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Phase-Modulated 2D Topological Physics in a One-Dimensional Ultracold System

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We propose a one-dimensional optical lattice model to simulate and explore two-dimensional topological phases with ultracold atoms, considering the phases of the hopping strengths as an extra dimension. It is shown that the model exhibits nontrivial phases, and corresponding two chiral-edge states. Moreover, we demonstrate the connections between changes in the topological invariants and the Dirac points. Furthermore, the topological order detected by the particle pumping approach in cold atoms is also investigated. The results obtained here provide a feasible and flexible method of simulating and exploring high-dimensional topological phases in low-dimension systems via the controllable phase of the hopping strength.

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Topological insulators which rely on the global features of a material's band structure are a novel form of matter. Compared with conventional insulators, they are characterized by a full insulating gap in the bulk, but possess gapless edge or boundary states, which are robust against perturbations.[1–4] This new classification of the phases of matter can be characterized with reference to topological invariants,[5–7] e.g., the Chern number; these can then be utilized to investigate two-dimensional (2D) topological insulators. It has been proposed that the Chern number corresponds to the existence or the absence of the topological localized state.[8–11]

On the experimental side, ultracold atoms have furnished a natural platform from which to explore fundamental and significant topological properties, due to their high controllability and versatility.[12–14] For example, researchers[15–32] have shown that spin-orbit coupling can be experimentally engineered via trapped cold atoms, both in free space and in optical lattices, where the topological invariant and localized states can also be detected. Moreover, researchers[33–40] have also demonstrated that the external periodic modulated parameters in ultracold atoms can be treated as an additional synthetic dimension; as such, one may be able to study the physics of topologies subjected to dimensions higher than their inherence. Among the key issues is the measurement of the topological invariant in exploring the topological properties of a system. In Refs. [22–24,41,42], methods relating to Bloch oscillations, atomic interferometry, and time-of-flight imaging by measuring atomic distribution in momentum space have been reported. On the other hand, the phase of coupling strength has been considered in recent diverse studies, in fields ranging from optical communication to quantum information. For example, authors of Refs. [43–46] showed that single-photon scattering spectra and routing properties can be modulated by this characteristic quantity. It is therefore natural to ask whether the quantity could be utilized to simulate the topological physics in one-dimensional cold atoms. The answer is in the affirmative.

In this Letter, we propose a one-dimensional optical lattice model to simulate and explore two-dimensional topological phases using ultracold atoms, where the extra dimension comes from the cyclically modulated phases of hopping strengths. We show that the topological order can be defined in a 2D space, spanned by momentum, and a continuously changing parameter. The properties of two nontrivial edge states may be obtained from the normalized eigenstates, which are a function of eigenvalues. Furthermore, we investigate how to detect the topological order experimentally in the proposed cold atomic system.

Firstly, we introduce a theoretical model of an ultracold atomic system with coupling phase. Secondly, we elaborate on the topological properties of this model by calculating the Chern invariant and gapless edge states. Thirdly, we propose a practical method for the experimental detection of the topological number in the proposed cold atomic system, via numerical simulations. Finally, a short conclusion is given.

Model and Theory. To simulate the 2D topological phase in a one-dimensional (1D) system, we consider the non-interacting spin $-\frac{1}{2}$ of ultracold fermionic gas

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(labeled as spins ↑ and ↓), loaded in a one-dimensional optical lattice:

$$H = \sum_j (t_1 e^{i\phi_1} a^\dagger_{j,\uparrow} a_{j,\downarrow} + t_2 e^{i\phi_2} a^\dagger_{j+1,\uparrow} a_{j,\downarrow} + t_3 e^{i\phi_3} a^\dagger_{j,\uparrow} a_{j+1,\downarrow} + t_4 e^{i\phi_4} a^\dagger_{j,\uparrow} a_{j+1,\uparrow} + t_5 e^{i\phi_5} a^\dagger_{j,\downarrow} a_{j+1,\downarrow}) + \text{H.c.}, \quad (1)$$

where $a_{j,\sigma}$ ($a^\dagger_{j,\sigma}$) denotes the annihilation (creation) operator on site j, with spin $\sigma = (\uparrow, \downarrow)$; $t_l e^{i\phi_l}$ ($l = 1, 2, 3, 4, 5$) are hopping amplitudes and corresponding phases, respectively. The amplitude and the corresponding phase can be induced by varying the strength and the position, and this Hamiltonian can also be engineered by other artificial systems.[19,20,47,48] Generally, the physical properties of the bulk should not rely on the edges in the thermodynamic limit. Therefore, the Hamiltonian can be selected to be periodic. Using $a_{k,\sigma} = \frac{1}{\sqrt{N}} \sum_j a_{j,\sigma} e^{ikj}$ in the momentum space, the Hamiltonian is equivalent to the form:

$$H = \sum_{k,\sigma,\sigma'} a^\dagger_{k,\sigma} [H(k)]_{\sigma,\sigma'} a_{k,\sigma'}.$$

Alternatively, $H(k)$ can also be expressed as $H(k) = h_{\rm I}(k) I + \boldsymbol{\sigma} \cdot d\boldsymbol{k}$. Here, I denotes an identity matrix, and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. These are provided by

$$h_{\rm I}(k) = t_4 \cos(\phi_4 + k) + t_5 \cos(\phi_5 + k), \quad (2)$$
$$d_x = t_1 \cos\phi_1 + t_2 \cos(k - \phi_2) + t_3 \cos(k + \phi_3), \quad (3)$$
$$d_y = -t_1 \sin\phi_1 + t_2 \sin(k - \phi_2) - t_3 \sin(k + \phi_3), \quad (4)$$
$$d_z = t_4 \cos(\phi_4 + k) - t_5 \cos(\phi_5 + k). \quad (5)$$

The energy spectrum can be obtained by diagonalizing the Hamiltonian,

$$E = h_{\rm I} \pm \sqrt{d_x^2 + d_y^2 + d_z^2}. \quad (6)$$

The topological quantum phase transition will not occur until the two energy bands touch. Based on Eqs. (3)–(5), if hopping phases are not present, it is understood that, in general cases, e.g., ($t_1 > 2t_2$), the energy bands never touch one another, meaning that the topological nature of the system does not change. As such, the introduction of phases will produce a much richer phase diagram than that for a case lacking these parameters.

Topological Invariant with Synthetic Dimension. In order to explore the influence of hopping phases on the topological properties of a 1D cold atom array, ϕ_3 is selected as the synthetic dimension apart from the momentum, k. In the following, we use the Chern number (CN), originating from the integral of Berry curvature over the first Brillouin zone, to describe the topological properties of this system:

$$\text{CN} = \frac{1}{4\pi} \int_{-\pi}^{+\pi} dk \int_{-\pi}^{+\pi} d\phi_3 \hat{d\boldsymbol{k}} \cdot [\partial_k \hat{d\boldsymbol{k}} \times \partial_{\phi_3} \hat{d\boldsymbol{k}}], \quad (7)$$

where $\hat{d\boldsymbol{k}} = d\boldsymbol{k}/|d\boldsymbol{k}|$. For concreteness, the parameters are then set by $t_2 = t_3 = t$. Figure 1 numerically depicts phase diagrams modified by changing the parameters, as given in Ref. [49]. In Figs. 1(a) and 1(b), the phase boundary is independent of the coupling strength of t_4 or t_5 when these two parameters are equal. Furthermore, comparing Figs. 1(c) with 1(d), we find that the phase boundary is associated with the ratios t, and t_1, but is neither t nor t_1. However, as seen in Figs. 1(a) and 1(d), when the hopping phases are changed, the boundaries of the phase diagrams will differ from one another. This conclusion, in which one trivial and two distinct topological phases can be obtained by tuning the hopping phases, has been confirmed numerically, and can also be illustrated via Eqs. (3)–(5) and (7).

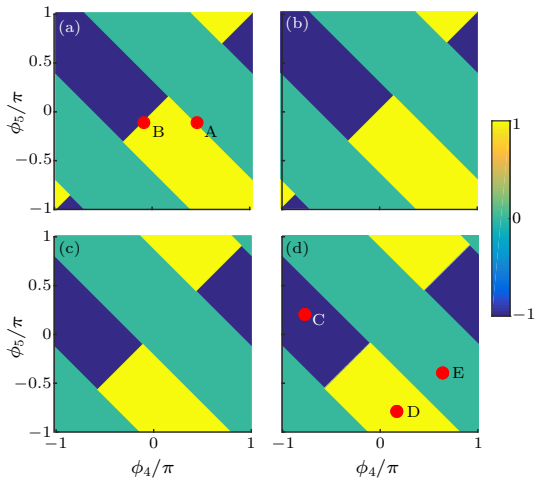

Fig. 1. Phase diagram of the synthetic system: (a) $t_1 = 1.5$, $t = 1$, $t_4 = t_5 = 1.2$, $\phi_1 = \frac{\pi}{4}$ and $\phi_2 = \frac{\pi}{3}$; (b), (c), and (d) for some different parameters from (a), i.e., (b) $t_4 = t_5 = 1.5$, (c) $t_1 = 1.65$, $t = 1.1$, $\phi_1 = \frac{\pi}{3}$ and $\phi_2 = \frac{2\pi}{3}$, (d) $\phi_1 = \frac{\pi}{3}$ and $\phi_2 = \frac{2\pi}{3}$.

The effects of the phases of the coupling strengths on the topological nature of a system may also be explained by the energy bands, based on the analytical expressions (3)–(5). Here, we take out a set of parameters in the first Brillouin zone in Fig. 1(a): $A(\frac{2\pi}{5}, \frac{8\pi}{5} - 2k_1)$, with $k_1 = \frac{\pi}{12} + \arccos(-\frac{3}{4})$, being the critical point between the trivial and non-trivial phases. As shown in Figs. 2(a) and 2(b), there is only one touching point, which is in harmony with the changes in Chern numbers. In this case, $B(\frac{8\pi}{5} - 2k_1, \frac{8\pi}{5} - 2k_1)$ stands for the critical point between two distinct non-trivial phases. Both Figs. 2(c) and 2(d) have two Dirac points in the parameter space, which is also consistent with changes in the topological invariants. Interestingly, comparing Figs. 2(a) and 2(b), or Figs. 2(c) and 2(d),

we find that the location of the contact point is clearly different, but that the numbers are the same, indicating that the term, $h_{\rm I}(k)I$, cannot affect the topological properties of the system.

To visualize the effect of the hopping phases in greater depth, the energy spectra and localized states can be numerically calculated, based on open boundary conditions (OBCs). Quantitatively, 2CN counts the total number of edge modes in the system. As shown in Fig. 3(a), with a continuous change in ϕ_3, from $-\pi$ to π, the spectrum exhibits two non-zero edge modes, crossing inside the energy gap connecting the valence with the conduction bands, i.e., the system is in the nontrivial phase, where CN = 1. To portray the edge modes more distinctly, the parameter of $\phi_3 = \frac{\pi}{2}$ can be adopted. Figure 3(b) shows all the normalized eigenstates as a function of eigenvalues. To render the image more visible, we set the unit cell to $N = 50$. Clearly, the eigenfunctions do not extend over the bulk, but are localized at either edge of the system at certain energies, and Fig. 3(c) and 3(d) clearly indicate those localized states, corresponding to the red points marked in Fig. 3(a). With the parameter ϕ_5 continuously increasing, the system will be in a critical state at the point $\phi_3 = \frac{\pi}{6}$, where the band gap is closed. If ϕ_5 keeps increasing, the gap will be reopened without emerging from the in-gap edge modes, indicating that the system is trivial, having the Chern number CN = 0, as shown in Fig. 3(f). The phase ϕ_4 exhibits the same effects as ϕ_5. Such a phase of hopping strength-induced transition between topologically distinct regimes is consistent with our previous calculations.

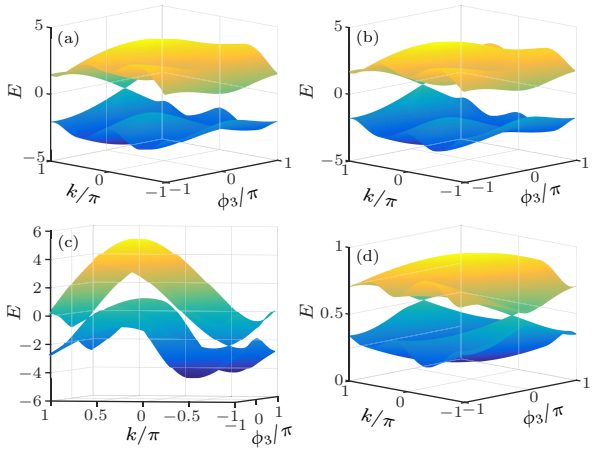

Fig. 2. Energy spectra in momentum space, where $t_1 = 1.5$, $t = 1$, $t_4 = t_5 = 1.2$, $\phi_1 = \frac{\pi}{4}$, and $\phi_2 = \frac{\pi}{3}$. For (a) and (b), $(\phi_4, \phi_5) = (\frac{2\pi}{5}, \frac{8\pi}{5} - 2k_1)$, the spectra exhibit one touching point, whereas for (c) and (d), $(\phi_4, \phi_5) = (\frac{8\pi}{5} - 2k_1, \frac{8\pi}{5} - 2k_1)$, the spectra have two touching points. We also consider the expression $h_{\rm I}(k) = 0$ for (b) and (d).

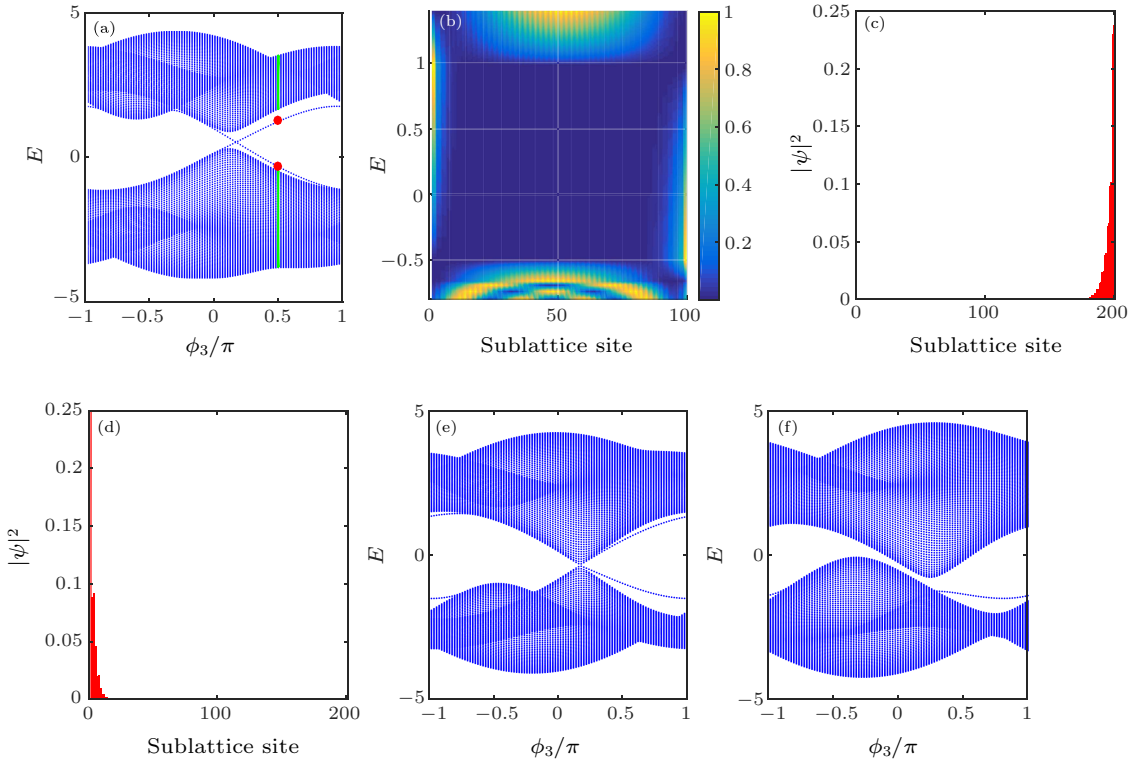

Fig. 3. Energy spectra and probability distributions of wave functions in real space, where $t_1 = 1.5$, $t = 1$, $t_4 = t_5 = 1.2$, $\phi_1 = \frac{\pi}{4}$ and $\phi_2 = \frac{\pi}{3}$. (a) Nontrivial case for $\phi_4 = \frac{3\pi}{5}$, $\phi_5 = -\frac{\pi}{2}$. (b) Energy eigenstates, corresponding to the green line of (a), $\phi_3 = \frac{\pi}{2}$. The eigenstates are normalized, such that the maximum is 1. (c) and (d) Density distributions of the wave functions, corresponding to the red dot in (a). (e) Critical case for $\phi_4 = \frac{3\pi}{5}$ and $\phi_5 = \frac{7\pi}{5} - 2k_1$. (f) Trivial case for $\phi_4 = \frac{3\pi}{5}$ and $\phi_5 = \frac{8\pi}{5} - 2k_1$.

Detection of the Topological Invariant. Here, an experimental scheme is proposed to measure the topological invariants, using a technique originating from the 1D topological pumping used in optical lattice systems.[50,51] The Berry curvature, which can be used to define the Chern number, has been theoretically proposed and experimentally implemented in many ways. As is well-known, the 1D insulator's polarization can be expressed as an integration of the Berry connection through the first Brillouin zone, and can further be expressed as the center of the hybrid Wannier functions (HWFs).[51,52] Note that our topological order in Eq. (7) is expressed in the spaces spanned by ϕ_3 and k. When the adiabatic parameter of ϕ_3, which can be regarded as the argument of the polarization, is slowly changed by 2π, the shift of the HWF center, i.e., the change in polarization, is proportional to the Chern number. The HWF center of our system in Hamiltonian (1) can be given by

$$\bar{n}(\phi_3) = \frac{\sum_n n\rho(n,\phi_3)}{\sum_n \rho(n,\phi_3)}, \tag{8}$$

in which $n = (1,1,2,2,\ldots,N,N)$, i.e., it is the sublattice index, and $\rho(n,\phi_3)$ denotes the density distribution of the HWF. In experiments, for a fixed pumping parameter ϕ_3, $\rho(\phi_3)$ can be detected using the hybrid time-of-flight images, referring to an *in situ* measurement.[35] As shown in Fig. 4, we numerically calculate $\bar{n}(\phi_3)$ of length $N = 100$ at three points marked in Fig. 1(d). At the point $C(-\frac{4\pi}{5}, \frac{\pi}{5})$, with the changing of the value of ϕ_3, in Fig. 4(a), the one-unit-cell jump illustrates that a single particle is pumped across the system, coincident with the expected topological invariant, CN $= -1$. We can also connect the point of $D(\frac{\pi}{5}, -\frac{4\pi}{5})$ with Fig. 4(b). The only difference is that the corresponding Chern number CN $= 1$. Figure 4(c) exhibits no jump of the HWF center, which is in agreement with the topological trivial phase of $E(\frac{3\pi}{5}, -\frac{2\pi}{5})$.

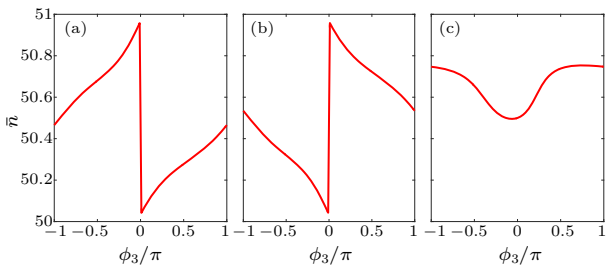

Fig. 4. HWF centers as a function of the parameter ϕ_3, obtained via numerical calculation of the Hamiltonian (1), $t_1 = 1.5$, $t = 1$, $t_4 = t_5 = 1.2$, $\phi_1 = \frac{\pi}{3}$, and $\phi_2 = \frac{2\pi}{3}$. (a) $\phi_4 = -\frac{4\pi}{5}$, and $\phi_5 = -\frac{\pi}{5}$. The HWF center jumps one unit cell, corresponding to the fact that this topological phase is nontrivial. (b) $\phi_4 = \frac{\pi}{5}$, $\phi_5 = -\frac{4\pi}{5}$. The HWF center also jumps one unit cell, and $C = 1$. (c) $\phi_4 = \frac{3\pi}{5}$, $\phi_5 = -\frac{2\pi}{5}$. The topological trivial phase with no jump is also depicted.

In summary, we have proposed a promising scheme, comprising one-dimensional ultracold atoms in an optical lattice. By calculating relevant topological properties, such as the Chern number, and all the states changed by the energies, we have shown that high-dimensional topological properties can be simulated and tuned via the phases of the coupling strengths, which can then support an extra synthetic dimension. Furthermore, we have presented a practical method of measuring the characteristic topological invariant in this 1D cold atom system. Building on all the results achieved in recent experiments, it should be possible to design and operate a 1D cold atom chain for the simulation of various 2D topological problems both reliably and with flexibility.

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