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Soliton Molecule and Breather-Soliton Molecule Structures for a General Sixth-Order Nonlinear Equation

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Starting from a general sixth-order nonlinear wave equation, we present its multiple kink solutions, which are related to the famous Hirota form. We also investigate the restrictions on the coefficients of this wave equation for possessing multiple kink structures. By introducing the velocity resonance mechanism to the multiple kink solutions, we obtain the soliton molecule solution and the breather-soliton molecule solution of the sixth-order nonlinear wave equation with particular coefficients. The three-dimensional image and the density map of these soliton molecule solutions with certain choices of the involved free parameters are well exhibited. After matching the parametric restrictions of the sixth-order nonlinear wave equation for having three-kink solution with the coefficients of the integrable bidirectional Sawada–Kotera–Caudrey–Dodd–Gibbons (SKCDG) equation, the breather-soliton molecule solution for the bidirectional SKCDG equation is also illustrated.

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In Ref. [1], K^{2}S^{2}T [A. Karasu-Kalkanlı, A. Karasu, A. Sakovich, S. Sakovich, R. Turhan] considered a sixth-order nonlinear wave equation

$$u_{6x}+au_xu_{4x}+bu_{xx}u_{3x}+cu_x^2u_{xx}+du_{tt}+eu_{xxxt}+fu_xu_{xt}+gu_tu_{xx}=0, \tag{1}$$

where a, b, c, d, e, f and g are arbitrary constants. By introducing the Painlevé test for integrability of partial differential equation to Eq. (1), K^{2}S^{2}T found that there are four cases of relations between the parameters that pass the Painlevé test

$$(1)\ a=30,\quad b=30,\quad c=180,\quad d=-\frac{g^2}{180},\quad e=\frac{1}{12}(f+g),\quad f=g, \tag{2}$$

$$(2)\ a=15,\ b=\frac{75}{2},\ c=45,\ d=\frac{1}{90}(f^2-fg-2g^2),\quad e=\frac{f+g}{6},\quad f=g, \tag{3}$$

$$(3)\ a=18,\ b=36,\ c=72,\ d=-2e^2+\frac{ef}{2}-\frac{f^2}{36},\quad f=g=0, \tag{4}$$

$$(4)\ a=20,\ b=40,\ c=120,\ d=0,\ e=\frac{1}{12}(f+g),\quad f=2g. \tag{5}$$

Equation (1) with the parametric restrictions (2) and (3), respectively, corresponds, up to a scale transformation of variables, to the known integrable bidirectional version of the Sawada–Kotera–Caudrey–Dodd–Gibbons (SKCDG) equation[2–7]

$$5\partial_x^{-1}v_{tt}+5v_{xxt}-15vv_t-15v_x\partial_x^{-1}v_t-45v^2v_x+15v_xv_{xx}+15vv_{xxx}-v_{5x}=0, \tag{6}$$

and the Kaup–Kupershmidt (KK) equation[8–10]

$$5\partial_x^{-1}v_{tt}+5v_{xxt}-15vv_t-15v_x\partial_x^{-1}v_t-45v^2v_x+\frac{75}{2}v_xv_{xx}+15v_xv_{xxx}-v_{5x}=0, \tag{7}$$

which describe the propagation of waves in two opposite directions. Equations (6) and (7) possess Lax pairs due to Dye and Parker's[4] construction and fall into the class (1) after the potential transformation $v=u_x$. The nonlinear wave Eq. (1) with the restrictions (4) and (5) are, respectively, associated with the Drinfel'd–Sokolov–Satsuma–Hirota system (DSSH)[11–13]

$$u_{tt}-u_{xxxt}-2u_{6x}+18u_xu_{4x}+36u_{xx}u_{xxx}-36u_x^2u_{xx}=0, \tag{8}$$

and the KdV6 equation[14–17]

$$u_{6x}+20u_xu_{4x}+40u_{xx}u_{xxx}+120u_x^2u_{xx}+u_{xxxt}+4u_tu_{xx}+8u_xu_{xt}=0. \tag{9}$$

The auto-Bäcklund transformations of Eqs. (8) and (9) were derived in Refs. [1,11] by the method of truncated singular expansion. In fact, these four integrable

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systems (6)–(9) have been studied extensively in the literature.[18–21]

In this Letter, starting from the original Eq. (1) with arbitrary constants, we try to construct its multiple kink solutions in the Hirota form. We then determine the corresponding restrictions on the coefficients of this equation with multiple kink solutions. By making use of the velocity resonance mechanism to the multiple kink solutions, we derive the soliton molecule solution, as well as the breather-soliton molecule solution to the sixth-order nonlinear wave equation with certain coefficient constraints. Soliton molecules, which were first forecasted theoretically in the nonlinear Schrödinger–Ginzburg–Landau equation [22] and the coupled nonlinear Schrödinger equations,[23] have been reported to appear in several physical fields in the last few decades, such as optics,[24–30] Bose–Einstein condensates[31,32] and fluid physics.[33] More recently, soliton molecules have also been derived within the framework of the (1+1)-dimensional nonlinear systems[34] by Lou via bringing in the velocity resonance condition to its multiple soliton solutions. Based on Lou's creative work, researchers broadened this method to the (2+1)-dimensional fifth-order KdV equation, the Sharma–Tasso–Olver–Burgers equation, the few-cycle-pulse optical model, the (1+1)-dimensional Sawada–Kotera equation, the extended Boiti–Leon–Manna–Pempinelli equation, etc. to obtain their soliton molecule solutions.[35–41] Moreover, Li and his collaborators found that soliton molecules can also be received by the Darboux transformation combined with the velocity resonance restrictions.[42,43]

In this Letter, the multiple kink solutions in the Hirota form for the sixth-order nonlinear wave Eq. (1) with appropriate coefficient constraints are constructed. By performing the velocity resonance restriction on the multiple kink solutions, the soliton molecule solution and the breather-soliton molecule solution of the sixth-order nonlinear wave equation are then derived. The specific pictures of the soliton molecule solution and the breather-soliton interaction solution with fixed parameters are then plotted. As a case in point, the breather-soliton molecule solution of the integrable bidirectional SKCDG equation is gained because the coefficients of the bidirectional SKCDG equation are consistent with those of the sixth-order nonlinear wave equation with three-kink solution. Finally, a summary and some discussions are given.

Soliton Molecule and Breather-Soliton Molecule Solutions for the Sixth-Order Nonlinear Wave Equation. To find the target multiple kink solutions to Eq. (1), we start from the transformation

$$u = r(\ln F)_x, \tag{10}$$

where r is an undetermined constant. The substituting of u into Eq. (1) and collecting all the coefficients of the same power of F yield

$$\begin{aligned}
&F^{-7}[2rF_x^7(r^2c-12ra-6rb+360)]\\
&-F^{-6}[7rF_x^5F_{xx}(r^2c-12ra-6rb+360)]\\
&+F^{-5}r[(r^2c-20ra-14rb+840)F_x^4F_{xxx}\\
&+2(4r^2c-45ra-21rb+1260)F_x^3F_{xx}^2\\
&-2(rf+rg-12e)F_x^4F_t]+F^{-4}r[(5ar+2br\\
&-210)F_x^3F_{4x}+(2fr+2gr-24e)F_x^3F_{xt}\\
&+(30ar-2cr^2+24br-1260)F_x^2F_{xx}F_{3x}\\
&+(30ar-3cr^2+9br-630)F_xF_{xx}^3\\
&+(3fr+3gr-36e)F_x^2F_tF_{xx}]+F^{-3}r[(42-ar)F_x^2F_{5x}\\
&-(5ar+3br-210)F_xF_{xx}F_{4x}+(12e-fr)F_x^2F_{xxt}\\
&+(140-4br)F_xF_{3x}^2+(cr^2-10ar-3br+210)F_{xx}^2F_{3x}\\
&+(8e-gr)F_xF_tF_{3x}+(24\times10^{-2}fr-3gr)F_xF_{xx}F_{xt}\\
&+(6e-fr)F_tF_{xx}^2+2dF_xF_t^2]+F^{-2}r[-7F_xF_{6x}\\
&+(ar-21)F_{xx}F_{5x}-4eF_xF_{xxxt}+(br-35)F_{3x}F_{4x}\\
&-eF_tF_{4x}+(fr-6e)F_{xx}F_{xxt}+(gr-4e)F_{xt}F_{3x}\\
&-dF_xF_{tt}-2dF_tF_{xt}]\\
&+F^{-1}r[F_{7x}+eF_{4xt}+dF_{xtt}]=0.
\end{aligned} \tag{11}$$

Neglecting the coefficients of F^{-6} and F^{-7} in Eq. (11), we immediately receive the first coefficient restriction condition

$$c=\frac{(12a+6b)r-360}{r^2}. \tag{12}$$

For the sake of further solving Eq. (11), we suppose that it possesses the following multiple soliton solution in the Hirota form

$$F=\sum_{\substack{\mu_i=0,1\\1\le i\le N}}\exp\left(\sum_{1\le i<j\le N}\varphi(i,j)\mu_i\mu_j+\sum_{i=1}^{N}\mu_i\xi_i\right), \tag{13}$$

with $e^{\varphi(i,j)}\equiv a_{ij}$. Here $\xi_i=k_ix+w_it+\xi_{i0}(i=1,2,\ldots,N)$ are the traveling wave variables, and $k_i,w_i,\xi_{i0}(i=1,2,\ldots,N)$ are free constants.

When different N is taken into account, abundant multiple kink structures of Eq. (1) can be found. In the interest of simplicity, we just consider the cases $N=1,2,3$ in the next paragraphs.

Case 1. One-Kink Solution. In the case of $N=1$, the one-kink solution to Eq. (1) can be expressed as

$$u=\frac{k_1re^{k_1x+w_1t+\xi_{10}}}{1+e^{k_1x+w_1t+\xi_{10}}} \tag{14}$$

with the dispersion relation

$$w_1=\frac{k_1^3}{2d}(s-e),s=\delta\sqrt{e^2-4d}\equiv\pm\sqrt{e^2-4d} \tag{15}$$

and $r=\frac{12(es-e^2+10d)}{2(a+b)d+(f+g)(s-e)}$, and the related kink figures are displayed in Fig. 1. Substituting r into Eq. (12), it can be concluded that the only coefficient

restriction for Eq. (1) owing one-kink solution of the form of Eq. (14) is

$$c = \{[10ad + (2ae + be - 5f - 5g)(s - e)][2(a + b)d + (f + g)(s - e)]\} \cdot \{2[10d + (s - e)e]^2\}^{-1}. \quad (16)$$

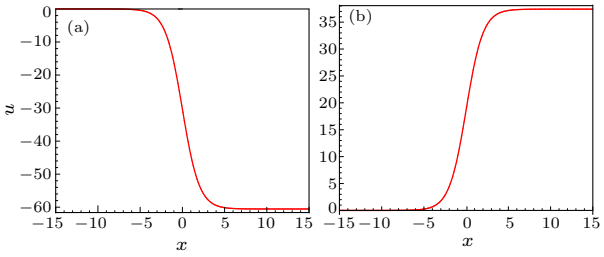

Fig. 1. The one-kink solution (14) for the parameters selected as $b = d = f = g = \delta = k_1 = 1$, $e = 3$, $\xi_{10} = 0$. (a) Kink solution with $a = 1$. (b) Anti-kink solution with $a = -1$.

Case 2. Soliton Molecule and Breather Solution. For $N = 2$, solution (10) with Eq. (13) becomes the two-kink solution

$$u = \{30[k_1 e^{k_1 x + w_1 t + \xi_{10}} + k_2 e^{k_2 x + w_2 t + \xi_{20}} + a_{12}(k_1 + k_2) e^{k_1 x + k_2 x + w_1 t + w_2 t + \xi_{10} + \xi_{20}}]\} \cdot \{a(1 + e^{k_1 x + w_1 t + \xi_{10}} + e^{k_2 x + w_2 t + \xi_{20}} + a_{12} e^{k_1 x + k_2 x + w_1 t + w_2 t + \xi_{10} + \xi_{20}})\}^{-1}, \quad (17)$$

where the dispersion relations are determined by

$$w_1 = \frac{k_1^3}{2d}(s - e), \quad w_2 = -\frac{k_2^3}{2d}(s + e), \quad (18)$$

and the wave parameter satisfies

$$a_{12} = \left\{\left[\left(k_1^2 - \frac{3}{2}k_1 k_2 + \frac{3}{2}k_2^2\right)s - \frac{3}{2}k_2 e(k_1 - k_2)\right] \cdot \left[\left(k_1^2 - k_1 k_2 + \frac{2}{3}k_2^2\right)s - k_1 e(k_1 - k_2)\right]\right\} \cdot \left\{\left[\left(k_1^2 + \frac{3}{2}k_1 k_2 + \frac{3}{2}k_2^2\right)s + \frac{3}{2}k_2 e(k_1 + k_2)\right] \cdot \left[\left(k_1^2 + k_1 k_2 + \frac{2}{3}k_2^2\right)s - k_1 e(k_1 + k_2)\right]\right\}^{-1}. \quad (19)$$

Under this circumstance, the restrictions on the coefficients of the original sixth-order nonlinear wave Eq. (1) are

$$a = b, \quad c = \frac{a^2}{5}, \quad f = g = \frac{ae}{5}, \quad (20)$$

and b, d and e are arbitrary constants. Figure 2(a) displays the kink-kink interaction structure (17). Figure 2(b) corresponds to the profile for the potential field $v = u_x$, where the parameters are chosen as

$$a = d = k_2 = \delta = 1, \ k_1 = 0.5, \ e = 4, \ \xi_{10} = \xi_{20} = 0. \quad (21)$$

By introducing the velocity resonance mechanism

$$\frac{k_1}{k_2} = \frac{w_1}{w_2} = \frac{\frac{k_1^3}{2d}(e - s)}{\frac{k_2^3}{2d}(e + s)}, \quad (k_1 \neq \pm k_2), \quad (22)$$

into the two-kink solution (17), which means that the two kinks of this solution travel in the same velocity, two kinks are then bounded to form a kink molecule. Solving the resonance restriction Eq. (22), we have the relation for k_1 and k_2,

$$k_2 = \delta k_1 \sqrt{\frac{e - s}{e + s}}. \quad (23)$$

Equation (23) implies that the kink molecule solution exists only for $\frac{e-s}{e+s} > 0$. If we set $\delta = 1$, $d = 2$, $e = 4$, which ensure that $\frac{e-s}{e+s} = \frac{4-\sqrt{8}}{4+\sqrt{8}} > 0$, and $a = 1$, $k_1 = 0.5$, $\xi_{10} = 0$, $\xi_{20} = 10$, the three-dimensional profile of the kink molecule is shown in Fig. 3(a), and the corresponding soliton molecule structure with the same parameters for the potential field v is depicted in Fig. 3(b).

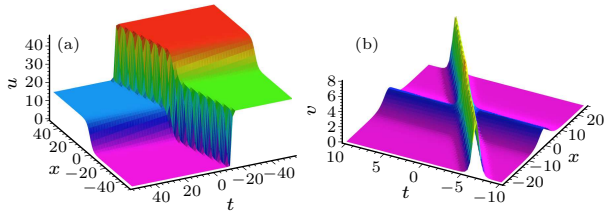

Fig. 2. The interaction of (a) two kinks for u and (b) two solitons for potential field v with parameters selected as in (21).

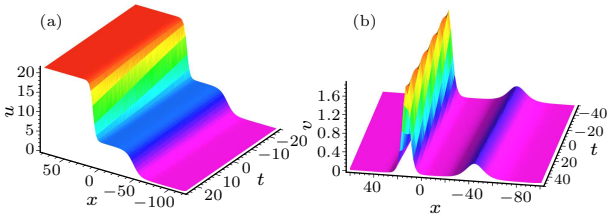

Fig. 3. The three-dimensional profile of the kink-kink molecule structure with parameters selection $a = \delta = 1$, $d = 2$, $e = 4$, $k_1 = 0.5$, $\xi_{10} = 0$, $\xi_{20} = 10$. (a) Kink molecule u. (b) Soliton molecule v.

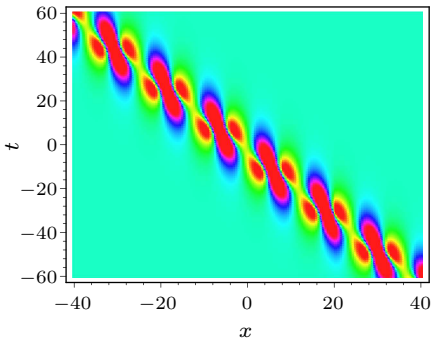

Fig. 4. The density plot of breather solution to Eq. (1) with the coefficient restrictions (20), where the parameters are selected as in (24).

In fact, the multiple kink solution (10) with Eq. (13) includes not only the usual soliton but also a breather, which can be obtained by paring solitons via taking one wave number as a complex conjugate of another, say, $k_i = k_j^*$. In the current case, we focus on the possible existence on breather solution to Eq. (1) by taking $k_1 = k_2^*$. Figure 4 displays the density map of the breather solution of v by choosing the parameters as

$$a = d = \delta = 1,\ e = 2,\ k_1 = k_2^* = 0.3 + 0.5i,$$
$$\xi_{10} = \xi_{20}^* = 0.1 + 0.2i. \tag{24}$$

Case 3. Breather-Soliton Molecule Solution. For further studying, we consider the three-kink solution to Eq. (1), which can be expressed as

$$\begin{aligned} u = \frac{30}{a}\Big[&k_1 e^{\xi_1} + k_2 e^{\xi_2} + k_3 e^{\xi_3} + a_{12}(k_1 + k_2)e^{\xi_1+\xi_2} \\ &+ a_{13}(k_1 + k_3)e^{\xi_1+\xi_3} + a_{23}(k_2 + k_3)e^{\xi_2+\xi_3} \\ &+ a_{123}(k_1 + k_2 + k_3)e^{\xi_1+\xi_2+\xi_3}\Big] \\ &\cdot \Big[1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + a_{12}e^{\xi_1+\xi_2} + a_{13}e^{\xi_1+\xi_3} \\ &+ a_{23}e^{\xi_2+\xi_3} + a_{123}e^{\xi_1+\xi_2+\xi_3}\Big]^{-1}, \end{aligned} \tag{25}$$

where

$$\begin{aligned} a_{12} =& \{\sqrt{5}\delta(k_1 + k_2)(k_1 - k_2)(k_1^2 - k_1k_2 + k_2^2) \\ &- (k_1^2 + k_2^2)(3k_1^2 - 5k_1k_2 + 3k_2^2)\} \\ &\cdot \{\sqrt{5}\delta(k_1 + k_2)(k_1 - k_2)(k_1^2 + k_1k_2 + k_2^2) \\ &- (k_1^2 + k_2^2)(3k_1^2 + 5k_1k_2 + 3k_2^2)\}^{-1}, \\ a_{13} =& \{(k_1 - k_3)^2[\sqrt{5}\delta(k_1^2 + k_1k_3 + k_3^2) \\ &- 3k_1^2 - k_1k_3 - 3k_3^2]\} \\ &\cdot \{(k_1 + k_3)^2[\sqrt{5}\delta(k_1^2 - k_1k_3 + k_3^2) \\ &- 3k_1^2 + k_1k_3 - 3k_3^2]\}^{-1}, \\ a_{23} =& \{\sqrt{5}\delta(k_2 - k_3)(k_2 + k_3)(k_2^2 - k_2k_3 + k_3^2) \\ &+ (k_2^2 + k_3^2)(3k_2^2 - 5k_2k_3 + 3k_3^2)\} \\ &\cdot \{\sqrt{5}\delta(k_2 - k_3)(k_2 + k_3)(k_2^2 + k_2k_3 + k_3^2) \\ &+ (k_2^2 + k_3^2)(3k_2^2 + 5k_2k_3 + 3k_3^2)\}^{-1}, \\ a_{123} =& a_{12}a_{13}a_{23}. \end{aligned} \tag{26}$$

Here the coefficient restrictions for Eq. (1) having the three-kink solution are

$$a = b,\quad c = \frac{a^2}{5},\quad d = -\frac{e^2}{5},\quad f = g = \frac{ae}{5}, \tag{27}$$

where coefficients a and e are arbitrary constants.

Figure 5(a) displays the interaction among three kinks (25) with (26), and the corresponding three-dimensional map for potential field v is given in Fig. 5(b) by fixing the parameters at

$$a = k_1 = \delta = 1,\quad e = 3,\quad k_2 = 1.5,\quad k_3 = 2,$$
$$\xi_{10} = \xi_{20} = \xi_{30} = 0. \tag{28}$$

Unfortunately, the coefficient restrictions (27) do not allow us to search for the velocity resonance of solitons for $N \geq 3$ because there is no real solution for the velocity resonance conditions

$$\frac{w_1}{k_1} = \frac{w_2}{k_2} = \frac{w_3}{k_3}. \tag{29}$$

What is more, we cannot even provide the interaction solution between one two-soliton molecule and one usual soliton, as the conditions (27) for the existence of the three-kink solution contain $d = -\frac{e^2}{5}$. This strict relation between d and e results in $\frac{e-s}{e+s} < 0$, which does not meet the requirement for the existence of the two-soliton molecule shown in Eq. (23).

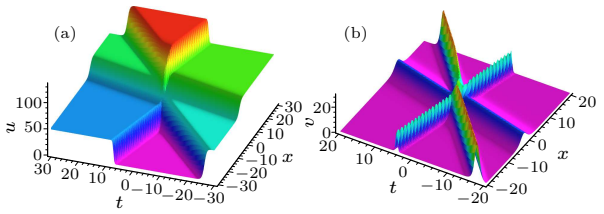

Fig. 5. The three-dimensional picture of interactions of (a) three-kink solution u and (b) three-soliton solution v with parameters selected as in (28).

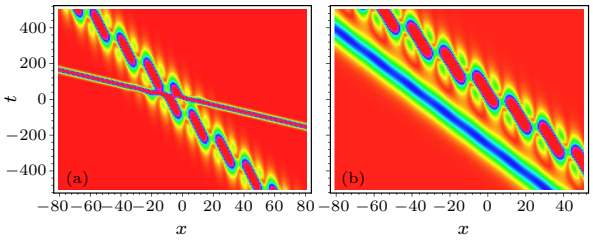

Fig. 6. (a) The density plot of the interaction between a breather and a soliton with parameters (30). (b) The density figure of the breather-soliton molecule with parameters (32).

Similar to the situation stated above, the three-kink solution (25) may also describe the interaction between one breather and one soliton as soon as the complex conjugates wave numbers are chosen. The density plot of the elastic collision between a breather and a soliton is shown in Fig. 6(a), where the parameters are given by

$$a = \delta = 1,\quad e = 3,\quad k_1 = k_3^* = 0.2 + 0.4i,\quad k_2 = 0.5,$$
$$\xi_{10} = \xi_{30}^* = 0.1 + 0.2i,\quad \xi_{20} = 0. \tag{30}$$

Furthermore, if one employs the velocity resonance mechanism to ensure that the velocity of the breather is resonant with the soliton, i.e.,

$$\begin{aligned} v_{\text{soliton}} &= \frac{k_2^2(3\sqrt{5}\delta + 5)}{2e} = v_{\text{breather}} \\ &= \frac{(3\sqrt{5}\delta - 5)(3k_{1i}^2 - k_{1r}^2)}{2e} \end{aligned} \tag{31}$$

with $k_1 = k_{1r} + ik_{1i}$, the three-kink solution (25) may depict a breather-soliton molecule. The density plot of a special breather-soliton molecule is displayed in Fig. 6(b) with the wave parameters selected as

$$a = \delta = 1, \quad e = 3, \quad k_1 = k_3^* = 0.2 + 0.4i,$$
$$k_2 = \frac{2\sqrt{55}}{25 + 15\sqrt{5}}, \ \xi_{10} = \xi_{30}^* = 0.1 + 0.2i, \ \xi_{20} = 8. \quad (32)$$

Breather-Soliton Molecule Solution for the Bidirectional SKCDG Equation. Comparing the restrictions (27) on the coefficients of the sixth-order nonlinear wave Eq. (1) for three-kink solution with four Painlevé integrable constraints (2)–(5) of the same equation, it can be found that only the case (1) $a = 30$, $b = 30$, $c = 180$, $d = -\frac{g^2}{180}$, $e = \frac{1}{12}(f + g)$, $f = g$ accords with the parametric restrictions (27). Particularly, if $g = -6$ is picked, Eq. (1) turns into

$$u_{6x} + 30u_x u_{4x} + 30u_{xx}u_{3x} + 180u_x^2 u_{xx} - \frac{1}{5}u_{tt} + u_{xxxt} + 6u_x u_{xt} + 6u_t u_{xx} = 0, \quad (33)$$

which is equivalent to the integrable bidirectional SKCDG equation mentioned above after transformation

$$v = -\frac{\partial}{\partial x}u\Big(\frac{1}{2}x, \frac{1}{40}t\Big). \quad (34)$$

Adopting the three-kink solution of Eq. (1) of the form (25) with (26) and the considering of transformation (34) yields the three-soliton solution for the bidirectional SKCDG equation. For simplicity, here we omit the tedious formula of the three-soliton solution. Figure 7(a) displays the density plot of three-soliton solution of the bidirectional SKCDG equation with the parameters fixed at

$$a = 30, \quad e = -1, \quad k_1 = \delta = 1, \quad k_2 = 1.2, \quad k_3 = 1.8, \quad \xi_{10} = \xi_{20} = \xi_{30} = 0. \quad (35)$$

Figure 7(b) shows the density map of the interaction between a breather and a soliton for Eq. (6) with the parameter selections

$$a = 30, \ e = -1, \ \delta = 1, \ k_1 = k_3^* = 1 + 1.2i, \ k_2 = 1.3, \ \xi_{10} = \xi_{30}^* = 1 - i, \ \xi_{20} = 0. \quad (36)$$

If one takes into consideration of the velocity resonant condition

$$v_{\text{soliton}} = \frac{k_2^2(3\sqrt{5}\delta + 5)}{40e} = v_{\text{breather}} = \frac{(3\sqrt{5}\delta - 5)(3k_{1i}^2 - k_{1r}^2)}{40e} \quad (37)$$

on the breather and soliton interaction solution, the breather-soliton molecule structure for the bidirectional SKCDG equation can be obtained. For example, once the wave parameters are selected as

$$a = 30, \quad e = -1, \quad \delta = 1, \quad k_1 = k_3^* = 0.8 + 1.5i, \quad k_2 = \frac{\sqrt{611}}{15 + 5\sqrt{5}}, \quad \xi_{10} = \xi_{30}^* = 1 - i, \quad \xi_{20} = 0, \quad (38)$$

the corresponding density plot of the breather-soliton molecule can be found in Fig. 7(c).

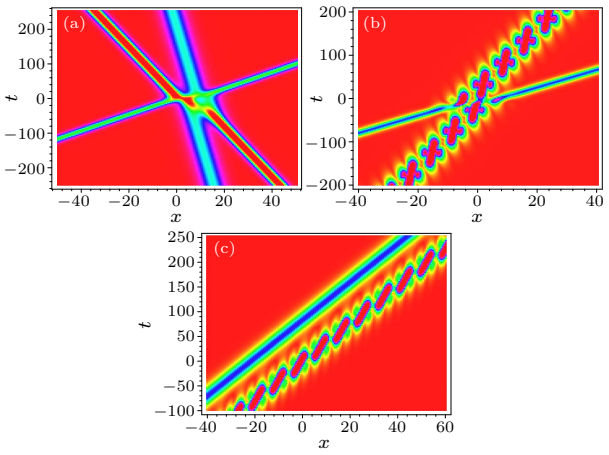

Fig. 7. The density plots of (a) the three-kink solution, (b) the breather and soliton interaction solution and (c) the breather-soliton molecule solution of the bidirectional SKCDG equation with parameters (35), (36) and (38), respectively.

Summary and Discussion. We have derived the multiple kink solutions relating to the Hirota bilinear form for the general sixth-order nonlinear equation with some particular restrictions on the six coefficients a, b, c, d, e, f and g. To vividly show the multiple kink solutions, the one-kink solution, the two-kink solution and the three-kink solution with certain wave parameters are illustrated in Figs. 1, 2(a) and 5(a). On the basis of the multiple kink solutions of the nonlinear wave Eq. (1), the breather solution and the breather-soliton interaction solution can also been acquired by paring solitons via taking one wave number as a complex conjugate of another. The density profiles of the breather solution and the soliton-breather interaction solution have been depicted in Figs. 4 and 6(a). By bringing in the velocity resonance restriction to the two-kink solution (17) and the three-kink solution (25), we also obtain the soliton molecule solution and the breather-soliton molecule solution to Eq. (1), whose three-dimensional plot and the density map can be seen in Figs. 3(b) and 6(b). By contrasting the Painlevé integrable constraints (2)–(5) of the sixth-order nonlinear wave equation with the parametric restrictions (27) for the three-kink solution, it is found that the coefficients of the bidirectional SKCDG equation meet the parameter constraints (27). For further studying, we give the three-soliton solution, the interaction solution between a breather and a soli-

ton and the breather-soliton molecule solution for the bidirectional SKCDG equation, and demonstrate the density plots of these solutions in Fig. 7. Since the bidirectional SKCDG equation has been applied to describe the propagation of waves in a fluid, we hope that the results obtained here may raise the possibility of relative experiments and potential applications. In consideration of the widely applications of soliton molecules in fluids, optics and plasmas, more studies regarding this type of traveling wave solutions of nonlinear partial differential equations in mathematical physics will be reported in our future research work.

References

[1] Karasu-Kalkanlı A, Karasu A, Sakovich A, Sakovich S and Turhan R 2008 *J. Math. Phys.* **49** 073516
[2] Sawada K and Kotera T 1974 *Prog. Theor. Phys.* **51** 1355
[3] Caudrey P J, Dodd R K and Gibbon J D 1976 *Proc. R. Soc. Lond. A* **351** 407
[4] Dye J M and Parker A 2001 *J. Math. Phys.* **42** 2567
[5] He J S, Cheng Y and Römer R A 2006 *J. High Energy Phys.* 2006(2006) 103
[6] Ma Y L and Geng X G 2012 *Appl. Math. Comput.* **218** 6963
[7] Lai X J and Cai X O 2010 *Z. Naturforsch. A* **65** 658
[8] Yang Y Q and Chen Y 2011 *Commun. Theor. Phys.* **55** 25
[9] Geng X G, Wu L H and He G L 2013 *J. Nonlinear Sci.* **23** 527
[10] Han P and Lou S Y 1993 *Chin. Phys. Lett.* **10** 257
[11] Karasu-Kalkanlı A and Yu S S 2001 *J. Phys. A* **34** 7355
[12] Heris J M and Lakestani M 2014 *Int. Sch. Res. Not.* **2014** **840689**
[13] Huang L L and Chen Y 2017 *Appl. Math. Lett.* **64** 177
[14] Kupershmidt B A 2008 *Phys. Lett. A* **372** 2634
[15] Mohammad M, Mostafa E and Anjan B 2015 *Nonlinear Dyn.* **80** 387
[16] Zhao Z L, Chen Y and Han B 2017 *Optik* **140** 10
[17] Geng X G and Xue B 2012 *Appl. Math. Comput.* **219** 3504
[18] Verhoeven C and Musette M 2003 *J. Phys. A* **36** L133
[19] Verhoeven C and Musette M 2001 *J. Phys. A* **34** L721
[20] Dye J M and Parker A 2002 *J. Math. Phys.* **43** 4921
[21] Verhoeven C 2004 *J. Phys. A* **37** 10625
[22] Malomed B A 1991 *Phys. Rev. A* **44** 6954
[23] Malomed B A 1992 *Phys. Rev. A* **45** R8321-R8323
[24] Tang D Y, Man W S, Tam H Y and Drummond P D 2001 *Phys. Rev. A* **64** 033814
[25] Tsatourian V, Sergeyev S V, Mou C B, Rozhin A, Mikhailov V, Rabin B, Westbrook P S and Turitsyn S K 2013 *Sci. Rep.* **3** srep03154
[26] Peng J, Zhan L, Luo S and Shen Q S 2013 *IEEE Photon. Technol. Lett.* **25** 948
[27] Wang P, Bao C Y, Fu B, Xiao X S, Grelu P and Yang C X 2016 *Opt. Lett.* **41** 2254
[28] Herink G, Kurtz F, Jalali B, Solli D R and Ropers C 2017 *Science* **356** 50
[29] Liu X M, Yao X K and Cui Y D 2018 *Phys. Rev. Lett.* **121** 023905
[30] Peng J, Boscolo S, Zhao Z and Zeng H 2019 *Sci. Adv.* **5** eaax1110
[31] Łakomy K, Nath R and Santos L 2012 *Phys. Rev. A* **85** 033618
[32] Łakomy K, Nath R and Santos L 2012 *Phys. Rev. A* **86** 013610
[33] Marin F and Ezersky A B 2008 *Eur. J. Mech. B* **27** 251
[34] Lou S Y 2019 arXiv:1909.03399 [nlin.SI]
[35] Xu D H and Lou S Y 2020 *Acta Phys. Sin.* **69** 014208 (in Chinese)
[36] Zhang Z, Yang S X and Li B 2019 *Chin. Phys. Lett.* **36** 120501
[37] Yan Z W and Lou S Y 2020 *Appl. Math. Lett.* **104** 106271
[38] Cheng X P, Lou S Y and Yang Y Q 2020 *Results Phys.* **18** 103184
[39] Jia M, Lin J and Lou S Y 2020 *Nonlinear Dyn.* **100** 3745
[40] Wang W, Yao R X and Lou S Y 2020 *Chin. Phys. Lett.* **37** 100501
[41] Ren B 2021 *Commun. Theor. Phys.* **73** 035003
[42] Zhang Z, Yang S X and Li B 2020 *Appl. Math. Lett.* **103** 106168
[43] Zhang Z, Yang X Y and Li B 2020 *Nonlinear Dyn.* **100** 1551