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Self-Similarity Breaking: Anomalous Nonequilibrium Finite-Size Scaling and Finite-Time Scaling

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Symmetry breaking plays a pivotal role in modern physics. Although self-similarity is also a symmetry, and appears ubiquitously in nature, a fundamental question arises as to whether self-similarity breaking makes sense or not. Here, by identifying an important type of critical fluctuation, dubbed 'phases fluctuations', and comparing the numerical results for those with self-similarity and those lacking self-similarity with respect to phases fluctuations, we show that self-similarity can indeed be broken, with significant consequences, at least in nonequilibrium situations. We find that the breaking of self-similarity results in new critical exponents, giving rise to a violation of the well-known finite-size scaling, or the less well-known finite-time scaling, and different leading exponents in either the ordered or the disordered phases of the paradigmatic Ising model on two- or three-dimensional finite lattices, when subject to the simplest nonequilibrium driving of linear heating or cooling through its critical point. This is in stark contrast to identical exponents and different amplitudes in usual critical phenomena. Our results demonstrate how surprising driven nonequilibrium critical phenomena can be. The application of this theory to other classical and quantum phase transitions is also anticipated.

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Symmetry breaking is well understood, and plays a pivotal role in modern physics. Although self-similarity is a kind of symmetry, and appears ubiquitously in nature,[1,2] a fundamental question arises as to whether self-similarity breaking makes sense or not, given that, in nature, self-similarity holds inevitably only within a certain range of scales,[2] in contrast to rigorous mathematical objects such as fractals, which are self-similar on all scales.[1]

Critical phenomena[3–5] are generic systems with self-similarity up to a diverging correlation length ξ.[6] Their self-similarity can be limited by a finite system size L. However, it remains up to L, and appears as finite-size scaling (FSS).[7–10] The FSS has been verified theoretically,[11,12] numerically,[9,10] and even to an extent experimentally,[13] and is the most widely used numerical method to extract critical properties.[14] The issue of FSS above upper critical dimensions has also recently been settled.[15,16] Dynamic FSS has also been confirmed.[17,18] Can FSS fail? If yes, self-similarity may be broken.

Critical phenomena are also temporally self-similar up to a diverging correlation time.[4,19,20] To avoid the resultant critical slowing down,[21,22] one can also restrict the self-similarity. This is achieved by driving a system through its critical point at a finite rate R.[23,24] This rate gives rise to a finite timescale, $R^{-z/r}$, which is experimentally controllable, and serves as the temporal analogue of L in FSS, where z is the dynamic critical exponent, and r is the exponent associated with R.[25] As a result, self-similarity is reflected in finite-time scaling (FTS).[23,24] Moreover, the system itself is driven off equilibrium when the finite driven timescale becomes shorter than the diverging correlation time. FTS has also been successfully applied to many systems, both theoretically[24,26–39] and experimentally.[40,41] Its renormalization-group theory[42] has been generalized to a case with a weak driving of an arbitrary form, leading to a series of driven nonequilibrium critical phenomena, such as negative susceptibility and the competition of various regimes and their crossovers, as well as violation of the fluctuation-dissipation theorem and hysteresis.[32] Equilibrium and nonequilibrium initial conditions have also been considered.[32,43] Again, violation of FTS may signal the breaking of self-similarity.

When both spatial and temporal limitations are present,[24] a special revised FTS, containing both L and the rate of cooling, has been suggested and verified exactly at the critical point.[28,29] This gives rise to a distinctive leading exponent for the order parameter M in cooling, but not in heating.[28] However, in this case, no violation of FSS and FTS was discovered. A natural question follows from this: what about the whole driving process, rather than just the critical point?

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matic Ising model, whose equilibrium critical properties are well known, we first find that the averaged magnetization and its absolute value, $\langle m\rangle$ and $\langle|m|\rangle$, together with their respective fluctuations, all defined in the following, exhibit different behavior. This implies that the system consists of large clusters with finite m values in either the up or down direction, like the checkerboard shown schematically in Fig. 1, although real clusters may have irregular boundaries, various sizes with the block size as their average, identical neighboring colors, and smaller clusters of the other spin direction in them with an average of roughly m. These large clusters can flip between the two directions, giving rise to the difference between $\langle m\rangle$ and $\langle|m|\rangle$. These can be regarded as the up and down phases comprising the ordered phase. We refer to these fluctuating phases as phases fluctuations. The plural form of phase here is both to emphasize that at least two phases are involved, owing to symmetry breaking, and also in order to distinguish the phenomenon from the usual phase of a complex field.

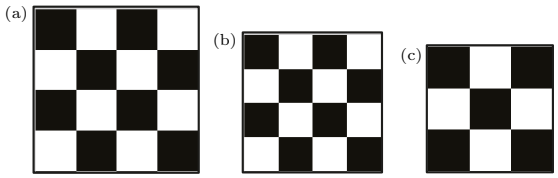

Fig. 1. Spatial self-similarity and its breaking. Black and white blocks represent the up and the down phases; their fluctuations represent phases fluctuations. The lateral sizes of each checkerboard and its blocks are L and $R^{-1/r}$, respectively. Here (a) and (b) are self-similar because they contain identical number of blocks, and thus share an identical $L^{-1}R^{-1/r}$, while (c) breaks the self-similarity because of its different $L^{-1}R^{-1/r}$.

We then show that when $L^{-1}R^{-1/r}$ or RL^r is fixed, the usual FTS and FSS of all observables studied are good, provided that corrections to scaling are negligible. However, once they are not fixed, the scalings for some observables in either the ordered phase or the disordered phase are violated in either heating or cooling. Given that the fixed variable is a ratio of two length scales, the lattice size L and the driven length $R^{-1/r}$, the above numerical results then indicate that the usual FTS is obeyed when different lattice sizes contain an identical number of clusters, viz., $L^{-1}R^{-1/r}$ is fixed. This is just the spatial self-similarity of the phases fluctuations illustrated in Fig. 1. It is then transparent that the violations of scaling, resulting from lifting the fixed value, originate from the breaking of such a self-similarity in the phases fluctuations. In FSS, the whole lattice contains on average only one fluctuating phase, which, however, turns over to the other phase in an average time of about L^z. The temporal self-similarity of the phases fluctuations and its breaking then relate to whether the driven timescale, $R^{-z/r}$, during which the driving changes appreciably, is fixed times of L^z for different lattice sizes, and thus whether RL^r is fixed or not. Note that these kinds of self-similarity are related to the lattice sizes and the external driving, and are different from the usual self-similarity of the critical fluctuations mentioned above. They are not time reversal symmetry, and hence not simply broken by the driving. They are not the similarity of morphology either, because the phases are fluctuating.

Moreover, we demonstrate that the breaking of self-similarity, abbreviated as Bressy, results in Bressy exponents that give rise to different leading exponents for either the ordered or the disordered phases, either upon heating or upon cooling, in stark contrast to identical exponents, but with different amplitudes for the two phases in equilibrium critical phenomena. Note that the phases here refer to the two phases involved in the transition, whereas the fluctuating phases in phases fluctuations compose the ordered phase. Further, the Bressy exponents are probably new critical exponents, except in the case of FSS under cooling. Although many questions, such as how the Bressy results in the Bressy exponents, and how different behaviors cross over, are yet to be studied, the results found here demonstrate how surprising driven nonequilibrium critical phenomena can be. The application of this theory to other classical and quantum phase transitions is thus greatly anticipated.

We first recapitulate the theories of FSS and FTS, and indicate where contradictions will emerge. Consider a system of size L, driven from one phase through a critical point at T_c to another phase by changing the temperature T with a finite $R\,(>0)$, such that

$$T - T_c \equiv \tau = \pm Rt, \tag{1}$$

where $+$ $(-)$ corresponds to heating (cooling). We have chosen $t = 0$ at T_c for simplicity. We start with the scaling hypothesis for the susceptibility χ,

$$\chi(\tau, R, L^{-1}) = b^{\gamma/\nu}\chi(\tau b^{1/\nu}, Rb^r, L^{-1}b), \tag{2}$$

which can be derived from the renormalization-group theory,[11,12,23,24,42] where b is a scaling factor, and γ, β, and ν are the critical exponents for χ, M, and ξ, respectively. We have replaced the time t with R because they are related by Eq. (1). Moreover, from the same equation, we find[25] $r = z + 1/\nu$ because t transforms as tb^{-z}.[4,19,20]

From Eq. (2), the selection of length scale $b = R^{-1/r}$ leads to the FTS form[23,24,28,42]

$$\chi = R^{-\gamma/r\nu}\mathcal{F}_{T\chi}(\tau R^{-1/r\nu}, L^{-1}R^{-1/r}), \tag{3}$$

while assuming that $b = L$ results in

$$\chi = L^{\gamma/\nu}\mathcal{F}_{S\chi}(\tau L^{1/\nu}, RL^r), \tag{4}$$

which is the FSS form subject to driving, and where $\mathcal{F}_{T\chi}$ and $\mathcal{F}_{S\chi}$ denote the universal scaling function.

Similarly, M must behave[23,24,28,42]

$$M = R^{\beta/r\nu} F_{\rm TM}(\tau R^{-1/r\nu}, L^{-1}R^{-1/r}) \quad (5)$$

in the FTS regime, while in the FSS regime, it becomes

$$M = L^{-\beta/\nu} F_{\rm SM}(\tau L^{1/\nu}, RL^r), \quad (6)$$

where $F_{\rm TM}$ and $F_{\rm SM}$ are also scaling functions. Here we find the RL^r and $L^{-1}R^{-1/r}$ terms relating to self-similarity.

Generally, scaling functions are analytic for vanishingly small scaled variables.[23,24,28] This implies that $R^{-1/r} \ll |\tau|^{-\nu}$ and $R^{-1/r} \ll L$ in the FTS regime, for example. In other words, the driven length scale $R^{-1/r}$ is the shortest of either $\xi \sim |\tau|^{-\nu}$ or L. Therefore, in the FTS (FSS) regime, $L^{-1}R^{-1/r}$ (RL^r) is negligible, and the leading singularity is merely the factor in front of each scaling function. However, in these specific regimes, we will see that the scaled variables cannot in fact be neglected for some observables affected strongly by the self-similarity of phases fluctuations. We note for completeness that when $L^{-1}R^{-1/r}$ (RL^r) is large, crossover to the FSS (FTS) regime occurs.[24,28]

Consider the standard Ising model with the Hamiltonian: $\mathcal{H} = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j$, which describes the interaction $J(>0)$ of a spin $\sigma_i = \pm 1$ on site i of a simple square or cubic lattice with its nearest neighbors. Periodic boundary conditions are applied throughout. We employ the single-spin Metropolis algorithm[44] and interpret it as dynamics.[14,45] The time unit is the standard MC step per site, which contains L^d attempts (d the spatial dimensionality) to randomly update the spins. In all simulations, we prepare the system in an ordered (disordered) initial configuration at a negative initial time then heat/cool it, respectively, through $T_{\rm c}$ at $t = 0$, according to a given R. We check that other details of the initial states produce no differences once they are sufficiently far away from $T_{\rm c}$, since they can then equilibrate quickly. We use 30000 samples to calculate an average. Tripling that number only smoothes the curves, with no appreciable displacements. We study primarily the two-dimensional (2D) model, whose $T_{\rm c} = 2J/\ln(1+\sqrt{2}) \approx 2.269J$, $\beta = 1/8$, $\nu = 1$, $\gamma = 7/4$,[4] and where $z = 2.167$.[46] For the 3D model, $T_{\rm c} = J/0.2216595(26) = 4.51142(6)J$,[47] $\nu = 0.6301(4)$, $\beta = 0.3265(3)$, $\gamma = 1.2372(5)$,[5,47,48] and $z = 2.055$.[28,49,50]

We then study the following two sets of observables:

$$\langle m\rangle = \Big\langle \frac{1}{L^d}\sum_{i=1}^{L^d}\sigma_i \Big\rangle, \quad \langle |m|\rangle = \Big\langle \Big|\frac{1}{L^d}\sum_{i=1}^{L^d}\sigma_i\Big| \Big\rangle, \quad (7)$$

$$\chi = L^d(\langle m^2\rangle - \langle m\rangle^2), \ \chi' = L^d(\langle m^2\rangle - \langle |m|\rangle^2), \quad (8)$$

where the angular brackets represent ensemble averages. We will generally refer to M and χ for both definitions, and stipulating a specific one when so indicated. The first set contains the usual definitions of the order parameter and its susceptibility, while the second set is employed when $\langle m\rangle = 0$ in the absence of symmetry breaking, and thus absolute values are needed. We will see that investigating both sets helps to reveal the phases fluctuations and their effects.

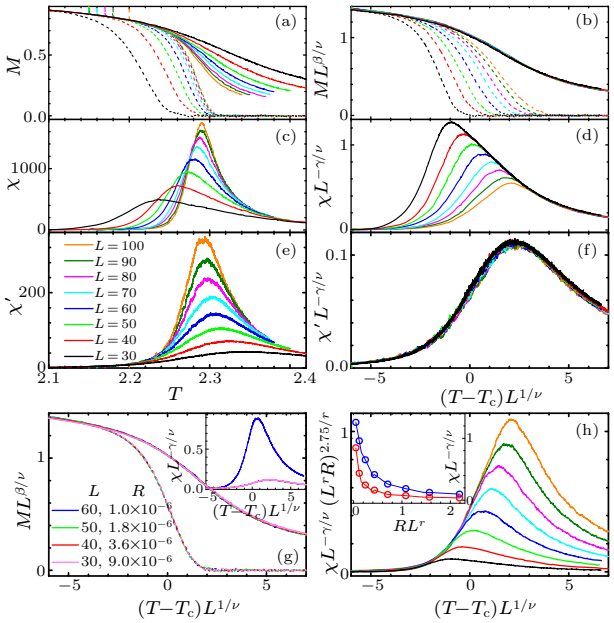

Fig. 2. FSS for a 2D Ising model during heating. (a) M and (b) its rescaled form, (c)–(f) χ and its rescaled form at $R = 0.000001$ for the different lattice sizes given in the legend in (e), which applies to all panels except (g). (g) FSS of M and χ (inset) for $RL^r = 10^{-6}\times 60^{3.167} \approx 0.433$, for four lattice sizes given in the legend in (g). (h) revised FSS for the ordered phase. The inset in (h) displays the dependence of $\chi L^{-\gamma/\nu}$ on RL^r at $\tau L^{1/\nu} = -1$ (blue) and -2 (red). The error bar of each datum is estimated to be no larger than the symbol size. Lines connecting symbols are only a guide to the eyes. In (a), (b) and (g), the dashed lines represent the results of $\langle m\rangle$ and χ, and the solid lines those of $\langle |m|\rangle$ and χ'.

Based on Fig. 2, it is evident that $\langle |m|\rangle$ (solid lines) and $\langle m\rangle$ (dashed lines) in heating differ and that the peak temperatures of χ and χ' even exhibit a qualitatively different dependence on R. However, what is most remarkable is that the FSS values of both $\langle m\rangle$ and χ at the low-T side, referred to roughly as the ordered phase in the following, are completely violated, though those of $\langle |m|\rangle$ and χ' and even χ at the high-T side or the disordered phase are good. In fact, $\langle |m|\rangle$ and χ' were utilized in the early verification of FSS.[14,51]

We now argue that the stark difference between $\langle m\rangle$ and $\langle |m|\rangle$ originates from the phases fluctuations. From Fig. 3, it can be seen that m fluctuates between up and down, and that its magnitude decreases as time elapses. Given that the fluctuations of m

are stochastic, $\langle m\rangle$ vanishes, but $\langle|m|\rangle$ is finite at a fixed $T > T_c$, in agreement with the same curves in Fig. 2(a). At $T < T_c$, the frequency of the fluctuations is temperature-dependent. Accordingly, in nonequilibrium heating, $\langle m\rangle$ can also be finite at sufficiently low temperatures, as was also the case in Fig. 2(a). In the FSS regime, clusters are on average of the size of L, such that their m is simply their magnetization. Since T increases with t in heating, the magnitude of m thus decreases with t. These results indicate that these large clusters roughly assume equilibrium in terms of magnetization. A further corroboration is that the $\langle|m|\rangle$ and $\langle m\rangle$ of different lattice sizes converge to an envelope at low temperatures, as illustrated in Fig. 2. These results permit us to refer to these large clusters of predominantly up or down spins as phases, and their temporal fluctuations as phases fluctuations. In FSS, these appear quite natural. In FTS, $L \gg R^{-1/r}$, and the large clusters have a size of $R^{-1/r}$ on average. The phases fluctuations can thus also be spatial, as illustrated in Fig. 1. These result in large critical fluctuations, because both originate from the dissymmetric phases of the symmetry-broken phase. This is apparent based on the large difference between χ and χ' in Figs. 2(c) and 2(e).

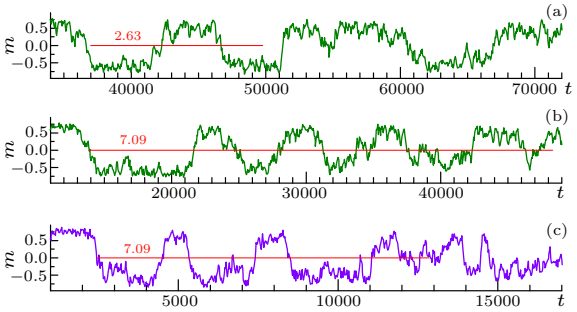

Fig. 3. Time evolution of m for (a) $L = 50$ and $R = 0.00001$, (b) $L = 50$ and $R \approx 0.00018$, and (c) $L = 30$ and $R \approx 0.000090$, the last two values fixing $L^{-1}R^{-1/r} \approx 0.433$ are as given in Fig. 2. The lengths of the red line segments, which are placed exactly at $m = 0$, are the driven timescale $R^{-z/r}$ in (a) and $4R^{-z/r}$ in (b) and (c). The red numbers over the lines are the results of the corresponding timescales divided by L^z. In all panels, $t = 0$ at T_c.

We next show that the violation of FSS stems from the breaking of the self-similarity of the phases fluctuations. As seen in Figs. 2(b), 2(d), and 2(f), the scaling is almost perfect for $\langle|m|\rangle$ and χ in the disordered phase, and is very good for χ'. Similar results are also found for other R values. These indicate that the system falls in the FSS regime as designed, and as such, RL^r can be neglected in accordance with Eqs. (4) and (6). Why, then, can the FSS of $\langle m\rangle$ and χ be bad in the same regime? We note from Fig. 2(g) that if RL^r is fixed, the violated FSS will recover completely. Therefore, the violations can only arise from RL^r. As pointed out above, fixing RL^r serves to ensure the temporal self-similarity of the phases fluctuations; thus, it is the breaking of this self-similarity that leads to the violation, and nothing else. This is illustrated in Fig. 3. Upon fixing RL^r, the two different-sized lattices, possessing different L^z, have roughly similar numbers of phases fluctuations, and thus temporal self-similarity.

We now find a breaking of self-similarity, or Bressy, exponent, defined as an extra singularity originating from the Bressy itself. Since the dependence differs in the ordered and the disordered phases, we distinguish these by means of the subscripts $\mp$, respectively, though a single $\mathcal{F}_{S\chi'}$ and $F_{S\langle|m|\rangle}$ suffice, as seen in Figs. 2(b) and 2(f). Based on the inset in Fig. 2(h), the dependence of $\mathcal{F}_{S\chi-}$ on RL^r is clearly singular. Indeed, the good collapse in Fig. 2(h) shows that $\mathcal{F}_{S\chi-}(\tau L^{1/\nu}, RL^r) \propto (RL^r)^{\sigma/r}$ with $\sigma = -2.75 \pm 0.15$ is consistent with the power-law exponent fitted out from the inset in Fig. 2(h). It can also be shown that the same σ accounts for the violated $\langle m\rangle$ scaling as well, corroborating to the self-similarity-breaking mechanism.[52] We note, however, that for large RL^r, the collapse becomes less good; for $R = 0$, the standard FSS ought to recover, and a crossover may occur. Nevertheless, with σ, the leading behavior of χ in heating is $L^{\gamma/\nu+\sigma}R^{\sigma/r}$ in the ordered phase, manifestly different from the usual $L^{\gamma/\nu}$ in the disordered phase, and in sharp contrast to the simple amplitude difference in equilibrium critical phenomena.

For the 2D Ising model, the $-\sigma$ value found may be $(2\gamma-6\beta)/\nu$ or $(\gamma+8\beta)/\nu$, or other combinations such as $d + 6\beta/\nu$ using $2\beta + \gamma = d\nu$.[3,4] However, these expressions are strange, in that they are unlikely to have direct explanations, and thus σ is more likely to be a new exponent, whose value coincides with that of the expressions. Further evidence of this is that, in the 3D model, we find $\sigma = -d \pm 0.15$,[52] which is markedly different from the above 2D expressions, although their numerical values slightly overlap within the errors. Theories and results from other models are thus urgently required.

Under cooling, $\langle m\rangle$ is vanishingly small, due to the absence of a symmetry-breaking field. As such, $\langle m\rangle$ and $\langle|m|\rangle$ are completely different, indicating once again the role of the phases fluctuations. Here, the FSS of all the observables studied appears good in the 2D model at first sight, as depicted in the leftmost curve in Fig. 4(a) for χ'. The quality of its scaling collapse is even similar to that of the middle curve, with a fixed RL^r, and therefore temporal self-similarity. However, where $\sigma = \beta/2\nu$, the collapse in the ordered phase appears better, and that in the disordered phase worse, as the rightmost curve in Fig. 4(a) manifests itself, although χ exhibits no such behavior. This again demonstrates possible different leading singularities in

the two phases.

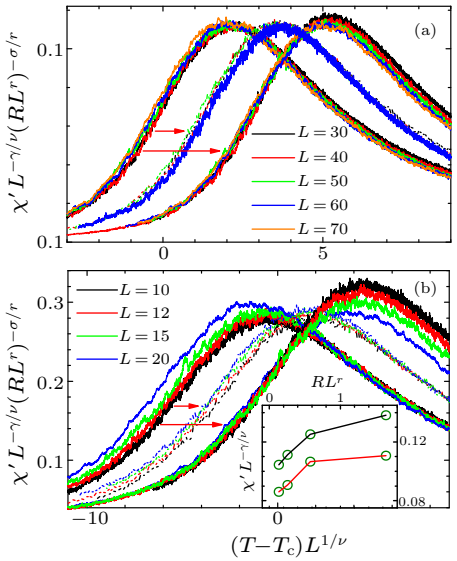

Fig. 4. FSS of χ' for (a) 2D and (b) 3D Ising models in cooling, with fixed $R = 0.000003$ in (a) and $R = 0.00003$ in (b) on the lattices indicated. The arrows indicate the displacements of the curves by 1.5 and 3 in (a), and 2.5 and 5 in (b), for clarity. The rightmost curves in both panels have $\sigma = \beta/2\nu$ while the others have $\sigma = 0$. The middle curves in (a) and (b) have fixed $RL^r \approx 1.30$ and 0.548, respectively, on four lattices given by the same legends. These are denoted by dashed lines, except for the same blue one on $L = 60$. The inset in (b) depicts $\chi' L^{-\gamma/\nu}$ at $\tau L^{1/\nu} = -8$ (black) and -9 (red) vs RL^r. The errors can be as much as double the symbol sizes, due to the small R, and the lines are only a guide to the eyes.

The FSS in cooling in two dimensions does not appear clear-cut. Nevertheless, as illustrated by the two left curves in Fig. 4(b) for the 3D model, the difference in the FSS of χ' in the ordered phase between having and lacking self-similarity is now evident, even though the scaled curve with self-similarity also exhibits a slightly systematic dependence on L. This must stem from corrections to scaling,[53] particularly in the case of small L. From the rightmost curve in Fig. 4(b), we see that a $\sigma = \beta/2\nu$ again collapses quite well in the ordered phase, and is consistent with the power-law exponent from the inset in Fig. 4(b). Note that in cooling, the 2D and 3D expressions of σ are identical, in contrast with the case for heating. The large 3D numerical value of σ should be responsible for its visibility, as compared with the 2D case.

In Fig. 5(a), we display a complete violation of FTS for $\langle|m|\rangle$ in cooling, although the FTS of χ appears reasonably good. Because $\langle m\rangle$ is vanishingly small and differs from $\langle|m|\rangle$, the phases fluctuations are pivotal. Indeed, FTS becomes almost perfect once lattices of different sizes hold an identical number of phases of size $R^{-1/r}$; hence the spatial self-similarity[38] of the phases fluctuations is ensured by fixing $L^{-1}R^{-1/r}$, as shown in the inset in Fig. 5(a). Moreover, the fluctuating phases must satisfy the central limit theorem, and thus behave as $L^{-d/2}$ for large L.[28] This implies that $F_{T\langle|m|\rangle} \propto (L^{-1}R^{-1/r})^{d/2}$ singularly,[28] which is confirmed by the left curve in Fig. 5(b). Such a singularity has been invoked to rectify the leading behavior, resulting in the distinctive leading exponent in cooling exactly at T_c,[28] although no violation of FSS and FTS was discovered there. Considerations regarding the singularity notwithstanding, the ordered phase still seems weakly singular, as shown in the inset in Fig. 5(b). Indeed, a further $\beta/2\nu$, consistent with the power-law exponent in the inset, renders the collapse better in the ordered phase, while concomitantly worse in the disordered phase, demonstrating again the different leading singularities in the two phases. However, in contrast with FSS in cooling, $\sigma = \beta/4\nu$ for the 3D model[52] thus again could possibly represent a new exponent.

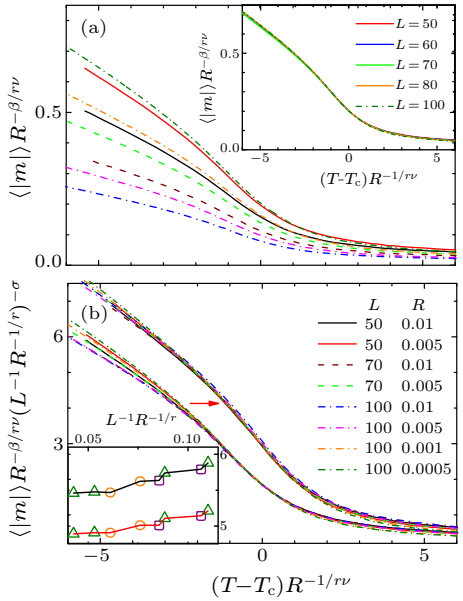

Fig. 5. (a) FTS of $\langle|m|\rangle$ and (b) its revised form in cooling on three fixed 2D lattices given in the legend in (b). The inset in (a) depicts the FTS of $\langle|m|\rangle$ for fixed $L^{-1}R^{-1/r} = 100^{-1} \times 0.0005^{-1/3.17} \approx 0.110$ on the lattices indicated. In (b), the left curve has $\sigma = d/2$, while the right curve has $\sigma = d/2 + \beta/2\nu$ and is shifted by 0.8, as indicated by the arrow. The inset displays $\langle|m|\rangle R^{-\beta/r\nu}(L^{-1}R^{-1/r})^{-d/2}$ at $\tau R^{-1/r\nu} = -4$ (red) and -5 (black) vs $L^{-1}R^{-1/r}$ for $L = 50$ (squares), 70 (circles), and 100 (triangles). Error bars and lines are similar to those in Fig. 2.

In Fig. 6(a), we shows the FTS of χ on heating, for several fixed lattices. It is apparent that χ and χ' differ at least in the disordered phase, and thus phases fluctuations are again indispensable, although the difference appears at higher temperatures for larger lattices. In contrast to FSS, it is the FTS of χ' in the disordered phase that is violated, while χ and $\langle m\rangle$ display good FTS for the R employed, and require only a single scaling function. The two sets of curves with fixed and non-fixed $L^{-1}R^{-1/r}$, hav-

ing and lacking, respectively, self-similarity, apparently differ, even though the curves with self-similarity appear not to collapse well in the disordered phase either. This is attributed to corrections of scaling, since the curves become closer to one another for larger L, hence smaller R, and smaller corrections. Again, the dependence on $L^{-1}R^{-1/r}$ is clearly singular, as illustrated in the inset in Fig. 6(b). Indeed, the selection of $\sigma = -0.625 \pm 0.025$, consistent with the power-law exponent in the inset, collapses the curves in the disordered phases, showing again the different leading behaviors in the two phases. Here, σ could be $(4\beta - \gamma)/2\nu$, among others, but again, is more likely to be a new exponent, whose value falls within the range found. Corroboration can once again be found in the 3D model $\sigma \approx (-0.58 \pm 0.05)/\nu$, which could arise from a completely different form $(2\beta - \gamma)/\nu$.[52] We note that the numerical values given for the two models overlap again within the errors.

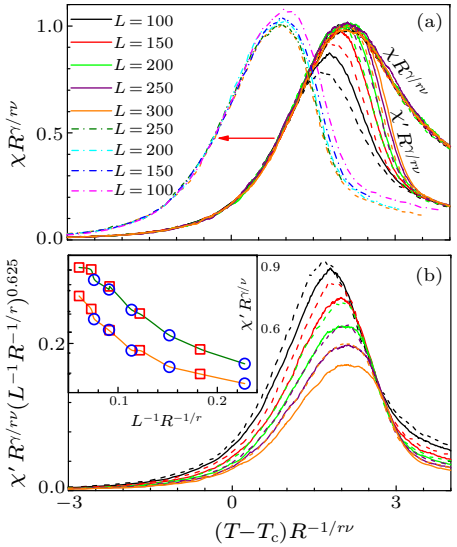

Fig. 6. (a) FTS of χ and χ', marked on the right, and (b) the revised FTS of χ' in heating on fixed 2D lattice sizes given by the first five items in the legend, which applies to both panels. The left curve of $\chi' R^{\gamma/r\nu}$ in (a) shifted by -1.2, as indicated by the arrow, has fixed $L^{-1}R^{-1/r} = 300^{-1} \times 0.00005^{-1/r} \approx 0.0758$, containing the last five lattice sizes in the legend. Solid and dashed lines of identical colors have $R = 0.0001$ and 0.00005, respectively. The inset in (b) displays the dependence on $L^{-1}R^{-1/r}$ of $\chi' R^{\gamma/r\nu}$ at $\tau R^{-1/r\nu} = 2.5$ (olive) and 2.7 (orange) for $R = 0.0001$ (squares) and 0.00005 (circles). Error bars and lines are similar to those in Fig. 2.

In conclusion, we have discovered a unique type of self-similarity breaking, by identifying an important kind of critical fluctuation, dubbed 'phases fluctuations', capable of exhibiting spatial or temporal self-similarity and its breaking. We have shown that such self-similarity breaking results in new breaking-of-self-similarity exponents, which lead to violation of either the well-known finite-size scaling or the less well-known finite-time scaling, and different leading critical exponents in either the ordered or the disordered phases of the Ising model when heated or cooled through its critical point, in stark contrast to identical leading critical exponents and different amplitudes in more usual critical phenomena. These results demonstrate how surprising driven nonequilibrium critical phenomena can be. The application of this phenomenon to other classical and quantum phase transitions is greatly anticipated.

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