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Exact Solution of a Non-Hermitian Generalized Rabi Model

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An integrable non-Hermitian generalized Rabi model is constructed. A twist matrix is introduced to the construction of Hamiltonian and generates the non-Hermitian properties. The Yang–Baxter integrability of the system is proven. The exact energy spectrum and eigenstates are obtained using the Bethe ansatz. The method given in this study provides a general way to construct integrable spin-boson models.

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The Rabi model[1] plays a central role in research of quantum optics. It describes a two-level atom interacting with a single mode of cavity field. Originally, the Rabi model is used to describe a spin-$\frac{1}{2}$ atom with electrical dipole moment in a resonator. If the magnetic field is modulated periodically, spin of the atom will flip periodically, which is called the Rabi oscillation. The Rabi model can be applied in many fields of physics due to the rich physical realization of two-level systems. For example, the qubits in quantum information and quantum computation[2] are the two-level systems. The Rabi model is also applied in semiconductor physics[3] and ion-trap technology.[4] Due to the rich physics in two-level atoms, it is natural to generalize the Rabi model from one two-level atom to many-atom cases, such as the Dick model,[5] of which the atoms are free from each other, and the Rabi-Hubbard model,[6,7] of which the atoms are put in a lattice and interact with each other with Hubbard type interaction. The properties of the cavity field is also worth studying, for example, the cavity field experiences a quantum phase transition from the normal phase to the superradiant phase by adjusting the coupling constant in the Dicke model in the thermodynamical limit.[8] There are also generalizations of Rabi models to multi-mode interaction and multi-photon interaction, such as the two-photon Rabi model,[9,10] the two-mode Rabi model[11] and the Rabi model with three-level atom and two-photon interaction.[12]

Exact solutions from the Rabi model have been missing for a long time. Braak obtained the exact solution of the Rabi model with the series method.[13,14] By introducing the Bargmann representation of the Bosonic operator, the eigenvalue of the Rabi Hamiltonian turns into an infinite series of some algebra functions. The energy spectrum can be obtained by solving the zero-point problem of certain transcendental functions.[15]

Among the various methods of exactly solving physical models, there is an important method called Bethe Ansatz or the quantum inverse scattering method. As is known, the Rabi model only has Z_2 symmetry which makes all the efforts trying to solving it by Bethe Ansatz unsuccessfully. However, we can still construct the Yang–Baxter integrable spin-boson models with extra interactions beyond the Rabi type. In fact, the Rabi model itself in different dynamical regimes can be approximated as the spin-boson models with various interaction terms. To construct exact solvable spin-boson models, one should introduce proper L-operator with elements satisfying the angular momentum theory and boson algebra. The L-operators are required to satisfy the RLL relation. There are many integrable spin-boson models constructed by this way. For example, various modified Tavis–Cummings,[16−20] spin-boson models with q-deformed boson[21,22] and generalized Rabi models with extra interactions,[23−28] among which the $\sigma^z a^\dagger a$ type and the terms containing four operators are the most common ones.

Generally, the Hamiltonian of a physical system is Hermitian. Recently the study of a quantum open system suggests the Hamiltonian of a system that does not have to be Hermitian, where the energy spectrum is complex. The imaginary part of the energy is interpreted as the exchanging of particles between the system and the environment. If the non-Hermitian Hamiltonian has the PT-symmetry, the energy spec-

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trum is real.[29–31] The non-Hermiticity can induce exotic phenomena without Hermitian counterparts, such as the exceptional points,[32] PT symmetry broken,[29] non-Hermitian skin effects,[33–35] and fantastic dynamics and transport properties.[36–38]

In this Letter, we propose a new integrable spin-boson model which is non-Hermitian. The Hamiltonian reads

$$H = \omega a^\dagger a + \Delta\sigma^z + g(\sigma^- a - \sigma^+ a^\dagger), \tag{1}$$

where ω is the frequency of the boson field, a and $a^\dagger$ are the annihilation and creation operator of the boson field, respectively, Δ is the Zeeman term, σ^z is the z-component of the Pauli matrix, g is the coupling strength, σ^- and σ^+ are the ladder operators of the spin-$\frac{1}{2}$ system. The difference between the Hamiltonian (1) and the Rabi model with counter rotating wave approximation is the minus sign in the interaction term, which makes the Hamiltonian (1) non-Hermitian.

The Hamiltonian (1) is generalized by the transfer matrix $t(u)$ as follows:

$$H = \frac{\partial^2 t(u)}{\partial u^2}|_{u=0} + c, \tag{2}$$

where c is a constant. The transfer matrix is defined as

$$\begin{aligned} t(u) &= \mathrm{tr}_0 T_0(u), \\ T_0(u) &= W_0 L_{0,s}(u-\theta_1) L_{0,b}(u-\theta_2). \end{aligned} \tag{3}$$

Here tr_0 means the partial trace in the auxiliary space "0". W_0 is the twist matrix,

$$W_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

$L_{0,s}(u)$ is the L-operator of spin-1/2,

$$L_{0,s}(u) = \begin{pmatrix} u + \frac{\eta}{2} + \eta\sigma^z & \eta\sigma^- \\ \eta\sigma^+ & u + \frac{\eta}{2} - \eta\sigma^z \end{pmatrix}, \tag{5}$$

where the subscript s denotes the quantum space of the spin-1/2 operators, η is the crossing parameter, σ^- and σ^+ are the spin-1/2 ladder operators, and σ^z is the z-component of Pauli matrix. $L_{0,b}(u)$ is the L-operator of spin-S,

$$L_{0,b}(u) = \begin{pmatrix} u + \frac{\eta}{2} + \eta S^z & \eta S^- \\ \eta S^+ & u + \frac{\eta}{2} - \eta S^z \end{pmatrix}, \tag{6}$$

where the subscript b denotes the quantum space of the spin-S operators, S^- and S^+ are the spin-S ladder operators, and S^z is the z-component of the spin-S operator; θ_1 and θ_2 are the inhomogeneous parameters.

It can be proven that the W-matrix (4), L-operators (5) and (6) satisfy the RLL-relations

$$\begin{aligned} &R_{0,0'}(u) W_0 W_{0'} = W_{0'} W_0 R_{0,0'}(u), \\ &R_{0,0'}(u-v) L_{0,s}(u) L_{0',s}(v) = L_{0',s}(v) L_{0,s}(u) R_{0,0'}(u-v), \\ &R_{0,0'}(u-v) L_{0,b}(u) L_{0',b}(v) = L_{0',b}(v) L_{0,b}(u) R_{0,0'}(u-v), \end{aligned} \tag{7}$$

where the associated R-matrix is

$$R_{0,0'}(u) = \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix}. \tag{8}$$

From the construction (3) and the RLL relations (7), one can check that the monodromy matrix $T_0(u)$ satisfies the RTT-relation

$$R_{0,0'}(u-v) T_0(u) T_{0'}(v) = T_{0'}(v) T_0(u) R_{0,0'}(u-v). \tag{9}$$

Taking partial trace of the above equation, we obtain that the transfer matrices with different spectral parameters commutate with each other, i.e., $[t(u), t(v)] = 0$. Thus $t(u)$ serves as the generating functional of all the conserved quantities including the model Hamiltonian, which ensures the integrability of the system.

Substituting Eqs. (3)–(6) into Eq. (2), we have

$$\begin{aligned} H =& (-2\theta_1\eta + \eta^2) S^z - (2\theta_2\eta - \eta^2)\sigma^z \\ &+ \eta^2(\sigma^- S^+ - \sigma^+ S^-), \end{aligned} \tag{10}$$

and $c = -\eta^2 + \theta_2\eta$. With the Holstein–Primakoff (HP) transformation

$$S^+ = \sqrt{2S - a^\dagger a}\, a, \quad S^- = a^\dagger \sqrt{2S - a^\dagger a}, \tag{11}$$

we can express the spin-S operators by the boson operators which satisfy the boson algebra

$$[a, a^\dagger] = 1. \tag{12}$$

When the spin S tends to infinity, the HP transformation takes the form of

$$S^+ = \sqrt{2S}\, a, \quad S^- = a^\dagger \sqrt{2S}, \quad S^z = -a^\dagger a. \tag{13}$$

Then the Hamiltonian Eq. (10) becomes

$$\begin{aligned} H =& (2\theta_1\eta - \eta^2) a^\dagger a - (2\theta_2\eta - \eta^2)\sigma^z \\ &+ \eta^2\sqrt{2S}(\sigma^- a - \sigma^+ a^\dagger). \end{aligned} \tag{14}$$

Comparing the Hamiltonian expresses (1) and (14), we can realize that the model (1) is Yang–Baxter integrable if the model parameters are chosen as the following values:

$$\omega = 2\theta_1\eta - \eta^2, \quad \Delta = -2\theta_2\eta + \eta^2, \quad g = \eta^2\sqrt{2S}. \tag{15}$$

Next, we utilize the algebraic Bethe Ansatz to compute the exact solution of the system. The Hamiltonian has $U(1)$ symmetry with conserved quantity $\hat{N} = S^z + \frac{1}{2}\sigma^z$, which suggests an obvious eigenstate

$$|\Omega\rangle = |\downarrow\rangle_s \otimes |-S\rangle_b, \tag{16}$$

where $|\downarrow\rangle_s$ means the spin-down state, $|-S\rangle_b$ denotes the $m=-S$ state of spin-S operator, and m is the magnetic quantum number.

The matrix form of the monodromy matrix is

$$T_0(u)=\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{17}$$

and the elements acting on the reference state $|\Omega\rangle$ gives

$$\begin{aligned} &A(u)|\Omega\rangle=a(u)|\Omega\rangle, \quad B(u)|\Omega\rangle=0, \\ &C(u)|\Omega\rangle\neq 0, \quad D(u)|\Omega\rangle=d(u)|\Omega\rangle, \end{aligned} \tag{18}$$

where

$$\begin{aligned} a(u)&=\Big(u-\theta_1-\frac{\eta}{2}\Big)\Big(u-\theta_2+\frac{\eta}{2}-S\eta\Big), \\ d(u)&=\Big(-u+\theta_1-\frac{3\eta}{2}\Big)\Big(u-\theta_2+\frac{\eta}{2}+S\eta\Big). \end{aligned} \tag{19}$$

From Eq. (18), we can see that the operators $A(u)$ and $D(u)$ acting on the reference state give the eigenvalues. The operator $B(u)$ acting on the reference state gives zero. The operator $C(u)$ acting on the reference state generates other states thus can be regarded as the creation operators. Thus we can construct the assumed eigenstates of the system with operator $C(u)$ as

$$|u_1,\cdots,u_M\rangle=\prod_{i=1}^{M}C(u_i)|\Omega\rangle, \tag{20}$$

where M is the number of C-operator and $\{u_i\}$ are the Bethe roots.

Using the RTT-relation given in Eq. (9), we can derive the commutation relations among the elements of monodromy matrix as follows:

$$\begin{aligned} A(u)C(v)&=\frac{u-v+\eta}{u-v}C(v)A(u)-\frac{\eta}{u-v}C(u)A(v), \\ D(u)C(v)&=\frac{\eta}{u-v}C(u)D(v)+\frac{u-v-\eta}{u-v}C(v)D(u), \\ [C(u),C(v)]&=0. \end{aligned} \tag{21}$$

Now, we act the transfer matrix $t(u)=A(u)+D(u)$ on the Bethe state (20). With the help of commutation relations in Eqs. (21), we obtain

$$\begin{aligned} t(u)|u_1,\cdots,u_M\rangle&=\Lambda(u)|u_1,\cdots,u_M\rangle+\sum_{i=1}^{M}\Lambda_i(u) \\ &\times C(u_1)\cdots C(u_{i-1})C(u)C(u_{i+1})\cdots C(u_M)|\Omega\rangle, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \Lambda(u)&=a(u)\prod_{i=1}^{M}\frac{u-u_i-\eta}{u-u_i}+d(u)\prod_{i=1}^{M}\frac{u-u_i+\eta}{u-u_i}, \\ \Lambda_i(u)&=\frac{\eta}{u-u_i}\Big[a(u_i)\prod_{j\neq i}^{M}\frac{u_i-u_j-\eta}{u_i-u_j} \\ &\quad+d(u_i)\prod_{j\neq i}^{M}\frac{u_i-u_j+\eta}{u_i-u_j}\Big]. \end{aligned} \tag{23}$$

The first term in Eq. (22) is the eigenvalue while the second term is unwanted. The requirement of Bethe states being the eigenstates of the transfer matrix asks for the coefficients of the unwanted term vanish, which means $\Lambda_i(u)=0$ and gives the Bethe Ansatz equations

$$\frac{(u_i-\theta_1-\frac{\eta}{2})(u_i-\theta_2+\frac{\eta}{2}-S\eta)}{(u_i-\theta_1+\frac{3\eta}{2})(u-\theta_2+\frac{\eta}{2}+S\eta)}=\prod_{j=1}^{M}\frac{u_i-u_j+\eta}{u_i-u_j-\eta}, \quad i=1,\ldots,M. \tag{24}$$

From the construction (2), we obtain the energy spectrum of the system (1) in terms of Bethe roots as

$$\begin{aligned} &E(u_1,\cdots,u_M) \\ &=2\prod_{i=1}^{M}\frac{u_i+\eta}{u_i}-2(\theta_1+\theta_2+S\eta)\sum_{i=1}^{M}\prod_{j\neq i}^{M}\frac{u_i+\eta}{u_i}\frac{\eta}{u_i^2} \\ &-\Big(\theta_1+\frac{\eta}{2}\Big)\Big(-\theta_2+\frac{\eta}{2}-S\eta\Big)\sum_{i=1}^{M}\sum_{j\neq i}^{M}\prod_{k\neq i,j}^{M}\frac{u_k+\eta}{u_k}\frac{\eta}{u_j^2}\frac{\eta}{u_i^2} \\ &+\Big(\theta_1+\frac{\eta}{2}\Big)\Big(-\theta_2+\frac{\eta}{2}-S\eta\Big)\sum_{k=1}^{M}\prod_{j\neq i}^{M}\frac{u_j+\eta}{u_j}\frac{2\eta}{u_i^3} \\ &+2\prod_{i=1}^{M}\frac{u_i-\eta}{u_i}+2[\theta_1+\theta_2-(S+2)\eta]\sum_{i=1}^{M}\prod_{j\neq i}^{M}\frac{u_i-\eta}{u_i}\frac{\eta}{u_i^2} \\ &+\Big(\theta_1-\frac{3\eta}{2}\Big)\Big(-\theta_2+\frac{\eta}{2}+S\eta\Big)\sum_{i=1}^{M}\sum_{j\neq i}^{M}\prod_{k\neq i,j}^{M}\frac{u_k-\eta}{u_k}\frac{\eta}{u_j^2}\frac{\eta}{u_i^2} \\ &-\Big(\theta_1-\frac{3\eta}{2}\Big)\Big(-\theta_2+\frac{\eta}{2}+S\eta\Big)\sum_{k=1}^{M}\prod_{j\neq i}^{M}\frac{u_j-\eta}{u_j}\frac{2\eta}{u_i^3}+c. \end{aligned}$$

Some remarks are in order. The Hamiltonian (14) commutates with the conserved quantity $\hat{N}=S^z+\frac{1}{2}\sigma^z$, thus the Bethe states (20) are also the eigenstates of operator $\hat{N}$. Acting $\hat{N}$ on the Bethe states, we obtain the eigenvalues $\{M-\frac{1}{2}-S\}$, where M is the number of Bethe roots. From the explicit form of C-operator,

$$C(u)=-\eta\sigma^+\Big(u+\frac{\eta}{2}+\eta S^z\Big)-\eta S^+\Big(u+\frac{\eta}{2}-\eta\sigma^z\Big), \tag{25}$$

and the construction of Bethe-type eigenstates (20), we can prove that the Hilbert space of the system can be divided into the direct summation of the invariant subspaces with different particle numbers $\{M-\frac{1}{2}-S\}$. The subspace is spanned by the bases $|\downarrow\rangle_s\otimes|M-S+1\rangle_b$ and $|\uparrow\rangle_s\otimes|M-S\rangle_b$. For example, if $M=2$, the Bethe state reads

$$\begin{aligned} |u_i,u_j\rangle=&\eta^2\Big[u_j(u_i+\frac{\eta}{2}-S) \\ &+(u_i+\eta)(u_i+\frac{\eta}{2}+1-S)\Big]|\uparrow\rangle_s\otimes|1-S\rangle_b \\ &-\eta^2(u_i+\eta)(u_j+\eta)|\downarrow\rangle_s\otimes|2-S\rangle_b, \end{aligned} \tag{26}$$

which is the superposition of states with the particle number $\frac{3}{2} - S$ and the coefficients are determined by the Bethe roots.

In summary, we have constructed a non-Hermitian generalization of the Rabi model that is Yang–Baxter integrable. Applying a twist matrix in monodromy matrix we obtain the Hamiltonian that is both non-Hermitian and without extra terms. We obtain the exact solution of the system using Bethe Ansatz method. The scheme given in this study can be generalized in constructing other kinds of non-Hermitian integrable spin-boson models.

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