CHIN. PHYS. LETT. Vol. 38, No. 5 (2021) 051101

Finding Short-Range Parity-Time Phase-Transition Points with a Neural Network

Songju Lei(雷松炬)[1], Dong Bai(柏栋)[2*], Zhongzhou Ren(任中州)[2,3*], and Mengjiao Lyu(吕梦蛟)[4,5]

[1]*School of Physics, Nanjing University, Nanjing 210093, China*
[2]*School of Physics Science and Engineering, Tongji University, Shanghai 200092, China*
[3]*Key Laboratory of Advanced Micro-Structure Materials (Ministry of Education), Shanghai 200092, China*
[4]*College of Science, Nanjing University of Aeronautics and Astronautics (NUAA), Nanjing 210016, China*
[5]*Key Laboratory of Aerospace Information Materials and Physics (NUAA), MIIT, Nanjing 211106, China*

(Received 7 January 2021; accepted 11 March 2021; published online 2 May 2021)

The non-Hermitian PT-symmetric system can live in either unbroken or broken PT-symmetric phase. The separation point of the unbroken and broken PT-symmetric phases is called the PT-phase-transition point. Conventionally, given an arbitrary non-Hermitian PT-symmetric Hamiltonian, one has to solve the corresponding Schrödinger equation explicitly in order to determine which phase it is actually in. Here, we propose to use artificial neural network (ANN) to determine the PT-phase-transition points for non-Hermitian PT-symmetric systems with short-range potentials. The numerical results given by ANN agree well with the literature, which shows the reliability of our new method.

DOI: 10.1088/0256-307X/38/5/051101

Parity-time (PT) symmetric non-Hermitian Hamiltonians have nontrivial physical properties.[1] They could have either real spectra or complex spectra coming in conjugate pairs. The non-Hermitian systems are said to live in the unbroken PT-symmetric phase in the case of real spectra, while they are said to live in the broken PT-symmetric phase in the case of complex conjugate spectra. Conventionally, one has to solve the Schrödinger equation to determine which phase the non-Hermitian Hamiltonian lives in. It is interesting to develop alternative methods to identify different phases.

In recent years, artificial intelligence has been extensively applied to physics research.[2−17] In this Letter, we design an artificial neural network (ANN) to predict whether a non-Hermitian PT-symmetric Hamiltonian is in unbroken or broken PT-symmetric phase without solving the corresponding Schrödinger equations explicitly. We assume the potential to be short-range. In other words, $V(x)$ vanishes as $|x| \to \infty$. For the parameterized non-Hermitian PT-symmetric Hamiltonians, we also use the trained ANN to determine the PT-phase transition point. This separates the unbroken and broken PT-symmetric phases and is an important physical quantity to characterize the parameterized non-Hermitian PT-symmetric Hamiltonians.

The rest of this article is organized as follows. First, the PT-symmetric non-Hermitian quantum mechanics is introduced. Then, the ANN and its training process are discussed briefly. Furthermore, we verify the reliability of ANN. Finally, a summary is given, as well as some short remarks on some future directions.

PT-Symmetric Quantum Mechanics. For a one-dimensional single-particle non-Hermitian Hamiltonian with a complex potential $V(x) = u(x) + iw(x)$, it is said to have the PT-symmetry if it satisfies[1]

$$u(x) = u(-x), \quad w(-x) = -w(x). \tag{1}$$

An early example of the non-Hermitian PT-symmetric Hamiltonians is

$$H = p^2 + x^2(ix)^\varepsilon. \tag{2}$$

Bender and Boettcher showed that the eigenvalues of this class of Hamiltonians are real, discrete, and positive for all $\varepsilon \geq 0$. When $\varepsilon < 0$, the eigenvalues of this class of Hamiltonians are, however, complex and come in conjugate pairs. Therefore, these Hamiltonians are said to live in the unbroken PT-symmetric phase for $\epsilon > 0$ and in the broken PT-symmetric phase for $\epsilon < 0$; $\epsilon = 0$ is referred to as the PT-phase-transition point. This picture is also applicable to other non-Hermitian PT-symmetric Hamiltonians. The PT-symmetry and the PT-phase-transition point have been studied from the experimental side. There are many optical systems that are suitable for realizing PT-symmetry in experiments, such as microcavities,[18] optically induced atomic lattices,[19] metamaterials,[20] and exciton-

Supported by the National Natural Science Foundation of China (Grant Nos. 11535004, 11975167, 11761161001, 11375086, 11565010, 11881240623 and 11961141003), the National Key R&D Program of China (Grant Nos. 2018YFA0404403 and 2016YFE0129300), the Science and Technology Development Fund of Macau (Grant No. 008/2017/AFJ), and the Fundamental Research Funds for the Central Universities (Grant Nos. 22120210138 and 22120200101).
*Corresponding author. Email: dbai@tongji.edu.cn; zren@tongji.edu.cn
© 2021 Chinese Physical Society and IOP Publishing Ltd

polariton condensates.[21−23] In particular, the PT-phase-transition point is observed in some optical waveguide and laser systems.[24−29] From the theoretical side, the PT-phase-transition points are conventionally determined by solving the Schrödinger equations directly. In this Letter, we propose an alternative approach based on ANN, which is a helpful complement to the conventional method.

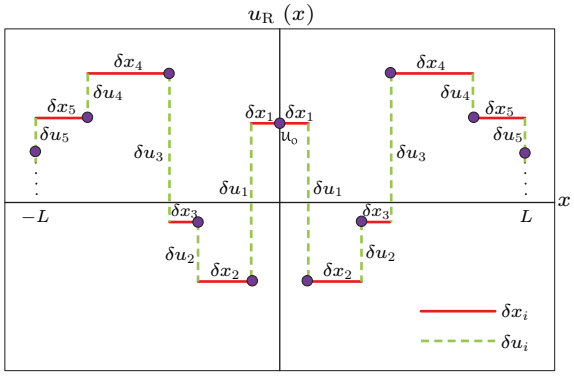

Fig. 1. Schematic description of how to generate the potential $u_{\rm R}(x)$ by the random walk process. Here δx_i and δu_i obey Gaussian distributions, and the purple dots represent the values of the random potential.

Artificial Neural Network. In this study, we attempt to determine the PT-phase-transition points using an ANN. We first discuss how the training data are prepared in our study. We generate a number of random short-range PT-symmetric potentials in the following forms:

$$u(x) = \begin{cases} u_{\rm R}(x), & -L < x < 0, \\ u_{\rm R}(x), & 0 < x < L, \end{cases} \tag{3}$$

and

$$w(x) = \begin{cases} -w_{\rm R}(x), & -L < x < 0, \\ w_{\rm R}(x), & 0 < x < L, \end{cases} \tag{4}$$

where $u(x)$ and $w(x)$ are nonzero only within the interval $[-L, L]$. The functions $u_{\rm R}(x)$ and $w_{\rm R}(x)$ are generated by using the random walk process. For example, $u_{\rm R}(x)$ is chosen to be

$$u_{\rm R}(x) = u_i, \quad \text{for } x_i < x < \min\{x_{i+1}, L\}, \tag{5}$$

with $i = 1, \ldots, n$. Here u_j and x_i are generated by the recurrence relations $u_i = u_{i-1} + \delta u_i$ and $x_i = x_{i-1} + \delta x_i$. The initial value u_0 at $x_0 = 0$ is chosen from some interval $[a, b]$ following the uniform distribution. δx_i and δu_i are all random variables satisfying the distributions $\theta(\delta x_i) N(\delta x_i, 4\,L/n, \sqrt{2}L/n)$ and $N(\delta u_i, 0, u_0/10)$. $N(x, \mu, \sigma)$ is the normal distribution with the expectation μ and the standard deviation σ, and $\theta(x)$ is the Heaviside function. In Fig. 1, a schematic description is given on how to generate $u_{\rm R}(x)$ from the random walk process. In Fig. 2, we give a concrete example of the generated $u_{\rm R}(x)$. Similarly, we can also generate the imaginary part $w(x)$ of the complex potential $V(x)$.

Using the algorithm given above, we generate 10^5 random PT-symmetric potentials for the training set and 10^4 PT-symmetric potentials for the test set. Their PT-symmetric phases are determined by solving the corresponding Schrödinger equations directly. The training and test data are labelled explicitly as "unbroken PT symmetry" and "broken PT symmetry". The structure of the ANN used in this study is shown in Fig. 3. We use the "scaled exponential linear units" (SELU) function as the activation function:[30]

$$\text{selu}(x) = \begin{cases} \lambda x, & \text{if } x > 0, \\ \lambda\alpha e^x - \alpha, & \text{if } x \le 0, \end{cases} \tag{6}$$

with $\alpha = 1.050700987355480493419334985294 6$ and $\lambda = 1.673263242354377284817042991671 7$. The number of neurons in the input layer is $N_{\rm in} = 258$. The number of neurons in each middle layer is simply set to be 50. An advantage of such an ANN is that it can give satisfying results within five epochs. When epochs are chosen to be greater than 30, the accuracy of the ANN could even be more than 99%. This suggests that ANN can be helpful in distinguishing whether a non-Hermitian PT-symmetric Hamiltonian is in the unbroken or broken PT-symmetric phase.

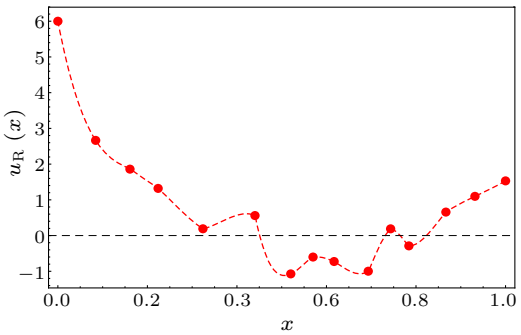

Fig. 2. An example of random potentials obtained by random walk. Here, we choose $L = 1$, $n = 64$ for $u_{\rm R}(x)$ at $x > 0$. After interpolation, the random potential becomes smooth.

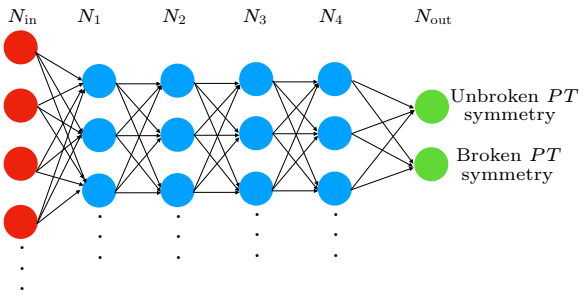

Fig. 3. A demonstration of the ANN used in this study. There are four middle layers in our ANN.

Results. We test the reliability of our ANN in two examples.

In the first example, we choose the following square

well potential[31]

$$\begin{aligned} &\mathrm{Re}\, V(x) = 0, \quad x \in (-1, 1), \\ &\mathrm{Im}\, V(x) = -g, \quad x \in (-l, 0), \\ &\mathrm{Im}\, V(x) = g, \quad x \in (0, l). \end{aligned} \tag{7}$$

This is closely related to the potential in Eq. (2) as $\varepsilon \to \infty$.[32–34] This potential is solvable. Its PT-phase-transition point is known to be $g_c \approx 4.475$[35] when $l = 1$. We start with $g = 0$ and increase g gradually. For each g, we determine the PT-symmetric phase using the ANN trained above. Initially, the Hamiltonian is in the unbroken PT-symmetric phase. The PT-phase-transition point is identified as the g value where the ANN starts to output "broken PT symmetry". The PT-phase-transition point is found to be $g_c = 4.48$ by our ANN, which is in good agreement with $g_c \approx 4.475$ in literature. To verify the robustness of our ANN, we change the value of l and calculate the corresponding g_c. As shown in Table 1, the results of ANN are in good agreement with the results of Ref. [35]. The relationship between l and g_c is also shown in Fig. 4. It is found that g_c increases rapidly as $l \to 0$.

Table 1. The PT-phase-transition point g_c for square-well potentials at different l values. The second row gives the results of Ref. [35]. The third row gives the results of ANN.

l	1	0.7	0.5	0.4	0.3	0.2
g_c (numerical results)	4.475	4.813	6.436	8.601	13.43	27.27
g_c (neural network)	4.48	4.81	6.35	8.81	13.39	26.51

Table 2. The critical value V_{2c} for the Scarf-Ⅱ potential with different V_1 values given by ANN. They are in good agreement with the theoretical results given by $V_{2c} = V_1 + 1/4$.[36]

V_1	1	2	3	5	8	10
V_{2c}(neural network)	1.26	2.26	3.25	5.26	8.25	10.25

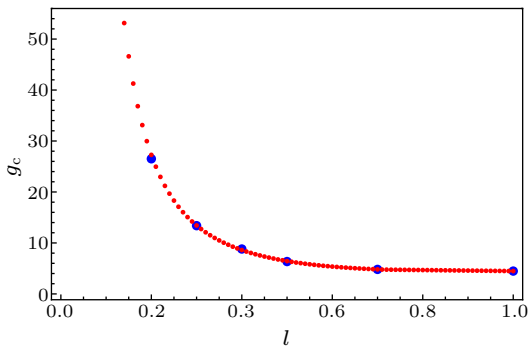

Fig. 4. Numerical results for the PT-phase-transition points g_c given by ANN and Ref. [35]. The red dots are the results of Ref. [35]. The blue dots are the results of ANN.

In the second example, we choose the complex PT-symmetric Scarf-Ⅱ potential

$$\begin{aligned} V(x) = &-V_1 \,\mathrm{sech}^2\, x + iV_2 \,\mathrm{sech}\, x \tanh x, \\ &V_1, V_2 > 0. \end{aligned} \tag{8}$$

When V_2 exceeds the critical value $V_{2c} = V_1 + 1/4$ from below, there is a PT-phase-transition from the unbroken phase to the broken phase.[36] We fix $V_1 = 1, 2, 3, 5, 8, 10$ and use our ANN to find the corresponding critical value V_{2c}. The results are shown in Table 2. The results of our ANN agree well with the theoretical results.

In summary, it is straightforward for human eyes to determine whether a complex potential is PT-symmetric or not from its shape. However, it is less straightforward for human eyes to determine whether a PT-symmetric non-Hermitian potential lives in the unbroken or broken PT-symmetric phase. In this Letter, we show that this task could be handled easily with an ANN, which is an artificial counterpart of neural network in the human brain. We construct an ANN with four middle layers and prepare the training and test sets with help of the random walk process. We test the reliability of ANN with two examples: the square-well potential and the Scarf-Ⅱ potential. It is found that the PT-phase-transition points given by ANN agree well with theoretical results in literature. Our method could also be applied to some other complex problems, such as the PT-symmetric non-Hermitian Bose–Hubbard system,[37] which is an interacting quantum system. It may be interesting to explore this direction in the future.

References

[1] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
[2] Carrasquilla J and Melko R G 2017 *Nat. Phys.* **13** 431
[3] Broecker P, Carrasquilla J, Melko R G and Trebst S 2017 *Sci. Rep.* **7** 1
[4] Ma Y, Su C, Liu J, Ren Z, Xu C and Gao Y 2020 *Phys. Rev. C* **101** 014304
[5] Ch'ng K, Carrasquilla J, Melko R G and Khatami E 2017 *Phys. Rev. X* **7** 031038
[6] Wu J and Zhang W 2019 arXiv:1906.11216 [cond-mat.dis-nn]
[7] Zhang Y and Kim E A 2017 *Phys. Rev. Lett.* **118** 216401
[8] Zhang Y, Melko R G and Kim E A 2017 *Phys. Rev. B* **96** 245119
[9] Ohtsuki T and Ohtsuki T 2016 *J. Phys. Soc. Jpn.* **85** 123706
[10] Schindler F, Regnault N and Neupert T 2017 *Phys. Rev. B* **95** 245134
[11] Ponte P and Melko R G 2017 *Phys. Rev. B* **96** 205146
[12] Wang L 2016 *Phys. Rev. B* **94** 195105
[13] Tanaka A and Tomiya A 2017 *J. Phys. Soc. Jpn.* **86** 063001
[14] Van Nieuwenburg E P, Liu Y H and Huber S D 2017 *Nat. Phys.* **13** 435
[15] Yao J, Wu Y, Koo J, Yan B and Zhai H 2020 *Phys. Rev. Res.* **2** 013287
[16] Liu G, Ma W P, Cao H and Lyu L D 2020 *Laser Phys. Lett.* **17** 045201
[17] Ohtsuki T and Mano T 2020 *J. Phys. Soc. Jpn.* **89** 022001
[18] Peng B, Özdemir Ş K, Lei F, Monifi F, Gianfreda M, Long G L, Fan S, Nori F, Bender C M and Yang L 2014 *Nat. Phys.* **10** 394
[19] Zhang Z, Zhang Y, Sheng J, Yang L, Miri M A, Christodoulides D N, He B, Zhang Y and Xiao M 2016 *Phys. Rev. Lett.* **117** 123601
[20] Castaldi G, Savoia S, Galdi V, Alù A and Engheta N 2013 *Phys. Rev. Lett.* **110** 173901

[21] Lien J Y, Chen Y N, Ishida N, Chen H B, Hwang C C and Nori F 2015 *Phys. Rev. B* **91** 024511
[22] Gao T, Estrecho E, Bliokh K, Liew T, Fraser M, Brodbeck S, Kamp M, Schneider C, Höfling S, Yamamoto Y *et al.* 2015 *Nature* **526** 554
[23] Chestnov I Y, Demirchyan S S, Alodjants A P, Rubo Y G and Kavokin A V 2016 *Sci. Rep.* **6** 19551
[24] Guo A, Salamo G J, Duchesne D, Morandotti R, Volatier-Ravat M, Aimez V, Siviloglou G A and Christodoulides D N 2009 *Phys. Rev. Lett.* **103** 093902
[25] Rüter C E, Makris K G, El-Ganainy R, Christodoulides D N, Segev M and Kip D 2010 *Nat. Phys.* **6** 192
[26] Regensburger A, Bersch C, Miri M A, Onishchukov G, Christodoulides D N and Peschel U 2012 *Nature* **488** 167
[27] Wimmer M, Regensburger A, Miri M A, Bersch C, Christodoulides D N and Peschel U 2015 *Nat. Commun.* **6** 7782
[28] Hodaei H, Miri M A, Heinrich M, Christodoulides D N and Khajavikhan M 2014 *Science* **346** 975
[29] Longhi S 2010 *Phys. Rev. A* **82** 031801
[30] Klambauer G, Unterthiner T, Mayr A and Hochreiter S 2017 arXiv:1706.02515 [cs.LG]
[31] Znojil M 2001 *Phys. Lett. A* **285** 7
[32] Cannata F, Junker G and Trost J 1998 *Phys. Lett. A* **246** 219
[33] Fernández F M, Guardiola R, Ros J and Znojil M 1999 *J. Phys. A Math. Theor.* **32** 3105
[34] Bender C M, Boettcher S, Jones H and Savage V M 1999 *J. Phys. A Math. Theor.* **32** 6771
[35] Bagchi B, Mallik S, Bíla H, Jakubskỳ V, Znojil M and Quesne C 2006 *Int. J. Mod. Phys. A* **21** 2173
[36] Ahmed Z 2001 *Phys. Lett. A* **282** 343
[37] Graefe E M, Korsch H J and Niederle A E 2008 *Phys. Rev. Lett.* **101** 150408