CHIN. PHYS. LETT. Vol. 38, No. 3 (2021) 034101

Effective Interaction Force between an Electric Charge and a Magnetic Dipole and Locality (or Nonlocality) in Quantum Effects of the Aharonov–Bohm Type

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(Received 4 December 2020; accepted 12 January 2021; published online 2 March 2021)

Classical electrodynamics foresees that the effective interaction force between a moving charge and a magnetic dipole is modified by the time-varying total momentum of the interaction fields. We derive the equations of motion of the particles from the total stress-energy tensor, assuming the validity of Maxwell's equations and the total momentum conservation law. Applications to the effects of Aharonov–Bohm type show that the observed phase shift may be due to the relative lag between interfering particles caused by the effective local force.

DOI: 10.1088/0256-307X/38/3/034101

The discussions about the nonlocality, or locality, of the quantum effects of the Aharonov–Bohm (AB)[1–5] type have been developing extensively through the decades.[6–17] Some authors claim that there are no forces acting on the particles in the AB effects.[6,7,9] However, other authors sustain diverse interpretations of the AB type of effects in terms of the local action of forces.[8,10–17] Moreover, it has been argued that nonlocality claims are inconsistent with Hilbert-space quantum mechanics.[18] Nonlocal quantum effects are relevant in a wide context of scientific and heuristic scenarios and, according to the supporters of quantum nonlocality, the effects of the AB type represent a new departure from classical physics because they cannot be interpreted by means of local forces acting on the particles. Traditional classical physics requires instead that observable physical effects arise as a result of a cause, generally the action of a force that produces the effect. Thus, supporters of classical physics claim that the phase shift of the effects of the AB type may well arise as the result of the action of local forces, which, for example, may produce a relative lag between particles passing on opposite sides of the line sources, resulting in an observable quantum phase shift $\Delta\phi$.[8,10–17]

As some other authors have pointed out,[9] we believe that there is no consensus regarding the nature and interpretation of the AB effect, despite of the several, ongoing discussions on the subject. Supporting the view of an interpretation involving the local action of a force, we consider here three effects of the AB type[1–5] where a beam of interfering particles interacts with an external electromagnetic (em) potential (or field) in a, supposedly, force-free region of space. In the AB effect[1] a charged particle interacts with the vector potential $\boldsymbol{A}$ of a solenoid, in the Aharonov–Casher (AC) effect[2] a neutral particle with a magnetic dipole moment $\boldsymbol{m}$ interacts with an electric field $\boldsymbol{E}$ of a line of charges, and in the Spavieri effect[3–5] a neutral particle with an electric dipole moment $\boldsymbol{d}$ interacts with the vector potential $\boldsymbol{A}$ of a distribution of magnetic dipoles. If $\boldsymbol{Q}$ is the canonical interaction momentum, the quantum phase of these effects is $\phi = \hbar^{-1}\int \boldsymbol{Q}\cdot d\boldsymbol{\ell}$. A unitary vision of these effects is given in Refs. [3–5], where $\boldsymbol{Q}$ is related to the momentum of the interaction fields, $\boldsymbol{Q} = \pm\boldsymbol{Q}_{\rm em}$. Some of the phase shifts $\Delta\phi$ arising in these effects have been verified experimentally: for the AB effect see Refs. [19,20] and, for tests of the AC effect, see Refs. [21–24].

The basic em interaction in the mentioned AB effects is the one between a particle with an electric charge q and a neutral particle possessing a magnetic dipole moment $\boldsymbol{m}$. As well known, the standard expression for the interaction force between a magnetic dipole $\boldsymbol{m}$ and a charge q, in motion with relative velocity $\boldsymbol{v}$, does not comply with the action and reaction principle. In fact, neglecting higher order relativistic terms, in the reference frame where the dipole is stationary, the em force acting on q is $\boldsymbol{f}_q^{\rm em} = q\boldsymbol{E} + (q/c)\boldsymbol{v}\times\boldsymbol{B} = (q/c)\boldsymbol{v}\times\boldsymbol{B}$, where the electric field is $\boldsymbol{E} = 0$ for a neutral dipole and $\boldsymbol{B}$ is the magnetic induction field produced by $\boldsymbol{m}$. Instead, the em force acting on $\boldsymbol{m}$ is $\boldsymbol{f}_m^{\rm em} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B}_q)$, where $\boldsymbol{B}_q = (\boldsymbol{v}/c)\times\boldsymbol{E}_q$ is the magnetic field produced by the moving charge, $\boldsymbol{E}_q$ being its electric field. Thus, for example, in the direction of motion, with

Supported partially by the CDCHT (ULA, Mérida, Venezuela).
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$\boldsymbol{v} = v\hat{i}$ in the x direction, we find $f^{\rm em}_{\rm qx} = 0$, while $f^{\rm em}_{\rm mx} = \partial_x(\boldsymbol{m}\cdot\boldsymbol{B}_q) \neq 0$. The action and reaction principle is not conserved even when q and $\boldsymbol{m}$ are stationary but the current in $\boldsymbol{m}$ varies with time ($\dot{\boldsymbol{m}} \neq 0$), as pointed out by Shockley–James,[25] who claim that their paradox indicates that even the conservation law for the linear momentum is not conserved.

However, it has been shown[6,26,27] that the effective force acting on a particle, and the corresponding equation of motion, might be modified if, besides the mechanical momentum of the particle, (mass) × (velocity), the physical system carries an additional momentum related to the interaction fields. In our case the fields of our physical system possesses a nonvanishing electromagnetic interaction momentum $\boldsymbol{Q}_{\rm em}$ and, moreover, it possesses also a momentum $\boldsymbol{Q}_h$ that is due either to the internal stresses[6,26,27] or to the charges induced by the field $\boldsymbol{E}_q$.[26,27] The purpose of our letter is to derive within classical electrodynamics the equations of motion of q and $\boldsymbol{m}$ after determining the effective interaction force between them, assuming the validity of conservation laws, the action and reaction principle, and the contribution of the momentum of interaction fields, as also required for the solution of the Shockley–James[25] paradox. The role of the momentum $\boldsymbol{Q}_h$ has been taken into account in the nonrelativistic interpretation of the atomic spin-orbit coupling[28,29] and we show that the (effective) interaction force here derived leads to an interpretation of the AB effects in terms of classical local forces. In fact, the derived interaction force is in agreement with the experimental evidence of the observed phase shift $\Delta\phi$ in the effects of the AB type.[19,22–24] Concerning the test of dispersionless forces through a long toroid,[9] we show that our effective force agrees with the result of the test. Other possible tests for the interaction force, realizable with present technology,[30,31] are discussed.

The Interaction Force between a Charge q and a Magnetic Dipole Moment $\boldsymbol{m}$. The relevant em interaction, taking place in the effects of the AB type, is the one between a charged particle q and a neutral particle with a magnetic dipole moment $\boldsymbol{m}$. The appropriate tensor describing the system composed by a charge q and a nonconducting dipole $\boldsymbol{m}$ is[6,26,27] $T^{\mu\nu} = \theta^{\mu\nu} + S^{\mu\nu} + \delta_0 U^\mu U^\nu$, complemented by the continuity equation $\partial_\mu T^{\mu\nu} = 0$, where $\theta^{\mu\nu}$ is the em tensor, $S^{\mu\nu}$ is the stress tensor, and δ_0 is the proper density of the proper mass. Relevant quantities are the em momentum $\boldsymbol{Q}_{\rm em}$ and the momentum $\boldsymbol{Q}_h$ due to stresses (referred to as the *hidden* momentum),[6,26,27] their components being given by

$$Q^i_{\rm em} = \frac{1}{4\pi c}\int (\boldsymbol{E}\times\boldsymbol{B})^i d\tau = \int \theta^{i0} d\tau$$
$$Q^i_h = \int S^{i0} d\tau. \tag{1}$$

Let $\boldsymbol{p}_q = M_q\boldsymbol{v}$ be the linear momentum of the charge of mass M_q and $\boldsymbol{p}_m = M_m\boldsymbol{v}_m$ that of the magnetic dipole $\boldsymbol{m}$ of mass M_m. Integration over the volume of the continuity equation, $\partial_\mu T^{\mu\nu} = \partial_\mu(\theta^{\mu\nu} + S^{\mu\nu} + \delta_0 U^\mu U^\nu)$, leads to,

$$\frac{d}{dt}(\boldsymbol{Q}_{\rm em}+\boldsymbol{Q}_h)+\int(\partial_j\theta^{ij}+\partial_j S^{ij})d\tau+\frac{d}{dt}(\boldsymbol{p}_q+\boldsymbol{p}_m) = 0, \tag{2}$$

$$\frac{d}{dt}(\boldsymbol{Q}_{\rm em}+\boldsymbol{Q}_h)+\frac{d}{dt}(\boldsymbol{p}_q+\boldsymbol{p}_m) = 0, \tag{3}$$

where, in Eq. (3), for a closed isolated system the volume integral of the divergences $\partial_j\theta^{ij}$ and $\partial_j S^{ij}$ vanishes, and expression (2) provides the linear momentum conservation law (3). In the case of the interaction between q and $\boldsymbol{m}$, in the dipole approximation $\boldsymbol{Q}_{\rm em}$ and $\boldsymbol{Q}_h$ can be expressed[6,10–13,26,27,32] as

$$\boldsymbol{Q}_{\rm em} = \int\frac{\rho}{c}\boldsymbol{A}(\boldsymbol{x})d\tau = \frac{q}{c}\boldsymbol{A},$$
$$\boldsymbol{Q}_h = -\frac{q}{c}\boldsymbol{A} = \frac{\boldsymbol{m}}{c}\times\boldsymbol{E}_q. \tag{4}$$

Moreover, for a finite stationary configuration, expression (3) implies[6,26,27]

$$\boldsymbol{Q}_{\rm em}+\boldsymbol{Q}_h = \frac{q}{c}\boldsymbol{A}+\frac{\boldsymbol{m}}{c}\times\boldsymbol{E}_q = 0. \tag{5}$$

Since the interaction momentum $\boldsymbol{Q}_h$ is a nonlocal quantity, it can be expressed indifferently in terms of $\boldsymbol{A}$ or $\boldsymbol{E}_q$ in Eq. (4). In the dipole approximation $\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}_m) = \boldsymbol{m}\times(\boldsymbol{x}-\boldsymbol{x}_m)/|\boldsymbol{x}-\boldsymbol{x}_m|^3$ is the vector potential of the magnetic dipole $\boldsymbol{m}$ and $\boldsymbol{E}_q = q(\boldsymbol{x}-\boldsymbol{x}_q)/|\boldsymbol{x}-\boldsymbol{x}_m|^3$ is the electric field of the charge q. In Eq. (4) $q\boldsymbol{A}$ has to be evaluated at the position of the charge $\boldsymbol{x}_q$ and $c^{-1}\boldsymbol{m}\times\boldsymbol{E}_q$ at $\boldsymbol{x}_m$.

Making use of the relation $\partial_\beta\theta^{\alpha\beta} = -c^{-1}F^{\alpha\lambda}J\lambda$, where $F^{\alpha\lambda}$ is the em field-strength tensor, after volume integration, for our closed system the em and stress force density can be expressed, respectively, as $\boldsymbol{f}^{\rm em} = -(d/dt)\boldsymbol{Q}_{\rm em}$ and $\boldsymbol{f}_h = -(d/dt)\boldsymbol{Q}_h$. Then, $\boldsymbol{f}^{\rm em}+\boldsymbol{f}_h = d(\boldsymbol{p}_q+\boldsymbol{p}_m)/dt$ or,

$$\boldsymbol{f}^{\rm em}-\frac{d}{dt}\boldsymbol{Q}_h = \frac{d}{dt}(\boldsymbol{p}_q+\boldsymbol{p}_m) = \boldsymbol{f}_q+\boldsymbol{f}_m, \tag{6}$$

where in Eq. (6) $\boldsymbol{f}_q = d\boldsymbol{p}_q/dt$ and $\boldsymbol{f}_m = d\boldsymbol{p}_m/dt$ are the effective forces acting on q and $\boldsymbol{m}$, respectively. We can see from Eq. (6) that the effective forces, $\boldsymbol{f}_q$ and $\boldsymbol{f}_m$, and the corresponding equations of motion are affected by the term $d\boldsymbol{Q}_h/dt$. When the magnetic dipole is made of conducting material, for the stationary system the electric field $\boldsymbol{E}_q$ induces charges on the magnetic dipole creating an electric field $\boldsymbol{E}_{\rm ind}$ that provides zero total electric field inside the dipole, $\boldsymbol{E}_{\rm ind}+\boldsymbol{E}_q = 0$. In this case, if the magnetic dipole is completely shielded, there are no internal stresses and no momentum due to stresses related to $\boldsymbol{m}$.[26,27] Still, the original $\boldsymbol{Q}_{\rm em}$ is modified now by the presence of the nonvanishing $\boldsymbol{E}_{\rm ind}$. In order to take into account the presence of $\boldsymbol{E}_{\rm ind}$, the relation $\boldsymbol{Q}_{\rm em}+\boldsymbol{Q}_h = 0$ of

Eq. (5) is replaced by the relation $\boldsymbol{Q}_{\rm em}+\boldsymbol{Q}_{\rm em-ind}=0$, where $\boldsymbol{Q}_{\rm em-ind}=\boldsymbol{Q}_h$ is the em momentum related to the induced charges and their field $\boldsymbol{E}_{\rm ind}=-\boldsymbol{E}_q$ inside the dipole. Thus, we assume here the validity of expression (5) with $\boldsymbol{Q}_h$ representing, depending on the case, either the momentum due to the stresses or the momentum due to the induced charges. The physical results that can be derived from Eqs. (3) and (6) are supposed to be model-independent and are shown to be holding for both the cases of conducting and nonconducting magnetic dipole[26,27] in solving the Shockley–James paradox.[25] Then, what Eq. (6) implies is that the action of the time-variation of $\boldsymbol{Q}_h$ (or $\boldsymbol{Q}_{\rm em-ind}=\boldsymbol{Q}_h$) has to be taken into account in Eq. (6) for determining the effective $\boldsymbol{f}_q$ and $\boldsymbol{f}_m$ that comply with the equilibrium condition and the momentum conservation law.

Action and Reaction Principle. For our closed system we are left to split Eq. (6) into two equations of motion, one for q and one for $\boldsymbol{m}$. For our purposes, we consider the case where $\boldsymbol{m}$ is stationary and q is moving relative to $\boldsymbol{m}$. The time derivative of $\boldsymbol{Q}_{\rm em}$ can be expressed as

$$\boldsymbol{f}^{\rm em}=-\frac{d}{dt}\boldsymbol{Q}_{\rm em}=\int(\rho\boldsymbol{E}+\boldsymbol{j}\times\boldsymbol{B})d\tau \tag{7}$$

$$=q\boldsymbol{E}+\frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}+\int(\boldsymbol{j}\times\boldsymbol{B}_q)d\tau \tag{8}$$

$$=-\frac{q}{c}\partial_t\boldsymbol{A}+\frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}+\nabla(\boldsymbol{m}\cdot\boldsymbol{B}_q), \tag{9}$$

where in Eq. (7), $\boldsymbol{j}\times\boldsymbol{B}=\boldsymbol{j}_q\times\boldsymbol{B}+\boldsymbol{j}\times\boldsymbol{B}_q$. In Eq. (9) the terms $-\frac{q}{c}\partial_t\boldsymbol{A}$ and $\frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}$ represent the standard force on q and the last term stands for the standard force[10–14,33] on the magnetic dipole $\boldsymbol{m}$ in the presence of the magnetic induction field $\boldsymbol{B}_q=c^{-1}\boldsymbol{v}\times\boldsymbol{E}_q$ of the moving charge. $\boldsymbol{Q}_{\rm em}$ depends on the interacting fields of both of q and $\boldsymbol{m}$ and its time derivative (7) contains terms representing forces localized and acting on q as well as forces localized and acting on $\boldsymbol{m}$. Being a nonlocal quantity, the momentum $\boldsymbol{Q}_{\rm em}$ is not localized on any of the field sources and thus its time derivative, $-(d/dt)(\boldsymbol{Q}_{\rm em})$, cannot be taken to represent uniquely the forces acting on one of the interacting particles, e.g., q (or $\boldsymbol{m}$). Since similar considerations hold for $\boldsymbol{Q}_h$, the variation $-(d/dt)(\boldsymbol{Q}_h)$ represents force terms, some acting on q and others on $\boldsymbol{m}$ (and not uniquely on q or on $\boldsymbol{m}$).

When $\boldsymbol{j}_q=(\rho/c)\boldsymbol{v}$ forms part of a current loop in a neutral wire, after integrating over the closed loop, the action and reaction principle holds for interaction forces between the loop and the dipole $\boldsymbol{m}$ even for time-varying fields. However, for our system the term $(q/c)\boldsymbol{v}$ represents a non-neutral "open current" element and, due to its nonvanishing electric field $\boldsymbol{E}_q$, the em momentum $\boldsymbol{Q}_{\rm em}\neq 0$. As mentioned above, in this case we have $(q/c)\boldsymbol{v}\times\boldsymbol{B}+\nabla(\boldsymbol{m}\cdot\boldsymbol{B}_q)\neq 0$, because the force $(q/c)\boldsymbol{v}\times\boldsymbol{B}$ is always perpendicular to the direction of motion, while $\nabla(\boldsymbol{m}\cdot\boldsymbol{B}_q)$ has also a nonvanishing longitudinal component in the direction of $\boldsymbol{v}$. Moreover, the effective interaction force (between q and $\boldsymbol{m}$) must be such as to solve the Shockley–James paradox,[25] where the radiation force on q, given by $-c^{-1}q\partial_t(\boldsymbol{A})$, requires to be balanced by an equal and opposite force on $\boldsymbol{m}$. As well known, this paradox can be solved by taking into account the momentum $\boldsymbol{Q}_h$[6,26,27] with the related force $\boldsymbol{f}_h=-(d/dt)\boldsymbol{Q}_h$, which, on account of expression (4), may conveniently written as

$$\begin{aligned}\boldsymbol{f}_h&=-\frac{d}{dt}\boldsymbol{Q}_h=-\frac{d}{dt}\Big(\frac{\boldsymbol{m}}{c}\times\boldsymbol{E}_q\Big)=\frac{d}{dt}\Big(\frac{q}{c}\boldsymbol{A}\Big)\\&=-\frac{\dot{\boldsymbol{m}}}{c}\times\boldsymbol{E}_q-\frac{\boldsymbol{m}}{c}\times(\partial_t\boldsymbol{E}_q)\\&=-\frac{\dot{\boldsymbol{m}}}{c}\times\boldsymbol{E}_q+\frac{q}{c}(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}.\end{aligned} \tag{10}$$

When $\boldsymbol{v}=0$, the force $\boldsymbol{f}_h$ in Eq. (10) is $-\partial_t(\boldsymbol{Q}_h)=-c^{-1}(\dot{\boldsymbol{m}}\times\boldsymbol{E}_q)=c^{-1}q\partial_t(\boldsymbol{A})$, effective when the current of the magnetic dipole is varying with time and em radiation fields are involved. Then, the Shockley–James paradox is solved because the force $-c^{-1}q\partial_t(\boldsymbol{A})$, acting on q in Eq. (9), is balanced by the equal and opposite force $-c^{-1}(\dot{\boldsymbol{m}}\times\boldsymbol{E}_q)=c^{-1}q\partial_t(\boldsymbol{A})$ acting on $\boldsymbol{m}$.

Neglecting higher order relativistic terms, expression (5) holds even when q moves with velocity $\boldsymbol{v}$ relative to $\boldsymbol{m}$. Thus, going beyond the effects of pure radiation fields, we assume that the idea behind the Shockley–James paradox (that em interaction must comply with the action and reaction principle and momentum conservation) can be extended to include the velocity dependent terms of Eqs. (9) and (10). For this purpose it is sufficient that the equal and opposite interaction forces be related to the gradients of the same interaction energy. The quantity $-\boldsymbol{m}\cdot\boldsymbol{B}_q$ represents the interaction potential energy between $\boldsymbol{m}$ and $\boldsymbol{B}_q$ and its negative ∇ leads to the force $\nabla(\boldsymbol{m}\cdot\boldsymbol{B}_q)$ acting on $\boldsymbol{m}$. In order for the action and reaction principle to be holding, the same potential energy must lead also to an equal and opposite force acting on q. In fact, with the help of Eqs. (4) and (5), we may write

$$\begin{aligned}-\boldsymbol{m}\cdot\boldsymbol{B}_q&=-\boldsymbol{m}\cdot\Big(\frac{\boldsymbol{v}}{c}\times\boldsymbol{E}_q\Big)\\&=\boldsymbol{v}\cdot\Big(\frac{\boldsymbol{m}}{c}\times\boldsymbol{E}_q\Big)=-\frac{q}{c}(\boldsymbol{v}\cdot\boldsymbol{A}),\end{aligned} \tag{11}$$

where $-(q/c)\boldsymbol{v}\cdot\boldsymbol{A}$ represents the interaction potential energy $-\int(\boldsymbol{j}_q\cdot\boldsymbol{A})d\tau$[33] of the open current element $(q/c)\boldsymbol{v}$ in the presence of the vector potential $\boldsymbol{A}$. Then, expression (11) implies that, in correspondence to the force term $\nabla(\boldsymbol{m}\cdot\boldsymbol{B}_q)$ acting on the dipole, there is an equal and opposite force $(q/c)\nabla(\boldsymbol{v}\cdot\boldsymbol{A})=(q/c)\boldsymbol{v}\times\boldsymbol{B}+(q/c)(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}$ acting on the charge, as shown below.

Equations of Motion for the Charge q and the Dipole $\boldsymbol{m}$. As a criterion for identifying which are

the effective forces on either q or $\boldsymbol{m}$, we assume the validity of the action and reaction principle. With the help of Maxwell's equation $\nabla\times\boldsymbol{B}_q=\frac{1}{c}\partial_t\boldsymbol{E}_q$, Eqs. (9), (10), and the identity

$$\nabla(\boldsymbol{m}\cdot\boldsymbol{B}_q)=(\boldsymbol{m}\cdot\nabla)\boldsymbol{B}_q+\frac{1}{c}\boldsymbol{m}\times(\partial_t\boldsymbol{E}_q),$$

we may rewrite Eq. (6) as

$$\boldsymbol{f}^{\rm em}+\boldsymbol{f}_h=-\frac{q}{c}\partial_t\boldsymbol{A}+\frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}+\frac{q}{c}(\boldsymbol{v}\cdot\nabla)\boldsymbol{A},$$

$$-\frac{\dot{\boldsymbol{m}}}{c}\times\boldsymbol{E}_q+\frac{\boldsymbol{m}}{c}\times(\partial_t\boldsymbol{E}_q)+(\boldsymbol{m}\cdot\nabla)\boldsymbol{B}_q=\boldsymbol{f}_q+\boldsymbol{f}_m. \quad (12)$$

The two terms of Eq. (12), $-\frac{q}{c}\partial_t\boldsymbol{A}$ and $-\frac{\dot{\boldsymbol{m}}}{c}\times\boldsymbol{E}$, are equal and opposite and, algebraically, may cancel. However, from a physical point of view, we may not suppress them if they represent, as they do in this context, the equal and opposite action and reaction forces on q and $\boldsymbol{m}$, respectively. About the terms $\frac{q}{c}(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}$ and $\frac{\boldsymbol{m}}{c}\times(\partial_t\boldsymbol{E}_q)$ of Eq. (12) (which are equal and opposite and, algebraically, may cancel) we consider the following possible interpretations.

(a) As done by Aharonov *et al.*,[6] we may assume that the momentum $\boldsymbol{Q}_h$ is localized on the dipole $\boldsymbol{m}$. Then, in this case the force (10) $-(d/dt)\boldsymbol{Q}_h=-\frac{\dot{\boldsymbol{m}}}{c}\times\boldsymbol{E}+\frac{q}{c}(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}=-\frac{\dot{\boldsymbol{m}}}{c}\times\boldsymbol{E}-\frac{\boldsymbol{m}}{c}\times(\partial_t\boldsymbol{E}_q)$ is entirely acting on $\boldsymbol{m}$ and $\frac{q}{c}(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}$ and $\frac{\boldsymbol{m}}{c}\times(\partial_t\boldsymbol{E}_q)$ in Eq. (12) are equal and opposite force terms that cancel because both act on $\boldsymbol{m}$. This way, the longitudinal components of the forces disappear and expression (12) becomes

$$\boldsymbol{f}_q+\boldsymbol{f}_m=-\frac{q}{c}\partial_t\boldsymbol{A}+\frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}-\frac{\dot{\boldsymbol{m}}}{c}\times\boldsymbol{E}+(\boldsymbol{m}\cdot\nabla)\boldsymbol{B}_q, \quad (13)$$

where the first two terms on the right-hand side represent $\boldsymbol{f}_q$ and the last two represent $\boldsymbol{f}_m$.

(b) Nevertheless, as mentioned above, the momentum is a nonlocal quantity and, as such, theoretically and experimentally, neither $\boldsymbol{Q}_{\rm em}$ or $\boldsymbol{Q}_h$ can be localized on any of the source of the interaction fields. What can be assumed as localized are the forces that are related to the time (or space) variations of $\boldsymbol{Q}_{\rm em}$ or $\boldsymbol{Q}_h$. Therefore, a priori, we cannot exclude that the term $\frac{q}{c}(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}$ may represent a force acting on the current element $(q/c)\boldsymbol{v}$. In this case, we may not cancel algebraically the two terms $\frac{q}{c}(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}$ and $\frac{\boldsymbol{m}}{c}\times(\partial_t\boldsymbol{E}_q)$ in Eq. (12) because they represent, respectively, a force acting on q and a force on $\boldsymbol{m}$. Thus, being $\boldsymbol{v}\times\boldsymbol{B}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}=\nabla(\boldsymbol{v}\cdot\boldsymbol{A})$, in the reference frame where the magnetic dipole is stationary, expression (12) leads to the following effective forces and corresponding equations of motion,

$$\boldsymbol{f}_q=-\frac{q}{c}\partial_t\boldsymbol{A}+\frac{q}{c}\nabla(\boldsymbol{v}\cdot\boldsymbol{A})=\frac{d}{dt}(\boldsymbol{p}_q) \quad (14)$$

$$\boldsymbol{f}_m=-\frac{\dot{\boldsymbol{m}}}{c}\times\boldsymbol{E}+\nabla(\boldsymbol{m}\cdot\boldsymbol{B}_q)=\frac{d}{dt}(\boldsymbol{p}_m). \quad (15)$$

The Lagrangian formulation for deriving Eqs. (14) and (15) will be given in a future contribution.

Experimental Evidence. The correct choice, (a) or (b), for the effective force on the charge q in motion, needs to be corroborated experimentally. In the experiment by Becker and Batelaan[9] the long macroscopic toroid, adopted for testing the time of flight of q, has not been used to observe the AB effect. Moreover, it is reasonable to assume that the vector potential $\boldsymbol{A}$ is nearly uniform inside the long toroid where, in the direction of motion, $(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}\simeq 0$. Thus, no action on the moving charge is exerted inside the toroid by the force term $\frac{q}{c}(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}$ of Eq. (14) and, consequently, no variation of its time of flight is foreseen, in agreement with the experimental results.[9] In fact, we expect our force to be acting briefly on the moving q at the beginning and end of the toroid only. The ideal experiment to more easily detect the longitudinal component of the force must adopt an arrangement where the perpendicular component $\boldsymbol{v}\times\boldsymbol{B}$ vanishes and, moreover, $\boldsymbol{A}$ is not uniform in the direction of motion, so that $(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_{\parallel}\neq 0$. Such an arrangement is obtained in the case of the AB effect with a standard toroid, where the force (14) is nonvanishing. Actually, in the experiment performed by Tomonura *et al.*,[20] which detects the Aharonov–Bohm phase shift $\Delta\phi_{\rm AB}$, a microscopic toroid has been used and the result corroborates the existence of a longitudinal force.

The effect of a local force on the phase shift of the AB system has been derived in Refs. [10–14], and we reconsider it here starting from the free-force phase $\phi=\hbar^{-1}(\boldsymbol{p}\cdot\boldsymbol{x}-\bar{E}t)$ of the interfering wave function of the electrons in the AB effect. In the presence of the small force $\boldsymbol{f}_q$, the particles moving on opposite sides of the solenoid acquire a relative lag that produces the phase shift $\Delta\phi$, either because of the particles relative variation $\delta\boldsymbol{v}$ or $\delta\boldsymbol{x}$.[10–14,26,27] Assuming that the particle of mass M_q slightly changes its momentum $\boldsymbol{p}_q=M_q\boldsymbol{v}$ under the action of the force $\boldsymbol{f}_q=M_q d\boldsymbol{v}/dt$, we have $\delta\boldsymbol{v}=M_q^{-1}\int_0^t\boldsymbol{f}_q dt$. If $x(t)$ is the position of the particle when $\boldsymbol{p}_q$ is constant, with $vdt=dx$ along the path of the particle (from $-\infty$ to $+\infty$), the force has the effect to change it by $\Delta x=\int(\delta v)dt=M_q^{-1}\int dt\int_0^t(\boldsymbol{f}_q)_x dt=(vM_q)^{-1}\int dt\int_0^t[\nabla(\frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{v})]_x dx$. Then, $\Delta x=(vM_q)^{-1}\int_{-\infty}^{\infty}\frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{v}dt$, leading to the phase shift $\delta\phi=\hbar^{-1}(\boldsymbol{p}_q\cdot\Delta\boldsymbol{x})=\hbar^{-1}\frac{q}{c}\int_{-\infty}^{\infty}\boldsymbol{A}\cdot d\boldsymbol{x}$. Because of the topological properties of the system and the symmetry of vector potential $\boldsymbol{A}$, the resulting relative phase shift between particles moving along the opposite sides of the solenoid is

$$\Delta\phi=-2\hbar^{-1}\frac{q}{c}\int_{-\infty}^{\infty}\boldsymbol{A}\cdot d\boldsymbol{x}=\hbar^{-1}\frac{q}{c}\oint_C\boldsymbol{A}\cdot d\boldsymbol{x}=\Delta\phi_{\rm AB}, \quad (16)$$

where $\Delta\phi_{\rm AB}$ is the observable Aharonov–Bohm phase shift.

Similar conclusions may be drawn for the Aharonov–Casher[2] and Spavieri[3–5] effects. In the case of the AC effect, $\boldsymbol{m}$ is moving with velocity $\boldsymbol{v}_m = -\boldsymbol{v}$ relative to a static electric charge distribution. Then, in Eq. (15) the term $\nabla(\boldsymbol{m}\cdot\boldsymbol{B}) = -c^{-1}\nabla[\boldsymbol{m}\cdot(\boldsymbol{v}_m\times\boldsymbol{E})] = \nabla(\boldsymbol{P}_m\cdot\boldsymbol{E}) = (\boldsymbol{P}_m\cdot\nabla)\boldsymbol{E}$ represents the force on the electric dipole moment $\boldsymbol{P}_m = c^{-1}\boldsymbol{v}_m\times\boldsymbol{m}$ of the moving $\boldsymbol{m}$. Thus, the force on the moving magnetic dipole $\boldsymbol{m}$ turns out to be given by the same expression as derived by Boyer in his classical interpretation of the AC effect.[10–14] The experimental evidence for the existence of the local force is given by the experiments cited in Refs. [21–24].

In summary, we have derived within classical electrodynamics the expressions (14) and (15) for the effective interacting force on the charge q and on the dipole $\boldsymbol{m}$, respectively. Starting from first principles, we describe the isolated system by means of the total tensor $T^{\mu\nu}$ assuming the validity of the continuity equation $\partial_\mu T^{\mu\nu} = 0$ and the conservation law of the total linear momentum. Our effective force expressions, which solve the Shockley–James paradox, account for a force term acting in the direction of motion. This term foresees that an observable phase shift must take place between interfering particles encircling the magnetic flux in the AB effect. The phase shift occurs because of the relative lag of the particles produced by the local action of the force. The best experimental evidence for the existence of the longitudinal force term in the AB and AC effects is given by the observed phase shift in the tests of Refs. [19–24].

The tradition in physics, based on cause and effect, requires observed effects to be explained by means of the local causes that produce them, be their origin due to electromagnetic interaction or even interacting quantum systems. With our approach, the observed AB phase shifts can be interpreted in terms of the action of local em effective forces, reinforcing the view of the classical origin of the effects of the AB type. For a conclusive interpretation of the Aharonov–Bohm effect it is essential, as suggested in Ref. [9], to close the loopholes that exist in the tests of the em interaction. Ideally, tests of the em forces acting in the q–$\boldsymbol{m}$ interaction (some are discussed in Refs. [30,31]) should aim at verifying the validity of the action and reaction principle, the conservation laws, and the conclusive expressions of both forces $\boldsymbol{f}_q$ and $\boldsymbol{f}_m$.

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