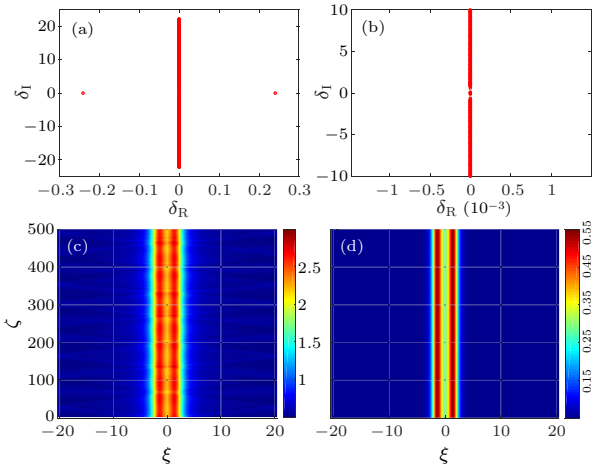

Fig. 3. The linear stability spectrum of symmetric soliton with (a) $\beta = 0.7$ and (b) $\beta = 1.25$, (c) and (d) the numerical simulation of symmetric soliton corresponding to (a) and (b). Parameters are the same as those in Fig. 1.

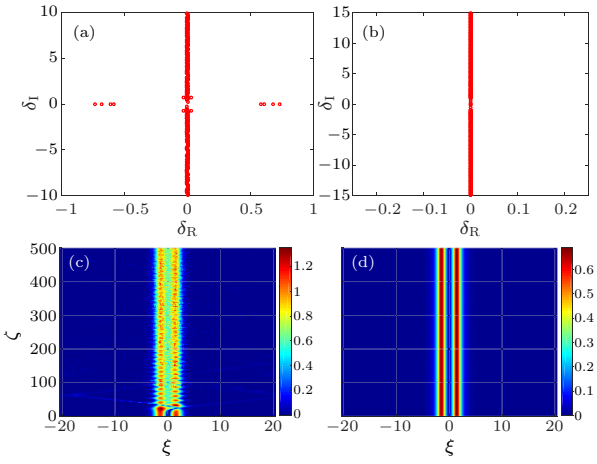

Fig. 4. The linear stability spectrum of antisymmetric soliton with (a) $\beta = 0.57$ and (b) $\beta = 1$, (c) and (d) the numerical simulation of symmetric soliton corresponding to (a) and (b). Parameters are the same as those in Fig. 1.

For the unstable interval, we choose the propagation constant $\beta = 0.57$. Figure 4(a) shows that the maximum value of the real part of the linear eigenvalue is not 0, and the soliton is unstable. The numerical simulation in Fig. 4(c) shows the instability of antisymmetric soliton, that is, the propagation of soliton has obvious fluctuation and finally soliton breaks up. For the stable interval, we choose the propagation

constant $\beta = 1.0$. Figure 4(b) shows that the real part of the linear eigenvalue is 0, and the soliton is stable. The numerical simulation in Fig. 4(d) also shows the stable propagation of antisymmetric soliton.

In summary, we have studied the fractional second- and third-order NLSE, and found the stable symmetric solution and the unstable antisymmetric solution using the square operator method. Comparing the two kinds of solitons, we realize that the stable symmetric soliton has a good anti-interference performance for the possible small disturbance in the actual transmission, and it shows a certain robustness. We also study the influence of the Lévy index on different soliton solutions. For the symmetric soliton, the Lévy index is positively correlated with the amplitude of soliton; while for the antisymmetric soliton, the Lévy index is inversely correlated with the amplitude of soliton. In addition, we investigate the maximum unstable growth rates of symmetric and antisymmetric solitons, and give the stable intervals of these solitons.

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