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Inelastic Interaction of Double-Valley Dark Solitons for the Hirota Equation

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For a scalar integrable model, it is generally believed that the solitons interact with each other elastically, for instance, multi-bright solitons from the nonlinear Schrödinger equation and the Korteweg-de Vries equation, etc. We obtain double-valley dark solitons from the defocusing Hirota equation by the Darboux transformation. Particularly, we report a remarkable phenomenon for the inelastic interaction of the double-valley dark solitons, in contrast to the solitons interacting with each other elastically for a scalar integrable model in previous works. Furthermore, we give the explicit conditions for the elastic collision based on the asymptotic analysis results. It is shown that the double-valley dark solitons could also admit elastic interaction under the special parameters settings.

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It is well known that the nonlinear Schrödinger equation (NLSE) plays an important role in many fields of nonlinear science such as water waves,[1] plasma,[2] Bose–Einstein condensates,[3] and nonlinear optics.[4,5] Particularly, in optical fibers the NLSE can describe a picosecond optical pulse propagation.[4,6] However, for propagation of a subpicosecond or femtosecond pulse, it is necessary to involve higher-order nonlinear and dispersive effects like third-order dispersion (TOD), self-steepening (SS), and intrapulse Raman scattering. When the TOD and SS are taken into account, the ultrashort light pulses in optical fibers can be governed by the integrable Hirota equation (HE):[7–10]

$$iq_t + \frac{1}{2}q_{xx} - \sigma|q|^2 q + i\alpha(-q_{xxx} + 6\sigma|q|^2 q_x) = 0, \quad (1)$$

where $\sigma = \pm 1$ means the focusing or defocusing type, and α denotes the coefficients of high order effects. When $\alpha = 0$, Eq. (1) can be reduced to the well-known NLSE.

Due to the integrability for the HE, many kinds of exact localized waves solutions, such as the multisolitons, rogue waves and breathers, were obtained by distinct methods.[11–18] For the defocusing case, generally, it admits single-valley dark solitons (SVDSs).[17,19–21] Very recently, multivalley dark soliton solutions were obtained in multicomponent Manakov systems.[22] To the best of our knowledge, double-valley dark solitons (DVDSs) of the HE have never been reported before. Actually the DVDS here is closely related to the soliton molecules,[23–25] especially the dark soliton molecules derived by Lou *et al.*[26] The details of their relationship will be disclosed in the following main text. What is more, we find that the interaction between SVDS and DVDS or between DVDSs is inelastic generally. We know that there have already been works showing the inelastic interaction in the coupled nonlinear equation,[27–31] but few works reported the inelastic collision in the scalar case. Recently, the inelastic interaction phenomena of solitons in the scalar Sasa–Satsuma equation corresponding to a 3×3 spectral problem has been reported in some works.[32,33] As far as we know, the inelastic interaction of double-valley dark solitons is revealed in the integrable systems with a 2×2 spectral problem for the first time.

In this Letter, we consider the defocusing HE (1) with $\sigma = 1$. Compared with the soliton in the NLSE,[34,35] the soliton in the HE has the different velocity due to the high order dispersion and time-delay corrections, and its integrable properties and solutions can be studied by the same way as treating the NLSE. Darboux transformation (DT) is an effective and powerful technique to formulate multi-soliton solutions for the nonlinear integrable equation.[36–40] We would like to utilize the DT method[41] to derive the multi-dark soliton solution, which can be reduced to the DVDS solution to Eq. (1) under the special parameter setting. The DVDS solution can be classified into the asymmetric and symmetric ones. By the complicated asymptotic analysis, we show the inelastic

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collision between two DVDSs or between one DVDS and one SVDS. Meanwhile, we obtain the special conditions to keep the elastic interaction.

Starting from the seed solution $q^{[0]} = c\,e^{i\theta}$, $\theta = ax - bt$, where $b = \alpha(a^2+6c^2)a + \frac{1}{2}a^2 + c^2$, the n-dark soliton solutions on the plane wave background can be derived by applying the n-fold DT.[41] Letting the spectral parameter $\lambda_i = -\frac{a}{2} + c\cos z_i$, $z_i \in (0,\pi)$, $i = 1,\ldots,n$, the n-dark soliton solution can be expressed as

$$
\begin{aligned}
q^{[n]} &= c\frac{\det(\boldsymbol{M}_1)}{\det(\boldsymbol{M})}e^{i\theta},\\
\boldsymbol{M} &= \Big(\frac{e^{i(X_j-\bar{X}_m)} + d\delta_{[m,j]}}{e^{iz_j} - e^{-iz_m}}\Big)_{1\le m,j\le n},\\
\boldsymbol{M}_1 &= \Big(\frac{e^{i(X_j+z_j-\bar{X}_m+z_m)} + d\delta_{[m,j]}}{e^{iz_j} - e^{-iz_m}}\Big)_{1\le m,j\le n},
\end{aligned}
\tag{2}
$$

with

$$
\begin{aligned}
X_j &= ic\sin(z_j)(x - v_j t),\\
\delta_{[m,j]} &= \begin{cases} 0, & m \neq j,\\ e^{2c\sin(z_m)\gamma_m}, & m = j, \end{cases} \quad j = 1,2,\ldots,n,
\end{aligned}
$$

where γ_m and d are real parameters related to the position of solitons, the velocity of solitons v_j is governed by

$$
\begin{aligned}
v_j =& \alpha[4c^2\cos(z_j)^2 - 6ac\cos(z_j) + 3a^2 + 2c^2]\\
&- c\cos(z_j) + a.
\end{aligned}
$$

Now we proceed to study the DVDS, which can be obtained by Eq. (2) under the setting $n = 2$ with the same velocity of two incoherent solitons. Due to $v_1 = v_2 \equiv v$, i.e.,

$$
\cos(z_2) = -\cos(z_1) + 3a/(2c) + 1/(4c\,\alpha),
$$

the range of z_1 belongs to $(0,\pi) \cap (\arccos[-1 + 3a/(2c) + 1/(4c\,\alpha)], \arccos[1 + 3a/(2c) + 1/(4c\,\alpha)])$. Then the expression of the DVDS can be simplified to

$$
q^{[2]} = c\frac{N}{D}e^{i\theta}, \tag{3}
$$

where

$$
\begin{aligned}
N =& \omega_{[1,2]}^2 e^{-\eta_1-\eta_2} + d^2 e^{\eta_1+\eta_2} + 2d\cosh(\eta_1 - \eta_2),\\
D =& \omega_{[1,2]}^2 e^{-\xi_1-\xi_2} + d^2 e^{\xi_1+\xi_2} + 2d\cosh(\xi_1 - \xi_2),
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
\omega_{[1,2]}(z_1,z_2) &= \frac{\sin[\frac{1}{2}(z_1 - z_2)]}{\sin[\frac{1}{2}(z_1 + z_2)]},\\
\xi_i &= c\sin(z_i)(x - vt + \gamma_i), \quad \eta_i = \xi_i + iz_i.
\end{aligned}
$$

Here the parameters γ_1, γ_2 should be close enough to make the two valleys of the DVDS not completely separate with each other. If $|\gamma_1 - \gamma_2|$ is large, the two valleys of the DVDS will be far from each other, in which we regard them as two SDVSs or the dark soliton molecule[26] rather than a DVDS. Furthermore, if we choose $d = \omega_{[1,2]}$ in the solution (3), then the formula (3) will be consistent with the dark soliton molecules derived by the bilinear method in Ref. [26]. Our choice $d = 1$ can ensure the denominator to be positive definite and to avoid the singularity.

The DVDS solution can admit the asymmetric one and symmetric one. By the tedious calculations, we obtain the condition $S(\gamma_1,\gamma_2) = 0$, where

$$
S(\gamma_1,\gamma_2) \equiv \gamma_1 - \gamma_2 + \frac{\sin(z_1) - \sin(z_2)}{2c\sin(z_1)\sin(z_2)}\ln|\omega_{[1,2]}|, \tag{5}
$$

which can ensure the soliton to be a symmetric one and the corresponding symmetric axis to be $x_1 = vt - \gamma_1 + \frac{\ln|\omega_{[1,2]}|}{2c\sin(z_1)}$. We show an example of asymmetric DVDS in Fig. 1(a) (blue solid line), and by choosing proper phase parameters γ_1 and γ_2 which satisfy Eq. (5), we can obtain the symmetric DVDS in Fig. 1(b) (blue dashed line).

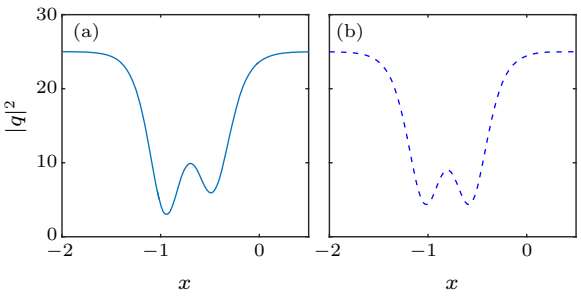

Fig. 1. The density profiles of DVDSs. (a) An asymmetric DVDS when $t = 0$, with $\gamma_1 = 0.5$ and $\gamma_2 = 0.6124$. (b) A symmetric DVDS when $t = 0$, with $\gamma_1 = 0.625$ and $\gamma_2 = 0.6577$. The other parameters are $a = 1$, $c = 5$, $\alpha = 0.1$, $z_1 = \arccos(3/5)$, $z_2 = \arccos(1/5)$.

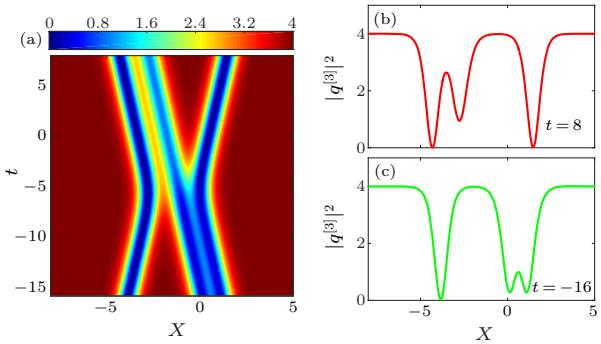

Fig. 2. (a) The collision dynamics of an SDVS and a DVDS. (b) The density profile of $q^{[3]}$ when $t = 8$. (c) The density profile of $q^{[3]}$ when $t = -16$. The parameters are chosen as $a = 1$, $c = 2$, $\alpha = -1/2$, $z_1 = \pi/2$, $z_2 = \pi/3$, $z_3 = 1.5$, $\gamma_1 = 1$, $\gamma_2 = 0.949$, $\gamma_3 = -0.5$.

Next we would like to investigate the interaction between the DVDSs. We consider the following two cases: (1) a collision between a DVDS and an SVDS, (2) a collision between two DVDSs. For the first case, we implement a three-fold DT with the spectral parameters $\lambda_1 = -\frac{a}{2} + c\cos z_1$, $\lambda_2 = -\frac{a}{2} + c\cos z_2$ (gen-

erating a DVDS), and $\lambda_3 = -\frac{a}{2} + c\cos z_3$ (generating an SVDS). We choose a symmetric soliton as the DVDS (q_{S_1}). Figure 2(a) shows the collision between a DVDS (q_{S_1}) and an SDVS (q_{S_2}). To show the collision clearly, we use the coordinate transformation $X = x - \frac{v_1+v_3}{2}t$, where v_1 and v_3 correspond to the velocities of DVDS (q_{S_1}) and SDVS (q_{S_2}), respectively. By plotting the density profile of the 3-dark soliton $q^{[3]}$ before ($t = -16$) and after the collision ($t = 8$) [Figs. 2(b) and 2(c)], we can find that the symmetric DVDS is not symmetric after the collision, but the SDVS does not change its shape. Thus the collision is inelastic for the DVDS, which has not been observed before in the scalar integrable system.

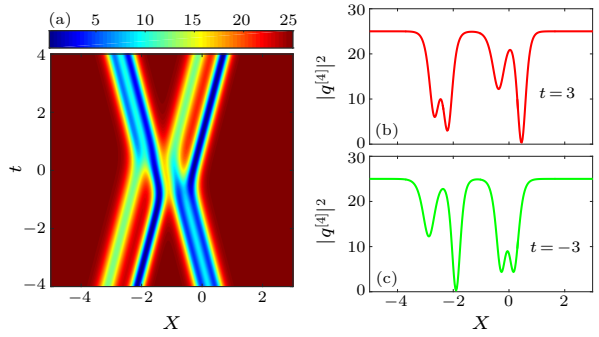

Fig. 3. (a) The collision dynamics of two DVDSs. (b) The density profile of $q^{[4]}$ when $t = 3$. (c) The density profile of $q^{[4]}$ when $t = -3$. Here $z_3 = \arccos(7/10)$, $z_4 = \arccos(1/10)$, $\gamma_3 = 0.8402$, $\gamma_4 = 0.3015$, and the other parameters are the same as those in Fig. 1(b).

Similarly, the collision between two DVDSs can be obtained by implementing a four-fold DT with the spectral parameters $\lambda_1 = -\frac{a}{2} + c\cos z_1$, $\lambda_2 = -\frac{a}{2} + c\cos z_2$ (generating a DVDS), and $\lambda_3 = -\frac{a}{2} + c\cos z_3$, $\lambda_4 = -\frac{a}{2} + c\cos z_4$ (generating another DVDS). We give an example in Fig. 3, it is shown that the symmetric DVDS (q_{S_1}) is not symmetric after the collision, and the DVDS (q_{S_2}) also changes its shape.

To analyze the mechanism of inelastic interaction deeply, we will give the asymptotic analysis for the exact soliton solutions $q^{[3]}$ and $q^{[4]}$. For the $q^{[3]}$, which is involved of a DVDS (q_{S_1}) and an SDVS (q_{S_2}), we take $0 < v_1 = v_2 < v_3$. Then, when $t \to \pm\infty$, the DVDS solution has the following asymptotic form:

$$q^{\pm}_{S_1} = c\frac{N_1^{\pm}}{D_1^{\pm}}e^{i\theta}, \tag{6}$$

where

$$\begin{aligned} N_1^{\pm} &= F^{\pm}e^{-\eta_1-\eta_2} + G^{\pm}e^{\eta_1-\eta_2} + H^{\pm}e^{\eta_2-\eta_1} \\ &\quad + e^{\eta_1+\eta_2}, \\ D_1^{\pm} &= F^{\pm}e^{-\xi_1-\xi_2} + G^{\pm}e^{\xi_1-\xi_2} + H^{\pm}e^{\xi_2-\xi_1} \\ &\quad + e^{\xi_1+\xi_2}, \end{aligned}$$

and

$$\begin{aligned} &\omega_{[i,j]} = \frac{\sin[\frac{1}{2}(z_i - z_j)]}{\sin[\frac{1}{2}(z_i + z_j)]}, \\ &F^- = \omega^2_{[1,2]},\ G^- = 1,\quad H^- = 1, \\ &F^+ = F^-G^+H^+,\quad G^+ = \omega^2_{[2,3]},\quad H^+ = \omega^2_{[1,3]}. \end{aligned}$$

Similarly, the asymptotic expression of the SVDS solution reads

$$q^{\pm}_{S_2} = c[1 - B_1 + B_1\tanh(Y_1)]e^{i\theta},\ t \to \pm\infty, \tag{7}$$

where

$$\begin{aligned} &B_1 = ie^{-iz_3}\sin z_3, \\ &Y_1 = c\sin z_3(x - v_3t + \gamma_3) - \ln(K^{\pm}), \\ &K^- = \prod_{i=1}^{2}|\omega_{[i,3]}|,\ K^+ = 1. \end{aligned}$$

It can be easily checked that

$$\left|q^-_{S_2}\left[x + \frac{\ln(K^-)}{c\sin(z_3)}\right]\right| = \left|q^+_{S_2}\left[x + \frac{\ln(K^+)}{c\sin(z_3)}\right]\right|,$$

so the SDVS (q_{S_2}) recovers its shape after the collision with a phase shift. However, by analyzing the expression (6) we can see that the collision may not be elastic for the DVDS (q_{S_1}) in most cases, and one of the conditions which make the collision be elastic is given by

$$\frac{\ln(G^+)}{\ln(H^+)} = \frac{\sin(z_2)}{\sin(z_1)}. \tag{8}$$

Therefore, when the parameters z_i ($i = 1, 2, 3$) satisfy Eq. (8), the DVDS and the SDVS would admit elastic collision. Fixing the DVDS with parameters z_1, z_2, we can adjust z_3 to satisfy the condition (8). Thus by setting the parameter $z_3 = 1.339$ and the other parameters as the same as the inelastic collision in Fig. 2, the new solution $q^{[3]}$ admits elastic collision which has been shown in Figs. 4(a)–4(c).

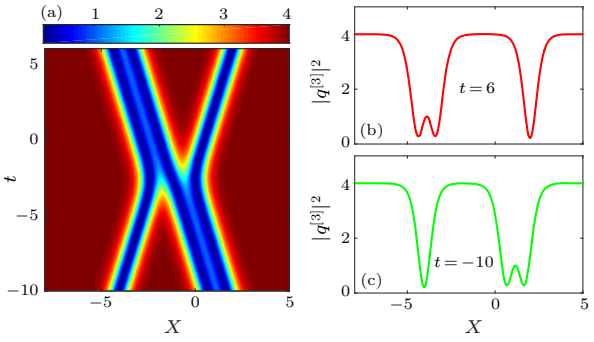

Fig. 4. (a) The elastic collision dynamics of an SDVS and a DVDS. (b) The density profile of $q^{[3]}$ when $t = 6$. (c) The density profile of $q^{[3]}$ when $t = -10$. Here $z_3 = 1.339$, and the other parameters are the same as those in Fig. 2.

Next, we study the 4-dark soliton solution $q^{[4]}$ which involves two DVDSs (q_{S_1}, q_{S_2}). Suppose that

the velocity of soliton is $0 < v_1 = v_2 < v_3 = v_4$. Then, when $t \to \pm\infty$, for the q_{S_k} we have

$$q^{\pm}_{S_k} = c\frac{N^{\pm}_k}{D^{\pm}_k}e^{i\theta},\ k = 1, 2, \tag{9}$$

with

$$\begin{aligned} N^{\pm}_k &= F^{\pm}_k e^{-\eta_{2k-1}-\eta_{2k}} + G^{\pm}_k e^{\eta_{2k-1}-\eta_{2k}} \\ &\quad + H^{\pm}_k e^{\eta_{2k}-\eta_{2k-1}} + e^{\eta_{2k-1}+\eta_{2k}}, \\ D^{\pm}_k &= F^{\pm}_k e^{-\xi_{2k-1}-\xi_{2k}} + G^{\pm}_k e^{\xi_{2k-1}-\xi_{2k}} \\ &\quad + H^{\pm}_k e^{\xi_{2k}-\xi_{2k-1}} + e^{\xi_{2k-1}+\xi_{2k}}, \end{aligned}$$

and

$$\begin{aligned} F^{-}_k &= \prod_{j=2k-1}^{2k}\prod_{i=1}^{j-1}\omega^2_{[i,j]},\quad F^{+}_k = \prod_{j=2k-1}^{2k}\prod_{i=j+1}^{4}\omega^2_{[i,j]}, \\ G^{-}_k &= \prod_{i=1}^{2k-2}\omega^2_{[i,2k]},\quad G^{+}_k = \prod_{i=2k+1}^{4}\omega^2_{[i,2k]}, \\ H^{-}_k &= \prod_{i=1}^{2k-2}\omega^2_{[i,2k-1]},\quad H^{+}_k = \prod_{i=2k+1}^{4}\omega^2_{[i,2k-1]}. \end{aligned}$$

From the asymptotic expression (9) we can also derive the elastic condition for the DVDS q_{S_k}:

$$\frac{\ln(G^{+}_k) - \ln(G^{-}_k)}{\ln(H^{+}_k) - \ln(H^{-}_k)} = \frac{\sin(z_{2k})}{\sin(z_{2k-1})}. \tag{10}$$

Therefore, when the parameters $z_i\,(i = 1, 2, 3, 4)$ satisfy Eq. (10) with $k = 1, 2$, both the DVDSs would admit elastic collision. However, both the conditions are not easy to satisfy at the same time, and we would like to explore them in the future. To give an example, we fix the DVDS (q_{S_1}) with parameters z_1, z_2, and adjust the rest parameters z_3, z_4 to satisfy the conditions (8) if only $k = 1$, thus the collision will be elastic for q_{S_1}. As shown in Fig. 5, we adjust the parameters z_3, z_4 in Fig. 3 to satisfy the condition (8) approximately, then the symmetric DVDS (q_{S_1}) still obtains its symmetry after the collision $(t = 2)$, thus the collision becomes elastic for the q_{S_1} now.

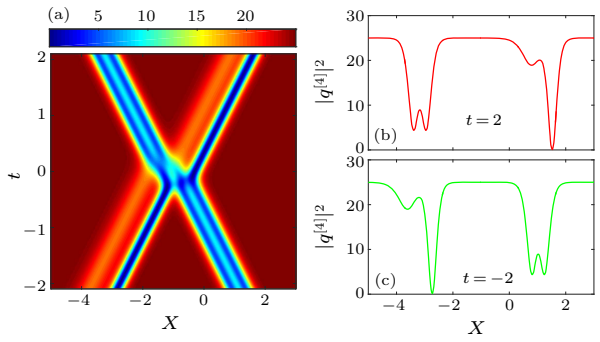

Fig. 5. (a) The elastic collision dynamics of two DVDSs. (b) The density profile of $q^{[4]}$ when $t = 2$. (c) The density profile of $q^{[4]}$ when $t = -2$. Here $z_3 = \arccos(109/125)$, $z_4 = \arccos(-9/125)$, and the other parameters are the same as those in Fig. 3.

Actually due to the specificity of symmetric DVDS, we can calculate the distance it moves as the collision is elastic for it. Taking the collision in Fig. 5 as an example, from Eq. (9) we can obtain

$$|q^{+}_{S_1}(x; \gamma_1, \gamma_2)|^2 = |q^{-}_{S_1}(x; \hat{\gamma}_1, \hat{\gamma}_2)|^2,$$

where

$$\hat{\gamma}_1 = \gamma_1 + \frac{\ln(H^{+}_1)}{2c\sin(z_1)},\quad \hat{\gamma}_2 = \gamma_2 + \frac{\ln(G^{+}_1)}{2c\sin(z_2)}. \tag{11}$$

Thus q_{S_1} would keep the symmetry after interaction when the new phase parameters $\hat{\gamma}_2$ and $\hat{\gamma}_1$ also satisfy the symmetric condition $S(\hat{\gamma}_1, \hat{\gamma}_2) = 0$. Substituting Eq. (11) into $S(\hat{\gamma}_1, \hat{\gamma}_2) = 0$ and comparing it with Eq. (5), it exactly turns out to be the elastic condition (10), which means that the collision is elastic for the symmetric DVDS when it keeps symmetric after the interaction. The symmetric axis has also changed after the collision, which turns to $x_2 = vt - \hat{\gamma}_1 + \frac{\ln|\omega_{[1,2]}|}{2c\sin(z_1)}$. Thus, the symmetric soliton does not change its shape but only moves along x-axis by $c = x_2 - x_1 = v(t^+ - t^-) + \frac{\ln(H^{+}_1)}{2c\sin(z_1)}$. Thus the symmetric DVDS (q_{S_1}) in Fig. 5 moves along the X-axis by $c = X_2 - X_1 = \frac{v_1 - v_3}{2}(t^+ - t^-) + \frac{\ln(H^{+}_1)}{2c\sin(z_1)} \approx -4.1979$. By moving the X-axis of the density profile after the collision by c, we can find the figure of the density profile of q_{S_1} before the collision [green solid line in Fig. 6(b)] overlap the figure after the collision [red dashed line in Fig. 6(b)] very well. We also move the X-axis in Fig. 3 and show the inelastic collision for q_{S_1} [Fig. 6(a)] to pose a contrast with the elastic condition in Fig. 6(b).

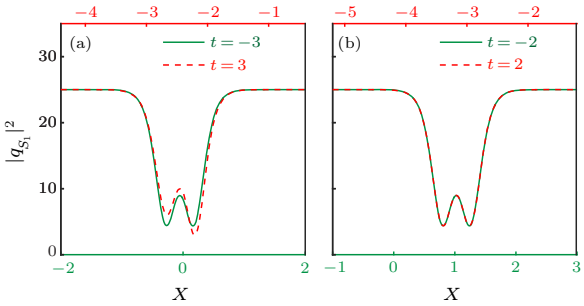

Fig. 6. (a) Density profile of q_{S_1} in Fig. 3 when $t = -3$ (green solid line) and $t = 3$ (red dashed line), for the inelastic collision. (b) Density profile of q_{S_1} in Fig. 5 when $t = -2$ (green solid line) and $t = 2$ (red dashed line), for the elastic collision.

In summary, we have given the DVDS solution for the scalar defocusing HE (1) using the DT and reported the inelastic collision between the DVDSs. The spectral parameter conditions for elastic collision have been discussed in detail by performing the asymptotic analysis. These results will deepen and enrich our understanding of the dark solitons of the HE. Furthermore, the dark soliton and soliton molecules have been

demonstrated experimentally in optical fibers[42−44] and especially in femtosecond pulse.[45−50] We expect that the inelastic phenomenon obtained in this study will be verified and observed in the physical experiments in future.

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