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Space-Curved Resonant Line Solitons in a Generalized (2 + 1)-Dimensional Fifth-Order KdV System

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On the basis of N-soliton solutions, space-curved resonant line solitons are derived via a new constraint proposed here, for a generalized $(2+1)$-dimensional fifth-order KdV system. The dynamic properties of these new resonant line solitons are studied in detail. We then discuss the interaction between a resonance line soliton and a lump wave in greater detail. Our results highlight the distinctions between the generalized (2+1)-dimensional fifth-order KdV system and the classical type.

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It is well-known that bilinear methods[1] are widely used to study soliton solutions for integrable systems. By means of various parameter constraints on the N-solitons, we can derive different types of solitons in the bound state, such as breathers and soliton molecules.[2−7] In particular, use of the bilinear method and the long wave limit method allows us to obtain lump waves[8−10] in some integrable systems.

The (2 + 1)-dimensional generalized fifth-order KdV system is a new type of integrable system. Compared with the (2 + 1)-dimensional fifth-order KdV system,[11,12] research relating to the new system is currently sparse. It is well-known that line solitons and Y-type resonant solitons are very common in most (2+1)-dimensional integrable systems. However, an investigation of this system brings to light an interesting phenomenon, whereby the resonant line solitons are gradually curved in space, via a mechanism similar to soliton fission and fusion.[13−16] Soliton fission and fusion phenomena have been studied by some scholars, whereas gradually curved resonant line solitons are rarely mentioned in the literature. In this work, we refer to them as space-curved resonant line solitons.

The space-curved resonant line solitons studied in this Letter are special cases of soliton fission and fusion phenomena. There are many kinds of waves in nature, including curved waves. Space-curved resonant line solitons can be used to describe their shapes, and as such, the study of space-curved resonant line solitons is of great physical significance.

The (2 + 1)-dimensional generalized fifth-order KdV equation is expressed as follows:

$$u_{xxxxx} + 15u_xu_{xx} + 15uu_{xxx} + 45u^2u_x + u_t + u_y = 0. \tag{1}$$

Equation (1) represents the general motion of long waves in shallow water, under a gravitational field, and in a two-dimensional nonlinear lattice. Its lump solutions and interaction solutions have been investigated by Liu *et al.* in Refs. [8,9]. Substituting $u(x,y,t) = 2(\ln f)_{xx}$ into Eq. (1) yields the following Hirota's bilinear form:

$$\mathbb{T}(f,f) = 0, \tag{2}$$

with

$$\mathbb{T}(f,g) = (D_x^6 + D_xD_t + D_xD_y)(f\cdot g) = 0, \tag{3}$$

where the operator D is Hirota's bilinear differential operator, as defined by

$$D_x^m D_t^n f\cdot g = \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big)^m \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\Big)^n f(x,y,t)\cdot g(x',y',t')\Big|_{x'=x,y'=y,t'=t}. \tag{4}$$

The N-soliton solution to Eq. (1) is obtained via the bilinear method:

$$u(x,y,t) = 2(\ln f)_{xx}, \tag{5}$$

where

$$f = \sum_{\mu=0,1} \exp\Big(\sum_{j<s}^{N} \mu_j\mu_s A_{js} + \sum_{j=1}^{N} \mu_j\xi_j\Big), \tag{6}$$

with

$$\xi_j = k_jx + w_jt + p_jy + \phi_j, k_j^5 + p_j + w_j = 0, \tag{7}$$

and

$$e^{A_{js}} = -[(w_j - w_s)(k_j - k_s) + (k_j - k_s)(p_j - p_s) + (k_j - k_s)^6]/[(w_j + w_s)(k_j + k_s) + (k_j + k_s)(p_j + p_s) + (k_j + k_s)^6]. \tag{8}$$

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In this study, we presume that $\exp(x) = 0$ is true if, and only if, $x = \ln(0)$. Based on our additional regulations, we can deduce the following results: $\exp(x+\ln(0)) = 0\exp(x) = 0$. The purpose of the supplement is to get rid of partial terms from Eq. (6). If all $A_{js} = \ln(0)$, or part $A_{js} = \ln(0)$, the space-curved resonant line solitons for the relevant equation can be obtained.

Proposition 1: On the basis of the N-soliton solution, the nonlinear superposition of M-space-curved resonant line solitons and L-space-curved resonant line solitons can be derived by means of the following constraints:

$$e^{A_{js}} = 0, \quad (1 \le j < s \le M, \quad N - L < j < s \le N, \quad N = M + L), \tag{9}$$

specifically,

$$k_j = k_s, \quad (1 \le j < s \le M, \quad N - L < j < s \le N, \quad N = M + L).$$

In Proposition 1, the M-space-curved resonant line solitons are obtained by setting $L = 0$, and regardless of the value of M, the wave height of an M-space-curved resonant line soliton is always $\frac{k^2}{2}$. This characteristic differs from the classical fifth-order KdV.[17] Curved waves in shallow water have various shapes and different curvatures, and the dynamic properties of space-curved resonant line solitons may be studied in detail via the selection of the appropriate parameters in Proposition 1. In order to study the properties of resonant line soliton in greater depth, we set $L = 0, M = 2$ in Eq. (9), and define the following parameter values: $k_1 = 1$, $k_2 = 1$, $p_1 = 1$. Figure 1(a) shows a 2-space-curved resonant line soliton, whose mathematical expression is as follows:

$$u = \frac{2[e^{(-p_2-1)t+x+p_2y} + e^{-2t+x+y}]}{1 + e^{t(-p_2-1)+x+p_2y} + e^{-2t+x+y}} - \frac{2[e^{(-p_2-1)t+x+p_2y} + e^{-2t+x+y}]^2}{[1 + e^{t(-p_2-1)+x+p_2y} + e^{-2t+x+y}]^2}. \tag{10}$$

The next step is to study the crest trajectory and wave width of the novel solution given above.

The trajectories of resonant line solitons are represented by the trajectories of wave crests in this work, together with the trajectory equation (10).

$$x = p_2t - p_2y + t - \ln[1 + e^{(y+p_2t-p_2y-t)}] \tag{11}$$

can be obtained by solving $u - \frac{1}{2} = 0$.

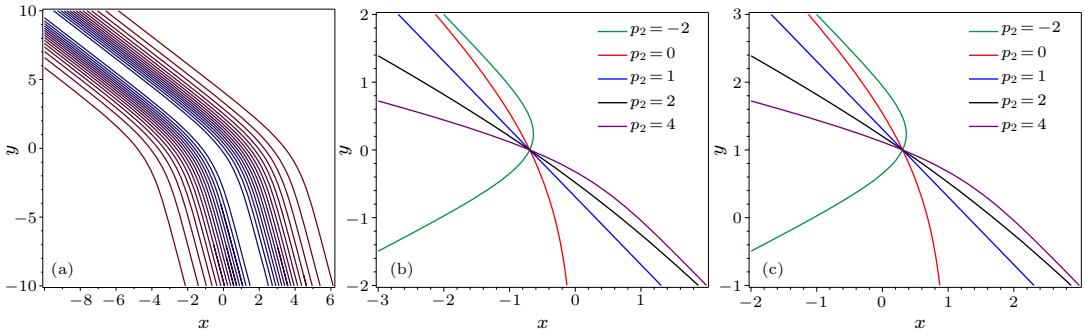

Fig. 1. (a) The 2-space-curved resonant line soliton, with the parameters $N = 2$, $k_1 = 1$, $k_2 = 1$, $p_1 = 1$, $p_2 = \frac{1}{5}$ at $t = 0$. [(b), (c)] Trajectories of Eq. (10) with different p_2 at (b) $t = 0$ and (c) $t = 1$.

To demonstrate that the resonant line soliton described by Eq. (10) is slowly curved in space, we give its corresponding curvature:

$$K = \left| \frac{y_{xx}}{(1 + y_x^2)^{\frac{3}{2}}} \right|, \tag{12}$$

with

$$y_x = -\frac{e^{tp_2-yp_2-t+y} + 1}{e^{tp_2-yp_2-t+y} + p_2},$$

$$y_{xx} = -\Big[(e^{(p_2-1)(t-y)}{p_2}^2 - 2\,p_2 e^{(p_2-1)(t-y)} + {p_2}^2 + e^{(p_2-1)(t-y)} - 2\,p_2 + 1)e^{(p_2-1)(t-y)}\Big] \cdot \Big[e^{3\,(p_2-1)(t-y)} + 3e^{2\,(p_2-1)(t-y)}p_2 + 3e^{(p_2-1)(t-y)}{p_2}^2 + {p_2}^3\Big]^{-1}.$$

The curvature with different p_2 is as follows:

(I) $p_2 = -2$,
$K_1 = \Big| -\big[9(|e^{3y} - 2|)^3(e^{3y} + 1)e^{3y}\big] \cdot \big[(2\,e^{6y} - 2\,e^{3y} + 5)^{3/2}(e^{9y} - 6\,e^{6y} + 12\,e^{3y} - 8)\big]^{-1}\Big|$.

(II) $p_2 = 0$, $K_2 = \Big| -\dfrac{(1 + e^y)e^{-2y}}{(2 + 2\,e^{-y} + e^{-2y})^{3/2}}\Big|$.

(III) $p_2 = 1$, $K_3 = 0$.

(IV) $p_2 = 2$, $K_4 = \Big| -\dfrac{(1 + e^y)e^{-1y}}{(5 + 6\,e^{-y} + 2\,e^{-2y})^{3/2}}\Big|$.

(V) $p_2 = 4$, $K_5 = \Big| -\dfrac{9(1 + e^{-3y})e^{-3y}}{(17 + 2\,e^{-6y} + 10\,e^{-3y})^{3/2}}\Big|$.

Figures 1(b) and 1(c) show the soliton trajectories of the 2-space-curved resonant line soliton under different values of p_2, leading to the following conclusion:

firstly, it will pass through a fixed point $[t-\ln(2), t]$ over time. Secondly, $p_2 = p_1$ is a critical condition, in which the curvature of the trajectory is zero, and the trajectory is arranged counterclockwise at the center of the fixed point with an increase in p_2. Lastly, the position of the fixed point shifts with time, but the shape of soliton remains unchanged, indicating the translational invariance of the space-curved resonant line soliton.

We consider that the width at half of the wave's height is the width of the solitary wave in this work; as such, the solitary wave has two boundary curves at $u = \frac{k^2}{4}$, and the distance between them is the width of a solitary wave. When p_2 is taken for different values, different wave widths can be obtained.

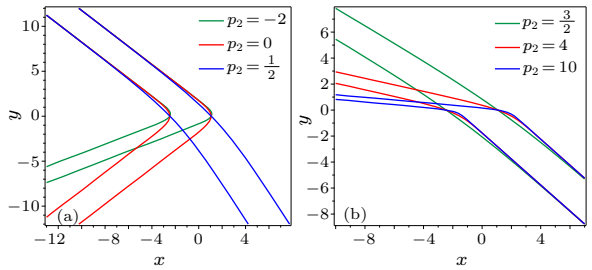

Fig. 2. Wave widths of space-curved resonant line solitons described by Eq. (10) with different P_2 values: (a) $p_2 = -2$, $p_2 = 0$, $p_2 = \frac{1}{2}$ at $t = 0$; (b) $p_2 = \frac{3}{2}$, $p_2 = 4$, $p_2 = 10$ at $t = 0$.

The two scenarios, i.e., $p_2 > 1$ and $p_2 < 1$. Figures 2(a) and 2(b) lead to the following results: when $p_2 < 1$, the trajectories of space-curved resonant line solitons almost coincide at $y \to \infty$, and the wave's width increases counter-clockwise with the increase in p_2 at $y \to -\infty$. When $p_2 > 1$, the trajectories of space-curved resonant line solitons almost coincide at $y \to -\infty$, and the wave's width decreases counterclockwise with the increase in p_2 at $y \to \infty$.

The phenomena of soliton fission and fusion can be obtained where $A_{js} = \ln(0)$ in many integrable systems.[17] In this equation, however, the phenomenon of soliton bending can be obtained. Therefore, it is assumed that similar types of space-curved resonant line soliton can be obtained in other (2+1)-dimensional integrable systems where $A_{js} = \ln(0)$ is equivalent to $k_j = k_s$.

Based on Proposition 1, where $M = 2$ and $L = 2$, an expression describing the interaction between 2-space-curved resonant line solitons takes the following form:

$$u(x, y, t) = 2(\ln f)_{xx}, \tag{13}$$

with

$$\begin{aligned} f =& 1 + e^{\xi_4} + e^{\xi_3} + e^{\xi_2} + e^{\xi_1} + e^{\xi_2+\xi_4+A_{24}} \\ &+ e^{\xi_2+\xi_3+A_{23}} + e^{\xi_1+\xi_4+A_{14}} + e^{\xi_1+\xi_3+A_{13}}, \end{aligned}$$

where the relevant parameters ξ_j, A_{js} are given by Eqs. (7) and (8).

By selecting the appropriate parameters, Eq. (13) can describe the three essential types of interaction between 2-space-curved resonant line solitons. Figures 3(a), 3(b) and 3(c) vividly and visually illustrate these types of interaction. It is understood that the interaction between line waves is elastic; thus, the interactions depicted in Fig. 3 are also elastic. More interesting phenomena can be obtained via the superposition of space-curved resonant line solitons, which fall outside the scope of this paper. It is certainly the case that the interactions between M-space-curved resonant line solitons and an L-order breather can also be obtained, if Eq. (6) satisfies the following terms:

$$\begin{aligned} &e^{A_{js}} = 0, \quad \xi_{M+2l-1} = {\xi_{M+2l}}^*, \\ &(1 \le j < s \le M,\ 1 \le l \le L,\ N = M + 2L). \end{aligned} \tag{14}$$

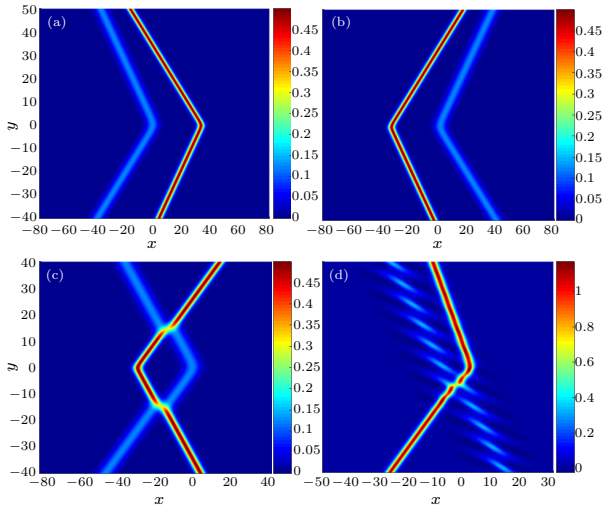

Fig. 3. Four different types of interactions at $t = 0$: (a) interaction between a resonant soliton, and a resonant soliton as described by Eq. (13), where $k_1 = 1$, $k_2 = 1$, $k_3 = \frac{1}{2}$, $k_4 = \frac{1}{2}$, $p_1 = \frac{6}{8}$, $p_2 = 1$, $p_3 = \frac{6}{16}$, $p_4 = \frac{1}{2}$, $\phi_1 = -30$, $\phi_2 = -30$, $\phi_3 = 0$, $\phi_4 = 0$; (b) interaction between a resonant soliton and a resonant soliton as described by Eq. (13), where $k_1 = -1$, $k_2 = -1$, $k_3 = -\frac{1}{2}$, $k_4 = -\frac{1}{2}$, $p_1 = \frac{6}{8}$, $p_2 = 1$, $p_3 = \frac{6}{16}$, $p_4 = \frac{1}{2}$, $\phi_1 = -30$, $\phi_2 = -30$, $\phi_3 = 0$, $\phi_4 = 0$; (c) interaction between a resonant soliton and a resonant soliton, as described by Eq. (13), where $k_1 = -1$, $k_2 = -1$, $k_3 = \frac{1}{2}$, $k_4 = \frac{1}{2}$, $p_1 = \frac{6}{8}$, $p_2 = 1$, $p_3 = \frac{6}{16}$, $p_4 = \frac{1}{2}$, $\phi_1 = -30, \phi_2 = -30$, $\phi_3 = 0$, $\phi_4 = 0$; (d) nonlinear superposition between a space-curved resonant line soliton and a breather wave, as described by Eq. (6), where $k_1 = \frac{2}{7} + \frac{3}{7}i$, $k_2 = \frac{2}{7} - \frac{3}{7}i$, $p_1 = \frac{1}{8} + i$, $p_2 = \frac{1}{8} - i$, $k_3 = \frac{3}{2}$, $k_4 = \frac{3}{2}$, $p_3 = \frac{1}{2}$, $p_4 = -1$, $\phi_1 = 1$, $\phi_2 = 1$, $\phi_3 = 0$, $\phi_4 = 0$.

We can also obtain lump-type solutions based on the long wave limit method,[18,19] i.e., a type of rational function solution, localized in all directions in space. Lump solutions for a generalized $(2+1)$-dimensional fifth-order KdV system have been studied in the literature.[8,9] Here, we obtain interactions between lump waves and L-space-curved resonant line solitons under specific conditions.

The research[20] indicated that this interaction is elastic. We can establish the trajectory of a single lump in a hybrid of a lump and 2-space-curved resonant line solitons in the same way. Primarily, we use

our newly proposed constraints to systematically remove some terms in the N-soliton solution. Assuming that $\exp(x) = 0$ is true for partial terms, we then use the long wave limit method to obtain hybrid solutions between a first-order lump and a 2-space-curved resonant line soliton. The trajectory and height of the lump can also be obtained subsequently to calculation, leading us directly to the following conclusions.

Proposition 2: On the basis of the N-soliton solution, the nonlinear superposition of a first-order lump and 2-space-curved resonant line solitons can be derived via the following constraints:

$$k_1 = k_2^*, \quad k_1 = K_1\epsilon, \quad k_2 = iK_1\epsilon, \quad p_1 = p_2^* = P_1\epsilon,$$
$$\phi_1 = \phi_2 = \pi i, \quad k_3 = k_4 = k, \quad \epsilon \to 0, \tag{15}$$

where we may assume that $k > 0$. In addition, the trajectories of the lump wave controlled by the parameters, before and after interaction with the 2-space-curved resonant line solitons, are given as

$$\{x_{\text{inf}} = 0,\ y_{\text{inf}} = t\}, \quad \left\{x_{-\text{inf}} = \frac{6}{k},\ y_{-\text{inf}} = t\right\}. \tag{16}$$

The wave height, h, of the lump changes over time, i.e.,

$$h_{\text{inf}} = \frac{1 - k^2 e^{-k^5 t + \phi_3}}{9e^{-k^5 t + \phi_3}},$$
$$h_{-\text{inf}} = \frac{k^2(e^{\phi_3 - k^5 t + 6} - 1)}{9}, \quad k > 0. \tag{17}$$

The hybrid solutions described by Proposition 2 can be expressed as

$$u(x, y, t) = 2(\ln f)_{xx}, \tag{18}$$

where

$$f = k^2\theta_1\theta_2 + (k\theta_1 - 6K_1)(k\theta_2 - 6iK_1)(e^{\xi_3} + e^{\xi_4}),$$

and the relevant parameters are $\xi_j,\ a_{js},\ \theta_j$, as follows:

$$\xi_j = k_j x + w_j t + p_j y + \phi_j, \quad k_j^5 + p_j + w_j = 0,$$
$$\theta_1 = K_1 x - P_1 t + P_1 y, \quad \theta_2 = iK_1 x - P_2 t + P_2 y.$$

We find that the height of the lump increases with time, indicating that the solution is unstable. Given the parameter limits $k_1 = 1 - i,\ k_2 = 1 + i,\ p_1 = 1 + i,\ p_2 = 1 - i,\ k_3 = k_4 = 1,\ p_3 = 1,\ p_4 = -1/2,\ \phi_3 = \phi_4 = 0$, the trajectories of the lump wave, before and after interaction, are $\{x_{\text{inf}} = 0,\ y_{\text{inf}} = t\}$, and $\{x_{-\text{inf}} = 6,\ y_{-\text{inf}} = t\}$, respectively. Before and after interaction, the respective heights are $\{h_{-\text{inf}} = \frac{56e^{-t}+4}{72e^{-t}} - \frac{8}{9}\}$ and $\{h_{\text{inf}} = \frac{2e^{t+6}}{9} - \frac{1}{9}\}$. Subject to repeated verification, a novel stable hybrid solution for a lump and a space-curved resonant line soliton can be generalized via an appropriate modification of Eq. (18). The new solution is as follows:

$$u(x, y, t) = 2(\ln f)_{xx}, \tag{19}$$

where

$$f = k^2\theta_1\theta_2 + \delta + ((k\theta_1 - 6K_1)(k\theta_2 - 6iK_1) + \delta) \cdot (e^{\xi_3} + e^{\xi_4}), \quad \delta > 0.$$

The new solution satisfies the equation. Given that the bilinear derivative of the same linear exponential function is zero, and our prior knowledge that if $\mathbb{T}(f, f) = 0$, let $g^1 = \delta + \delta(e^{\xi_3} + e^{\xi_4})$, then $f_1 = f + g^1$, the proof can be expressed as follows:

$$\begin{aligned}
\mathbb{T}(f_1, f_1) &= \mathbb{T}(f, f) + \mathbb{T}(f, g^1) + \mathbb{T}(g^1, f) + \mathbb{T}(g^1, g^1) \\
&= (D_x^6 + D_xD_t + D_xD_y)(f \cdot g^1) \\
&\quad + (D_x^6 + D_xD_t + D_xD_y)(g^1 \cdot f) \\
&= 2(f_{6x}g^1 - 6f_{5x}g_x^1 + 15f_{4x}g_{2x}^1 - 20f_{3x}g_{3x}^1 \\
&\quad + 15f_{2x}g_{4x}^1 - 6f_xg_{5x}^1 + fg_{6x}^1 + f_{\text{xt}}g^1 + fg_{\text{xt}}^1 \\
&\quad - f_xg_t^1 - f_tg_x^1 + f_{xy}g^1 + fg_{xy}^1 \\
&\quad - f_xg_y^1 - f_yg_x^1) \\
&= 0.
\end{aligned}$$

where $\mathbb{T}$ is defined by Eq. (2).

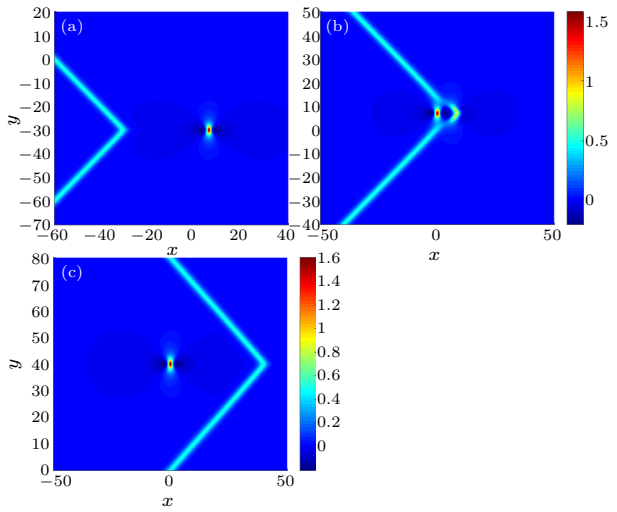

Fig. 4. (a) A nonlinear superposition described by Eq. (19) with $\{K_1 = 1 - i,\ K_2 = 1 + i,\ P_1 = 1 + i,\ P_2 = 1 - i,\ k_3 = 1,\ k_4 = 1,\ p_3 = 1,\ p_4 = -1/2,\ \phi_1 = i\pi,\ \phi_2 = i\pi,\ \phi_3 = 0,\ \phi_4 = 0\}$ at $t = -30$; (b) $\{K_1 = 1 - i,\ K_2 = 1 + i,\ P_1 = 1 + i,\ P_2 = 1 - i,\ k_3 = 1,\ k_4 = 1,\ p_3 = 1,\ p_4 = -1/2,\ \phi_1 = i\pi,\ \phi_2 = i\pi,\ \phi_3 = 0,\ \phi_4 = 0\}$ at $t = 7$; (c) $\{K_1 = 1 - i,\ K_2 = 1 + i,\ P_1 = 1 + i,\ P_2 = 1 - i,\ k_3 = 1,\ k_4 = 1,\ p_3 = 1,\ p_4 = -1/2,\ \phi_1 = i\pi,\ \phi_2 = i\pi,\ \phi_3 = 0,\ \phi_4 = 0\}$ at $t = 40$.

In order to illustrate this phenomenon clearly, we show in Fig. 4 that where $\delta = 5,\ k_1 = 1 - i,\ k_2 = 1 + i,\ p_1 = 1 + i,\ p_2 = 1 - i,\ k_3 = k_4 = 1,\ p_3 = 1,\ p_4 = -1/2,\ \phi_3 = \phi_4 = 0$, therefore, $f = [2t^2 - 4ty + 2y^2 + 2(x - 6) + 5](e^{-2t+x+y} + e^{x-y}) + 2t^2 - 4ty + 2y^2 + 2x^2 + 5$. The trajectories of the lump wave, before and after interaction, are $\{x_{\text{inf}} = 0,\ y_{\text{inf}} = t\}$, and $\{x_{-\text{inf}} = 6,\ y_{-\text{inf}} = t\}$, respectively, which are the same as given above; the heights before and after collision are, respectively, $\{h_{-\text{inf}} = \frac{40e^{2t} + 1892^{-t} - 2144}{(154 + 5e^t)^2}\}$ and $\{h_{\text{inf}} = \frac{160e^{-2t+12} + 1892e^{-t+6} - 536}{(10e^{-t+6} + 77)^2}\}$.

Table 1 verifies the accuracy of this new solution.

Table 1. Accuracy of this new solution.

t	x_{theorem}	y_{theorem}	h_{actual}	$h_{\text{actual}}/h_{\text{theorem}}$	u_x	u_y
−25.0	6.00	−25.00	1.6000	1.0000	0.000×10^{-10}	0.00
−20.0	6.00	−20.00	1.6000	1.0000	0.000×10^{-10}	0.00
−15.0	6.00	−15.00	1.5999	0.9999	1.300×10^{-8}	0.00
−20.0	6.00	−20.00	1.5999	0.9999	2.481×10^{-6}	0.00
−5.0	6.00	−5.00	1.5999	0.9999	3.680×10^{-4}	0.00
5.0	0.00	5.00	1.4443	0.9271	-4.399×10^{-1}	0.00
10.0	0.00	10.00	1.5989	0.9993	-3.993×10^{-3}	0.00
15.0	0.00	15.00	1.5999	0.9999	-2.696×10^{-5}	0.00
20.0	0.00	20.00	1.5999	0.9999	-1.817×10^{-7}	0.00
25.0	0.00	25.00	1.6000	0.9999	-1.224×10^{-9}	0.00

The results given in this work arise from the particularity of the equations studied; other integrable systems using the same method generally describe the phenomena of soliton fission and fusion. It is understood that split waves occur in nature; as such, bending waves also exist. We hope that our results will be proved to be valuable in the field of nonlinear science.

In conclusion, we have taken the generalized (2+1)-dimensional fifth-order KdV equation as an example, and systematically removed some terms from the N-soliton solution so as to obtain space-curved resonant line solitons. The properties of solitary wave trajectories have been examined, based on curvature and wave width, together with three essential interactions in Proposition 1. Combined with the long wave limits methods, the hybrid solutions of lump and space-curved resonant line solitons are constructed. However, the solution is extremely unstable, so we construct a new, more stable solution by adding a constant. In addition, we have conducted an in-depth study of the dynamic properties of the interaction between a first-order lump and a 2-space-curved resonant line soliton, and have clearly pointed out that their interaction is elastic. In addition, the method of deriving space-curved resonant line solitons proposed in this study could be extended to other integrable systems.

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