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Ideal Spin Hydrodynamics from the Wigner Function Approach

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Based on the Wigner function in local equilibrium, we derive hydrodynamical quantities for a system of polarized spin-1/2 particles: the particle number current density, the energy-momentum tensor, the spin tensor, and the dipole moment tensor. Compared with ideal hydrodynamics without spin, additional terms at the first and second orders in the Knudsen number Kn and the average spin polarization χ_s have been derived. The Wigner function can be expressed in terms of matrix-valued distributions, whose equilibrium forms are characterized by thermodynamical parameters in quantum statistics. The equations of motion for these parameters are derived by conservation laws at the leading and next-to-leading order Kn and χ_s.

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Large orbital angular momenta (OAM) perpendicular to the reaction plane can be generated in non-central relativistic heavy ion collisions. Part of initial OAM are converted to spin angular momenta of quarks through the spin-orbit coupling,[1,2] similar to the Einstein-de Hass and Barnett effects.[3,4] At the freeze-out stage, these polarized quarks combine to form polarized hadrons. The polarization of hyperons along the global OAM (global spin polarization) has recently been measured in the STAR experiment.[5,6] According to quantum statistics,[7–9] the spin polarization is connected to the thermal vorticity $\omega^{\rm th}_{\mu\nu} \equiv -(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)/2$, where $\beta_\mu = u_\mu/T$ is the ratio of the flow velocity u_μ to the temperature T. Hydrodynamical and transport models[10–14] can describe the experimental data of the global Λ polarization. However, the longitudinal polarization of Λ's calculated in hydrodynamical and transport models[15] has an opposite sign to experimental results,[16] which is called a "sign problem". Several attempts have been made in explaining the sign problem, including different choices of spin potentials,[17] non-relativistic connection between the polarization and vorticity,[18] off-equilibrium contributions according to transport model simulations,[19] the contribution from the shear-tensor,[20–23] and the equation of states and freeze-out temperatures.[24] However, at present no definite conclusion has been drawn. Meanwhile, it is expected for a parity violation to exist in a hot QCD matter, which may be observed through the chiral magnetic effect (CME).[25–27] Isobar collisions have been carried out by the STAR collaboration[28–30] but no CME signatures have been observed.[31] In order to better understand the sign problem and search for the CME signature, more theoretical efforts and numerical simulations for the dynamical evolution of spin degrees of freedom are needed, see e.g., Refs. [32–35] for recent reviews of spin effects in heavy ion collisions.

One way to describe the evolution of spin degrees of freedom in the quark gluon plasma is through relativistic spin hydrodynamics. Relativistic viscous hydrodynamics (without spin) is a successful framework to describe the evolution of strong interaction matter in heavy-ion collisions and therefore is widely used in numerical simulations.[36–41] Compared with the spinless case, the spin hydrodynamics includes an additional conservation law for the total angular momentum, which allows a conversion between spin and OAM. The spin hydrodynamics was proposed many years ago.[42] Recent developments in the relativistic hydrodynamics include the effective theory approach,[43–45] the derivation using the local equilibrium distribution functions[46–49] introduced in Ref. [9]. The second order hydrodynamical equations can be derived in many approaches, such as the generating function method,[50] relaxation time approximation,[51,52] and entropy current analysis based on the second law of thermodynamics.[53–56]

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In this Letter, we will derive relativistic spin hydrodynamics for ideal fluids from Wigner functions. The conventional ideal hydrodynamics can be recovered in the spinless case. The Wigner function is expressed in terms of matrix-valued distributions, whose local equilibrium forms are given by quantum statistics.[9] We first obtain the Wigner function up to the second order in space-time derivatives by solving the equation of the Wigner function. From the Wigner function, we derive hydrodynamical quantities as functions of thermodynamical parameters: the temperature, the flow velocity, the chemical potential and the spin potential. The equations of motion for these parameters are obtained from conservation laws.

Throughout this work, we use natural units $\hbar = c = k_{\rm B} = 1$. The antisymmetric tensor is defined as $A^{[\mu}B^{\nu]} \equiv (A^\mu B^\nu - A^\nu B^\mu)/2$, while the symmetric one is defined as $A^{\{\mu}B^{\nu\}} \equiv (A^\mu B^\nu + A^\nu B^\mu)/2$.

Power Counting. Before we start our discussion, we want to clarify the power counting scheme used throughout this study. In an ordinary Navier–Stokes-type fluid, it is usually assumed that the hydrodynamical limit can be reached when microscopic details in the system can be safely neglected. This limit is quantified as $Kn \equiv l/L \ll 1$, where the Knudsen number Kn is defined as the ratio of the typical microscopic length scale l to the macroscopic length scale L. In our discussion, l is associated with the microscopic mean free path, while L is described by $1/L \sim |\partial_\mu O/O|$, the typical scale of the gradient of the macroscopic quantity O, such as the charge and energy densities. Therefore, a gradient expansion is equivalent to an expansion with respect to the Knudsen number.

For systems with spin degrees of freedom, we introduce an additional parameter χ to quantify spin polarization. This parameter can be given by either the average spin polarization per particle or the average magnetic dipole moment,

$$\chi_s \sim \frac{|\mathcal{S}^{\lambda,\mu\nu}|}{n} \sim \frac{|\mathcal{M}^\mu|}{n}, \tag{1}$$

where $\mathcal{S}^{\lambda,\mu\nu}$ is the spin tensor density, $\mathcal{M}^\mu$ is the magnetic dipole moment density, and n is the particle number density. If spin effects are induced by the thermal vorticity as in global equilibrium, we expect $\chi_s \sim Kn$ and therefore spin effects are at least first order in the Knudsen number. In this study we consider a weakly polarized system by demanding $\chi_s \lesssim Kn$ with two expansion parameters χ_s and Kn. For physical observables, we keep zeroth order terms $\mathcal{O}(1)$, first order terms $\mathcal{O}(Kn)$, $\mathcal{O}(\chi_s)$, as well as second order terms $\mathcal{O}(Kn^2)$, $\mathcal{O}(Kn\,\chi_s)$, and $\mathcal{O}(\chi_s^2)$.

Wigner Function in Thermal Equilibrium. The distribution function $f(t,\boldsymbol{x},\boldsymbol{p})$ for a classical particle is the probability of finding a particle with the momentum $\boldsymbol{p}$ at the space-time point $(t,\boldsymbol{x})$. However, the position and the momentum of a quantum particle cannot be determined simultaneously according to the uncertainty principle. Thus the quantum analogue of $f(t,\boldsymbol{x},\boldsymbol{p})$ is the covariant Wigner function defined as a two-point correlation function[57,58]

$$W(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \Big\langle : \bar\psi\Big(x+\frac{y}{2}\Big) \otimes \psi\Big(x-\frac{y}{2}\Big) : \Big\rangle, \tag{2}$$

where $\langle :: \rangle$ denotes the ensemble average and $\otimes$ denotes the tensor product of two matrices, i.e., $[A\otimes B]_{\alpha\beta} = A_\beta B_\alpha$. One can verify that x^μ is commutable with p^μ because x^μ is the central position of two points, while p^μ is the momentum conjugate to the relative distance between two points. This indicates that the Wigner function is well-defined in phase space. Note that we neglected gauge fields such as electromagnetic fields throughout this study. If a gauge field is considered, one should put a gauge link between $\bar\psi(x+y/2)$ and $\psi(x-y/2)$ as in Refs. [57,59].

The kinetic equation of the Wigner function can be derived from the Dirac equation,[58,59]

$$\Big[\gamma_\mu\Big(p^\mu + \frac{i}{2}\partial^\mu\Big) - m\Big] W(x,p) = 0. \tag{3}$$

At zeroth order in gradient, the Wigner function has the following solution:

$$\begin{aligned} W_0(x,p) = {}& \frac{1}{(2\pi)^3}\delta(p^2-m^2) \\ & \times \sum_{rs} \{\theta(p^0)[\bar u_s(\boldsymbol{p}) \otimes u_r(\boldsymbol{p})] f^+_{rs}(x,\boldsymbol{p}) \\ & - \theta(-p^0)[\bar v_s(-\boldsymbol{p}) \otimes v_r(-\boldsymbol{p})] f^-_{rs}(x,-\boldsymbol{p})\}, \end{aligned} \tag{4}$$

which can be obtained by using the quantized form of the free Dirac field $\psi(x)$ and $\bar\psi(x)$ in Eq. (2). The matrix-valued distributions $f^\pm_{rs}(x,\boldsymbol{p})$ are constructed by ensemble averages of creation and annihilation operators for particles and antiparticles.[60,61] The trace $\sum_s f^\pm_{ss}(x,\boldsymbol{p})$ is identified as the number density in phase space for particles/antiparticles. A projection of $f^\pm_{rs}(x,\boldsymbol{p})$ onto Pauli matrices gives the polarization density in the local rest frame. These distributions satisfy the kinetic equations

$$\bar p \cdot \partial f^\pm_{rs}(x,\boldsymbol{p}) = 0, \tag{5}$$

where $\bar p^\mu \equiv (E_{\boldsymbol{p}}, \boldsymbol{p})$ with $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2+m^2}$. Equations (3) and (5) do not contain collision terms that have been derived in Ref. [62–65]. In this study, collision terms are neglected because we focus on systems in global or local thermal equilibrium.

The first and second order corrections in space-time gradient for the Wigner function can be obtained by solving the kinetic Eq. (3) (see Section I in the Sup-

plemental Material),

$$\delta W(x,p) = \frac{i}{4\,m}[\gamma^\mu, \partial_\mu W_0(x,p)] + \frac{1}{16m^2}(\gamma\cdot\partial)W_0(x,p)(\gamma\cdot\overleftarrow{\partial}) + \frac{\gamma\cdot p + m}{8m(p^2-m^2)}\partial^2 W_0(x,p). \quad (6)$$

Thus, the full solution of Eq. (3) is $W = W_0 + \delta W$. The expansion in space-time gradient is equivalent to that in the Planck constant $\hbar$. The first order contribution agrees with the results in Refs. [66–69].

The appearance of δW is a result of the uncertainty principle for quantum particles. If a system consists of point particles, the Wigner function will be given by W_0 only, with the momentum being constrained by the normal mass-shell condition $p^2 = m^2$. However, for quantum particles with finite space volume due to the constraint of the uncertainty principle, even though we neglect interactions among particles, the Wigner function contains corrections from non-local correlation, described by δW in Eq. (6). These corrections include the electric dipole moment induced by an inhomogeneous charge distribution, the magnetization current, and the off-mass-shell correction. The first two corrections are widely known and can be found even in classical electrodynamics.[70] The off-mass-shell correction takes the form of a second order space-time derivative, which can also be obtained from the mass-shell equation of the Wigner function $(p^2 - m^2 - \partial^2/4)W(x,p) = 0$. It has been recognized many years ago[58,59] but is rarely studied when the system varies rapidly within the Compton wavelength, i.e., when $Kn \ll 1$. In our power counting scheme, we identify them as second order corrections $\mathcal{O}(Kn^2)$.

In thermal equilibrium, we assume that $f^{\pm}_{rs}(x,\boldsymbol{p})$ have the following form as in Ref. [9],

$$f^{+}_{\mathrm{eq},rs}(x,\boldsymbol{p}) = \frac{1}{2\,m}\bar{u}_r(\boldsymbol{p}) \times (e^{\beta\cdot\bar{p}-\xi-\frac{1}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}+1)^{-1}u_s(\boldsymbol{p}),$$
$$f^{-}_{\mathrm{eq},rs}(x,\boldsymbol{p}) = -\frac{1}{2\,m}\bar{v}_r(\boldsymbol{p}) \times (e^{\beta\cdot\bar{p}+\xi+\frac{1}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}+1)^{-1}v_s(\boldsymbol{p}), \quad (7)$$

where $\bar{p}^\mu \equiv (E_{\boldsymbol{p}}, \boldsymbol{p})$ for both particles and antiparticles, $\sigma^{\mu\nu} \equiv (i/2)[\gamma^\mu, \gamma^\nu]$ with γ^μ being gamma matrices. The parameters in Eq. (7) are given by $\beta^\mu \equiv \beta u^\mu$, $\xi \equiv \beta\mu$, and $\omega^{\mu\nu} \equiv \beta\bar{\omega}^{\mu\nu}$, with β, u^μ, μ, and $\bar{\omega}^{\mu\nu}$ being the inverse temperature, the flow velocity, the chemical potential, and the spin potential, respectively. In global equilibrium, these thermodynamical parameters are space-time constants and then Eq. (7) can be derived rigorously from quantum statistics.[9] We assume that in local equilibrium, the distributions have the same expressions as those in global equilibrium but with space-time dependent $\beta^\mu(x)$, $\xi(x)$, and $\omega^{\mu\nu}(x)$.

We note that the spin degrees of freedom are dissipative in the quark-gluon plasma.[65,71,72] If the relaxation time of spin is much shorter than the typical time scale of the fluid evolution, the considered system will be more close to an instantaneous thermal equilibrium state in Eq. (7). However, the system can be off-equilibrium if the relaxation time of spin is relatively large. The relaxation time of spin is recently calculated in strongly coupled regime using the AdS/CFT correspondence,[71] or the perturbative QCD.[72] In this study, we assume that the spin relaxation is sufficiently fast, so the system is always in local equilibrium. This is in analogy to the case of ideal hydrodynamics, but equilibrium distributions in Eq. (7) contain spin polarizations described by the spin potential $\bar{\omega}^{\mu\nu}$.

In later calculation, we focus on a weakly polarized system satisfying $\chi_s \lesssim Kn \ll 1$. Then we can make expansions for thermodynamical parameters:

$$\omega^{\mu\nu} = \omega^{(1)\mu\nu} + \omega^{(2)\mu\nu} + \cdots,$$
$$\beta^\mu = \beta^{(0)\mu} + \beta^{(1)\mu} + \cdots,$$
$$\xi = \xi^{(0)} + \xi^{(1)} + \cdots, \quad (8)$$

where (i) with $i = 0, 1, 2, \cdots$ in superscripts denote orders in Kn and χ_s. Note that the order of magnitude of χ_s is determined by $\omega^{\mu\nu}$ in thermal equilibrium, which will be shown in later calculations, therefore the spin potential is at least first order $\omega^{\mu\nu} \sim \mathcal{O}(\chi_s)$ and its space-time derivative $\partial^\alpha\omega^{\mu\nu} \sim \mathcal{O}(Kn\,\chi_s)$ is at least second order. We also expand the distributions in Eq. (7) up to second order in χ_s in the Boltzmann limit,

$$f^{+}_{\mathrm{eq},rs}(x,\boldsymbol{p}) = \exp(-\beta\cdot\bar{p}+\xi) \times \Big[\Big(1+\frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\Big)\delta_{rs} + \frac{1}{8\,m}\omega^{\mu\nu}\bar{u}_r(\boldsymbol{p})\sigma_{\mu\nu}u_s(\boldsymbol{p})\Big],$$
$$f^{-}_{\mathrm{eq},rs}(x,\boldsymbol{p}) = \exp(-\beta\cdot\bar{p}-\xi) \times \Big[\Big(1+\frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\Big)\delta_{rs} + \frac{1}{8\,m}\omega^{\mu\nu}\bar{v}_r(\boldsymbol{p})\sigma_{\mu\nu}v_s(\boldsymbol{p})\Big]. \quad (9)$$

The Wigner function in local equilibrium can be obtained by inserting Eq. (9) into the zeroth order Wigner function (4) and higher order ones (6).

Hydrodynamical Quantities. According to its definition, the Wigner function satisfies $W^\dagger = \gamma^0 W \gamma^0$ and thus can be decomposed in terms of the generators of the Clifford algebra $\Gamma_i = \{1, i\gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^{\mu\nu}\}$,

$$W = \frac{1}{4}\Big(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\Big), \quad (10)$$

where 1 is the 4×4 unit matrix, γ^μ are gamma matrices, $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, and $\sigma^{\mu\nu} \equiv (i/2)[\gamma^\mu, \gamma^\nu]$. The expansion coefficients $\mathcal{F}$, $\mathcal{P}$, $\mathcal{V}_\mu$, $\mathcal{A}_\mu$, and $\mathcal{S}_{\mu\nu}$ are the scalar, pseudoscalar, vector, axial vector, and tensor components of the Wigner function, respectively, and they are all real.[59] The current density $J^\mu(x)$, the

energy-momentum tensor (density) $T^{\mu\nu}(x)$, and the spin tensor (density) $S^{\lambda,\mu\nu}$ can be obtained from $\mathcal{V}_\mu$ and $\mathcal{A}_\mu$ as follows:

$$J^\mu(x) = \int d^4p\, \mathcal{V}^\mu(x,p), \tag{11}$$

$$T^{\mu\nu}(x) = \int d^4p\, p^\nu \mathcal{V}^\mu(x,p), \tag{12}$$

$$S^{\lambda,\mu\nu}(x) = -\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\int d^4p\, \mathcal{A}_\rho(x,p). \tag{13}$$

Note that when including spin degrees of freedom, the energy-momentum tensor and the spin-tensor have the pseudo-gauge ambiguity.[73–75] Here we adopt the canonical definition in Eqs. (12) and (13) for $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$. Generally the canonical form of $T^{\mu\nu}$ will contain an antisymmetric part if the spin polarization is nonzero.

In local equilibrium, the Wigner function can be expressed in terms of the distributions (9). It is straightforward to extract the vector component of the Wigner function to obtain $J^\mu(x)$ and $T^{\mu\nu}(x)$ through Eqs. (11) and (12):

$$\begin{aligned}
J^\mu_{\rm eq}(x) =& \Big(1+\frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\Big)K_1 u^\mu \sinh\xi \\
&- \frac{1}{16m^2}\partial^2\Big[(2\beta K_2 - 5K_1)u^\mu \sinh\xi\Big] \\
&+ \frac{1}{4m^2}\partial_\nu\Big\{\Big[2u^{[\mu}\omega_\alpha^{\nu]}u^\alpha(K_2+\beta^{-1}K_1) \\
&+ \omega^{\mu\nu}(K_2-\beta^{-1}K_1)\Big]\sinh\xi\Big\}, \\
T^{\mu\nu}_{\rm eq}(x) =& \Big(1+\frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\Big)\Big(u^\mu u^\nu K_2 - \Delta^{\mu\nu}\beta^{-1}K_1\Big) \\
&\times\cosh\xi + \frac{1}{4m^2}\partial_\rho\Big(2\omega_\alpha^{[\mu}I^{\rho]\nu\alpha} + \omega^{\mu\rho}u^\nu m^2 K_1\Big) \\
&\times\cosh\xi + \frac{1}{16m^2}\partial^2\{[g^{\mu\nu}(2K_2 - 5\beta^{-1}K_1) \\
&- u^\mu u^\nu(K_2 + 2\beta m^2 K_1 + \beta^{-1}K_1)]\cosh\xi\}.
\end{aligned} \tag{14}$$

Here, u^μ is the flow velocity and K_n are functions of β as follows:

$$K_n(\beta) \equiv \frac{8}{(2\pi)^3}\int\frac{d^3\boldsymbol{p}}{2E_{\boldsymbol{p}}}E^n_{\boldsymbol{p}}e^{-\beta E_{\boldsymbol{p}}}. \tag{15}$$

They satisfy the following recursive relation

$$K_n = \frac{n+1}{\beta}K_{n-1} + m^2K_{n-2} - \frac{n-2}{\beta}m^2K_{n-3}, \tag{16}$$

which allows us to express K_n in terms of any K_i and K_j with $i \neq j$. In Eq. (14) and in the following, we choose K_1 and K_2 as basis functions to express K_n, since K_1 and K_2 are related to the particle number and energy density when neglecting second order terms, while other K_n do not have explicit physical meanings. The rank-3 moment $I^{\mu\nu\alpha}$ in $T^{\mu\nu}_{\rm eq}$ in Eq. (14) is given by

$$\begin{aligned}
I^{\mu\nu\alpha} =& u^\mu u^\nu u^\alpha K_3 + \frac{1}{3}(\Delta^{\mu\nu}u^\alpha \\
&+ \Delta^{\mu\alpha}u^\nu + \Delta^{\nu\alpha}u^\mu)(m^2K_1 - K_3),
\end{aligned} \tag{17}$$

where the projection operator is defined as $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$.

One can see in Eq. (14) that the spin dependent terms are at least second order in the space-time gradient. At zeroth order in Kn and χ_s, $J^\mu_{\rm eq}$ and $T^{\mu\nu}_{\rm eq}$ agree with the results in ideal hydrodynamics.[39] With the help of the fluid velocity u^μ, Eq. (14) can be rewritten as

$$\begin{aligned}
J^\mu_{\rm eq} &= n_{\rm eq}u^\mu + \delta j^\mu, \\
T^{\mu\nu}_{\rm eq} &= \epsilon_{\rm eq}u^\mu u^\nu - P_{\rm eq}\Delta^{\mu\nu} + \delta T^{\mu\nu}_S + \delta T^{\mu\nu}_A,
\end{aligned} \tag{18}$$

where $n_{\rm eq}$, $\epsilon_{\rm eq}$, and $P_{\rm eq}$ are the particle number density, the energy density, and the pressure, respectively, and they can be extracted as

$$\begin{aligned}
n_{\rm eq} \equiv& u_\mu J^\mu_{\rm eq} \\
=& \Big(1+\frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\Big)K_1\sinh\xi \\
&+ \frac{1}{4m^2}(\nabla_\mu u_\nu)\omega^{\mu\nu}(K_2+\beta^{-1}K_1)\sinh\xi \\
&- \frac{1}{2m^2}u_\mu\nabla_\nu[\omega^{\mu\nu}\beta^{-1}K_1\sinh\xi] \\
&- \frac{1}{16m^2}\partial^2[(2\beta K_2 - 5K_1)\sinh\xi], \\
\epsilon_{\rm eq} \equiv& u_\mu u_\nu T^{\mu\nu}_{\rm eq} \\
=& \Big(1+\frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\Big)K_2\cosh\xi \\
&+ \frac{1}{4m^2}(\nabla_\mu u_\nu)\omega^{\mu\nu}[5\beta^{-1}(K_2+\beta^{-1}K_1) \\
&+ m^2K_1]\cosh\xi - \frac{1}{2m^2}u_\mu\nabla_\nu[\omega^{\mu\nu}\beta^{-1}(K_2 \\
&+ \beta^{-1}K_1)\cosh\xi] + \frac{1}{16m^2}\partial^2[(K_2 - 6\beta^{-1}K_1 \\
&- 2\beta m^2K_1)\cosh\xi],
\end{aligned} \tag{19}$$

where $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$, and the pressure is obtained by $P_{\rm eq} \equiv -(1/3)\Delta_{\mu\nu}T^{\mu\nu}_{\rm eq}$, giving the trace anomaly

$$\begin{aligned}
\epsilon_{\rm eq} - 3P_{\rm eq} =& \Big(1+\frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\Big)m^2K_0\cosh\xi \\
&+ \frac{1}{16}\partial^2[(7K_0 - 2\beta K_1)\cosh\xi].
\end{aligned} \tag{20}$$

The residue terms δj^μ, $\delta T^{\mu\nu}_S$, and $\delta T^{\mu\nu}_A$ in Eq. (18) describe deviations from ideal hydrodynamics without spin. They are at least second order in the space-time gradient. We identify δj^μ as the magnetization current induced by the inhomogeneity of the spin polarization. Here $\delta T^{\mu\nu}_S$ and $\delta T^{\mu\nu}_A$ are symmetric and antisymmetric respectively for an interchange of μ and ν.

It is straightforward to read out the explicit expressions for δj^μ, $\delta T_S^{\mu\nu}$, and $\delta T_A^{\mu\nu}$ from Eq. (14). They contain the heat flow and viscous tensor correction induced by the spin polarization that have been discussed in the quantum spin vorticity theory.[76] In our power counting scheme, these spin-induced terms are of order $\mathcal{O}(Kn\,\chi_s)$, which is much smaller than first-order viscous corrections, and comparable to second-order corrections in regular viscous hydrodynamics without spin. That is because in the quark-gluon plasma produced in heavy-ion collisions, the average spin polarization of particles would be $\sim$0.02, as indicated by the Λ's global polarization.[5,6] The weakly polarized condition $\chi_s \lesssim Kn$ therefore can be fulfilled (Kn in heavy-ion collisions has been calculated in Ref. [77]). Note that δj^μ and $\delta T_S^{\mu\nu}$ also contain second order terms $O(Kn^2)$. These terms are contributions from long-range correlations, which do not appear in regular ideal hydrodynamics.

Even though we deal with hydrodynamics in local equilibrium, we still obtain viscous terms in Eq. (14). Similar to the viscous hydrodynamics, the definition of the rest frame and u^μ is subtle because the charge and energy currents are in general not parallel when we include higher order corrections. One can either choose the Eckart frame with $u_E^\mu \equiv J_{\rm eq}^\mu/\sqrt{J_{\rm eq}^\nu J_{{\rm eq},\nu}}$, or the Landau frame with $T_{\rm eq}^{\mu\nu}u_{L,\nu} = \epsilon_{\rm eq}u_L^\mu$. The difference between these two choices is in the second order in Kn and χ_s, i.e., $u_E^\mu - u_L^\mu \simeq \mathcal{O}(Kn^2, Kn\,\chi_s, \chi_s^2)$.

For the angular momentum, we look at the spin tensor $S^{\lambda,\mu\nu}(x)$ and the dipole-moment tensor $M^{\mu\nu}(x)$. The spin tensor $S^{\lambda,\mu\nu}(x)$ is one part of the total angular momentum tensor $J^{\lambda,\mu\nu}(x) = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + S^{\lambda,\mu\nu}$. Note that the separation of spin and orbital angular momentum is not unique and subject to pseudogauge transformation.[73–75] Here we adopt the canonical definition for $S^{\lambda,\mu\nu}(x)$ as in Eq. (13). In local equilibrium, we can express it in terms of β, ξ, u^μ, and $\omega^{\mu\nu}$ as

$$S_{\rm eq}^{\lambda,\mu\nu}(x) = \frac{1}{4}(u^\lambda\omega^{\mu\nu} + 2u^{[\mu}\omega^{\nu]\lambda})K_1\cosh\xi, \tag{21}$$

where K_1 is defined in Eq. (15). The spin potential $\omega^{\mu\nu}$ contains six independent variables. However, $S_{\rm eq}^{\lambda,\mu\nu}$ only contains three independent variables because it only depends on $\tilde{\omega}^\mu \equiv \epsilon^{\mu\nu\alpha\beta}u_\nu\omega_{\alpha\beta}/2$,

$$S_{\rm eq}^{\lambda,\mu\nu}(x) = \frac{1}{4}\epsilon^{\lambda\mu\nu\beta}\tilde{\omega}_\beta K_1\cosh\xi. \tag{22}$$

In order to study the remaining three degrees of freedom of $\omega^{\mu\nu}$, we look at the dipole momentum tensor defined as

$$D^{\mu\nu}(x) = \int d^4p\,\mathcal{S}^{\mu\nu}(x,p), \tag{23}$$

where $\mathcal{S}^{\mu\nu}$ is the tensor component of the Wigner function (10). The dipole moment tensor $D^{\mu\nu}$ is antisymmetric with respect to its indices, which allows the following decomposition

$$D^{\mu\nu} = \mathcal{E}^\mu u^\nu - \mathcal{E}^\nu u^\mu - \epsilon^{\mu\nu\alpha\beta}u_\alpha\mathcal{M}_\beta. \tag{24}$$

Such a decomposition depends on the choice of the frame velocity u^μ. We identify $\mathcal{E}^\mu = D^{\mu\nu}u_\nu$ as the electric dipole moment vector and $\mathcal{M}^\mu = -(1/2)\epsilon^{\mu\nu\alpha\beta}u_\nu D_{\alpha\beta}$ as the magnetic dipole moment vector, because they are coupled with electric field and magnetic field respectively if we consider an external electromagnetic field as in Ref. [61]. In local equilibrium, we obtain

$$\begin{aligned}\mathcal{E}_{\rm eq}^\mu =& -\frac{1}{m\beta}\Big(\omega^{\mu\nu}u_\nu + \frac{1}{2}\beta u_\nu\partial^\nu u^\mu\Big)K_1\sinh\xi \\ &+ \frac{1}{2\,m}\nabla^\mu(K_1\sinh\xi), \\ \mathcal{M}_{\rm eq}^\mu =& -\frac{1}{2\,m}\tilde{\omega}^\mu(K_2 - \beta^{-1}K_1)\sinh\xi \\ &- \frac{1}{2\,m}\epsilon^{\mu\nu\alpha\beta}u_\nu(\partial_\alpha u_\beta)K_1\sinh\xi.\end{aligned} \tag{25}$$

The magnetic dipole moment consists of two parts, one is proportional to $\tilde{\omega}^\mu$, which is the contribution of intrinsic spin degrees of freedom; the other part arises from the rotation of the fluid or the orbital angular momentum proportional to the vorticity $\epsilon^{\mu\nu\alpha\beta}u_\nu(\partial_\alpha u_\beta)$. The electric dipole moment, on the other hand, has three parts: the electric dipole moment of particles, the acceleration of the fluid, and the contribution of the inhomogeneous particle distribution, which are proportional to $\omega^{\mu\nu}u_\nu$, $u_\nu\partial^\nu u^\mu$, and $\nabla^\mu n_{\rm eq}$, respectively. In local equilibrium, a non-vanishing electric dipole moment is allowed.

EOMs for Thermodynamical Parameters. For the ideal hydrodynamics without spin, the current density $J^\mu(x)$ and the energy-momentum tensor $T^{\mu\nu}(x)$ are conserved, respectively,

$$\begin{aligned}\partial_\mu J^\mu(x) &= 0, \\ \partial_\nu T^{\mu\nu}(x) &= 0.\end{aligned} \tag{26}$$

These conservation laws still hold in the presence of spin degrees of freedom. Inserting the equilibrium expressions of $J^\mu(x)$ and $T^{\mu\nu}(x)$ in Eq. (14) into the conservation laws (26), we derive the equations of motion for β, ξ, and u^μ,

$$\begin{aligned}\dot{\beta} &= \frac{K_2 + \beta^{-1}K_1\cosh^2\xi}{K_1K_3\cosh^2\xi - K_2K_2\sinh^2\xi}K_1\theta, \\ \dot{\xi} &= \frac{(K_2 + \beta^{-1}K_1)K_2 - K_1K_3}{K_1K_3\cosh^2\xi - K_2K_2\sinh^2\xi}\theta\sinh\xi\cosh\xi, \\ \dot{u}^\mu &= \frac{K_1}{K_1 + \beta K_2}\tanh\xi\nabla^\mu\xi - \frac{1}{\beta}\nabla^\mu\beta,\end{aligned} \tag{27}$$

where dot represents the time derivative in comoving frame defined as $d/d\tau \equiv u_\mu\partial^\mu$, and $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ is the space derivative. With the help of Eqs. (19) and

(20), we find that $K_1 \simeq n_{\rm eq}/\sinh\xi \simeq \beta P_{\rm eq}/\cosh\xi$, $K_2 \simeq \epsilon_{\rm eq}/\cosh\xi$, with corrections to K_1 and K_2 are at least second order in Kn and χ_s, and thus $K_3 = \frac{3}{\beta}(K_2 + \beta^{-1}K_1) + m^2 K_1$ according to the recurrence relation (16). Therefore, equations in (27) can be expressed in term of the equilibrium quantities $n_{\rm eq}$, $P_{\rm eq}$, and $\epsilon_{\rm eq}$. Once we do this, we will find that a nonzero $\omega^{\mu\nu}$ does not contribute at the leading and next-to-leading order in Kn and χ_s. Such a conclusion can be obtained from another point of view from $J^\mu_{\rm eq}$ and $T^{\mu\nu}_{\rm eq}$: the deviation from ideal hydrodynamics without spin is at least $\mathcal{O}(Kn^2, Kn\,\chi_s, \chi_s^2)$.

On the other hand, the conservation of the total angular momentum gives

$$\partial_\lambda S^{\lambda,\mu\nu}(x) = T^{\nu\mu}(x) - T^{\mu\nu}(x). \tag{28}$$

Inserting equilibrium expressions in Eqs. (14) and (21) into the above equation, we obtain the equation of motion for $\omega^{\mu\nu}$ in $\dot\omega^{\mu\nu}$, which can be decomposed into

$$\dot\omega^{\mu\nu} = \Delta^\mu_\alpha\Delta^\nu_\beta\dot\omega^{\alpha\beta} - u^\mu\dot\omega^{\nu\alpha}u_\alpha + u^\nu\dot\omega^{\mu\alpha}u_\alpha, \tag{29}$$

where the first term and $\dot\omega^{\mu\nu}u_\nu$ appearing in last two terms are given by

$$\begin{aligned}
\Delta^\mu_\alpha\Delta^\nu_\beta\dot\omega^{\alpha\beta} =& C_3\Delta^\mu_\alpha\Delta^\nu_\beta\omega^{\alpha\beta} + C_4\Delta^{[\mu}_\beta\sigma_h^{\nu]\rho}\omega^\beta_\rho \\
&- \frac{1}{2}C_4(\nabla^{[\mu}\omega_\rho^{\nu]})u^\rho + C_2C_4u^\rho\omega_\rho^{[\mu}\nabla^{\nu]}\xi, \\
\dot\omega^{\mu\nu}u_\nu =& C_1\omega^{\mu\nu}u_\nu + C_2\Delta^\mu_\rho\omega^{\rho\nu}\nabla_\nu\xi \\
&+ \sigma_h^{\mu\nu}\omega_{\nu\rho}u^\rho + \frac{1}{2}\Delta^\mu_\rho(\nabla^\nu\omega^\rho_\nu),
\end{aligned} \tag{30}$$

where $\sigma_h^{\mu\nu} \equiv (\nabla^\mu u^\nu + \nabla^\nu u^\mu)/2 - \Delta^{\mu\nu}\theta/3$. The transport coefficients C_i $(i = 1, 2, 3, 4)$ are given by

$$\begin{aligned}
C_1 =& -\frac{5}{3}\theta + \frac{5}{\beta}\dot\beta + \frac{\beta m^2 K_1}{K_1 + \beta K_2}\dot\beta - \tanh\xi\dot\xi, \\
C_2 =& \frac{1}{2}\Big(1 - \frac{5K_1}{K_1 + \beta K_2} - \frac{m^2\beta^2 K_1 K_1}{(K_1 + \beta K_2)^2}\Big)\tanh\xi, \\
C_3 =& -\Big[5 - \frac{2\beta^2 m^2 K_1}{2(K_1 + \beta K_2) + \beta^2 m^2 K_1}\Big]\frac{\theta}{3} \\
&- \tanh\xi\dot\xi + \Big[5 - \frac{m^2\beta^2(3K_1 - \beta K_2)}{2(K_1 + \beta K_2) + \beta^2 m^2 K_1}\Big]\frac{\dot\beta}{\beta}, \\
C_4 =& \frac{\beta^2 m^2 K_1}{2(K_1 + \beta K_2) + \beta^2 m^2 K_1} - 1.
\end{aligned} \tag{31}$$

In our power counting scheme, we have $\omega^{\mu\nu} \sim \mathcal{O}(\chi_s)$ and thus both sides of Eq. (29) are $\mathcal{O}(Kn\,\chi_s)$. Equation (29) has a trivial solution $\omega^{\mu\nu} = 0$, corresponding to the spinless case. Note that similar results have been derived in Ref. [51], but the transport coefficients are slightly different. That is because we include spin-induced energy current, which leads to an antisymmetric part in $T^{\mu\nu}_{\rm eq}$, while $T^{\mu\nu}_{\rm eq}$ used in Ref. [51] is symmetric.

In summary, we formulate ideal hydrodynamics with spin degrees of freedom from the Wigner function approach. A general solution to the kinetic equation for the Wigner function has been found up to the second order in space-time gradient. The solution contains a zeroth order part W_0 and a high-order correction part δW. The zeroth order part W_0 can be expressed in terms of matrix-valued distributions, while δW can be expressed by space-time derivatives of W_0. We adopt the equilibrium form of the matrix-valued distributions for particles and antiparticles proposed in Ref. [9].

Based on the Wigner function in local equilibrium, we derive hydrodynamical quantities: the current density, the energy-momentum tensor, the spin tensor, and the electric/magnetic dipole moment. Compared with ideal hydrodynamics without spin, the current density and the energy-momentum tensor contain additional contributions at the second order in the Knudsen number Kn and the average spin polarization χ_s from space-time gradients and from spin degrees of freedom respectively. We also find that the spin tensor and magnetic dipole moment depend on $\epsilon^{\mu\nu\alpha\beta}u_\nu\omega_{\alpha\beta}$ but does not depend on $\omega^{\mu\nu}u_\nu$, while the electric dipole moment is related to $\omega^{\mu\nu}u_\nu$. We also derive equations of motion for thermodynamical parameters β, u^μ, ξ, and $\omega^{\mu\nu}$ from conservation laws. These equations can be applied to numerical simulations for the spin evolution in the quark-gluon plasma. We emphasize that this work is based on the assumption of local equilibrium. In more realistic cases, whether the relaxation time of spin is short enough or not is still under debate. On the other hand, ideal hydrodynamics without spin is acausal and not well-defined, and ideal spin hydrodynamics may also have similar problems.[78] To avoid these problems, dissipative effects induced by particle collisions must be included.

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References

[1] Liang Z T and Wang X N 2005 *Phys. Rev. Lett.* **94** 102301
[2] Voloshin S A 2004 arXiv:nucl-th/0410089
[3] Einstein A and de Hass W J 1915 *Dtsch. Phys. Ges. Verh.* **17** 152
[4] Barnett S J 1935 *Rev. Mod. Phys.* **7** 129
[5] Adamczyk L *et al.* (STAR collaboration) 2017 *Nature* **548** 62
[6] Adam J *et al.* (STAR collaboration) 2018 *Phys. Rev. C* **98** 014910
[7] Becattini F, Piccinini F and Rizzo J 2008 *Phys. Rev. C* **77** 024906
[8] Becattini F, Csernai L and Wang D J 2013 *Phys. Rev. C* **88** 034905
[9] Becattini F, Chandra V, Del Z L and Grossi E 2013 *Ann. Phys.* **338** 32
[10] Karpenko I and Becattini F 2017 *Eur. Phys. J. C* **77** 213
[11] Xie Y, Wang D and Csernai L P 2017 *Phys. Rev. C* **95**

031901
[12] Li H, Pang L G, Wang Q and Xia X L 2017 *Phys. Rev. C* **96** 054908
[13] Sun Y and Ko C M 2017 *Phys. Rev. C* **96** 024906
[14] Wei D X, Deng W T and Huang X G 2019 *Phys. Rev. C* **99** 014905
[15] Becattini F and Karpenko I 2018 *Phys. Rev. Lett.* **120** 012302
[16] Adam J *et al.* (STAR collaboration) 2019 *Phys. Rev. Lett.* **123** 132301
[17] Wu H Z, Pang L G, Huang X G and Wang Q 2019 *Phys. Rev. Res.* **1** 033058
[18] Florkowski W, Kumar A, Ryblewski R and Mazeliauskas A 2019 *Phys. Rev. C* **100** 054907
[19] Liu S Y F, Sun Y and Ko C M 2019 arXiv:1910.06774 [nucl-th]
[20] Liu S Y F and Yin Y 2021 arXiv:2103.09200 [hep-ph]
[21] Fu B, Liu S Y F, Pang L, Song H and Yin Y 2021 arXiv:2103.10403 [hep-ph]
[22] Becattini F, Buzzegoli M and Palermo A 2021 arXiv:2103.10917 [nucl-th]
[23] Becattini F, Buzzegoli M, Palermo A, Inghirami G and Karpenko I 2021 arXiv:2103.14621 [nucl-th]
[24] Yi C, Pu S and Yang D L 2021 arXiv:2106.00238 [hep-ph]
[25] Kharzeev D 2006 *Phys. Lett. B* **633** 260
[26] Kharzeev D E, McLerran L D and Warringa H J 2008 *Nucl. Phys. A* **803** 227
[27] Fukushima K, Kharzeev D E and Warringa H J 2008 *Phys. Rev. D* **78** 074033
[28] Voloshin S A 2010 *Phys. Rev. Lett.* **105** 172301
[29] Wang F Q and Zhao J 2018 *Nucl. Sci. Tech.* **29** 179
[30] Adam J *et al.* (STAR collaboration) 2021 *Nucl. Sci. Tech.* **32** 48
[31] Abdallah M *et al.* (STAR collaboration) 2021 arXiv:2109.00131 [nucl-ex]
[32] Liu Y C and Huang X G 2020 *Nucl. Sci. Tech.* **31** 56
[33] Gao J H, Ma G L, Pu S and Wang Q 2020 *Nucl. Sci. Tech.* **31** 90
[34] Gao J H, Liang Z T, Wang Q and Wang X N 2020 *Global polarization effect and spin-orbit coupling in strong interaction*
[35] Gao J H, Liang Z T and Wang Q 2021 *Int. J. Mod. Phys. A* **36** 2130001
[36] Romatschke P 2010 *Int. J. Mod. Phys. E* **19** 1
[37] Heinz U and Snellings R 2013 *Ann. Rev. Nucl. Part. Sci.* **63** 123
[38] Gale C, Jeon S and Schenke B 2013 *Int. J. Mod. Phys. A* **28** 1340011
[39] Romatschke P and Romatschke U 2019 *Relativistic Fluid Dynamics in and out of Equilibrium* (Cambridge: Cambridge University Press)
[40] Shen C and Yan L 2020 *Nucl. Sci. Tech.* **31** 122
[41] Wu S, Shen C and Song H 2021 arXiv:2104.13250 [nucl-th]
[42] Weyssenhoff J and Raabe A 1947 *Acta Phys. Polon.* **9** 7
[43] Becattini F and Tinti L 2010 *Ann. Phys.* **325** 1566
[44] Montenegro D, Tinti L and Torrieri G 2017 *Phys. Rev. D* **96** 056012
[45] Montenegro D, Tinti L and Torrieri G 2017 *Phys. Rev. D* **96** 076016
[46] Florkowski W, Friman B, Jaiswal A and Speranza E 2018 *Phys. Rev. C* **97** 041901
[47] Florkowski W, Friman B, Jaiswal A, Ryblewski R and Speranza E 2018 *Phys. Rev. D* **97** 116017
[48] Florkowski W, Kumar A and Ryblewski R 2019 *Prog. Part. Nucl. Phys.* **108** 103709
[49] Florkowski W 2019 *Acta Phys. Polon. B* **50** 1047
[50] Gallegos A D, Gürsoy U and Yarom A 2021 arXiv:2101.04759 [hep-th]
[51] Bhadury S, Florkowski W, Jaiswal A, Kumar A and Ryblewski R 2021 *Phys. Rev. D* **103** 014030
[52] Bhadury S, Florkowski W, Jaiswal A, Kumar A and Ryblewski R 2021 *Phys. Lett. B* **814** 136096
[53] Hattori K, Hongo M, Huang X G, Matsuo M and Taya H 2019 *Phys. Lett. B* **795** 100
[54] Fukushima K and Pu S 2021 *Phys. Lett. B* **817** 136346
[55] Li S, Stephanov M A and Yee H U 2020 *Phys. Rev. Lett.* **127** 082302
[56] She D, Huang A, Hou D and Liao J 2021 arXiv:2105.04060 [nucl-th]
[57] Heinz U W 1983 *Phys. Rev. Lett.* **51** 351
[58] Elze H T, Gyulassy M and Vasak D 1986 *Nucl. Phys. B* **276** 706
[59] Vasak D, Gyulassy M and Elze H T 1987 *Ann. Phys.* **173** 462
[60] Sheng X L 2019 arXiv:1912.01169 [nucl-th]
[61] Sheng X L, Wang Q and Huang X G 2020 *Phys. Rev. D* **102** 025019
[62] Yang D L, Hattori K and Hidaka Y 2020 arXiv:2002.02612 [hep-ph]
[63] Weickgenannt N, Speranza E, Sheng X L, Wang Q and Rischke D H 2020 arXiv:2005.01506 [hep-ph]
[64] Sheng X L, Weickgenannt N, Speranza E, Rischke D H and Wang Q 2021 arXiv:2103.10636 [nucl-th]
[65] Wang Z and Zhuang P 2021 arXiv:2105.00915 [hep-ph]
[66] Gao J H and Liang Z T 2019 *Phys. Rev. D* **100** 056021
[67] Weickgenannt N, Sheng X L, Speranza E, Wang Q and Rischke D H 2019 *Phys. Rev. D* **100** 056018
[68] Hattori K, Hidaka Y and Yang D L 2019 *Phys. Rev. D* **100** 096011
[69] Wang Z, Guo X, Shi S and Zhuang P 2019 *Phys. Rev. D* **100** 014015
[70] Jackson J D 1998 *Classical Electrodynamics* (New York: Wiley)
[71] Li S and Yee H U 2018 *Phys. Rev. D* **98** 056018
[72] Li S and Yee H U 2019 *Phys. Rev. D* **100** 056022
[73] Becattini F and Tinti L 2013 *Phys. Rev. D* **87** 025029
[74] Fukushima K and Pu S 2020 arXiv:2001.00359 [hep-ph]
[75] Speranza E and Weickgenannt N 2021 *Eur. Phys. J. A* **57** 155
[76] Fukuda M, Ichikawa K, Senami M and Tachibana A 2016 *AIP Adv.* **6** 025108
[77] Niemi H and Denicol G S 2014 arXiv:1404.7327 [nucl-th]
[78] Speranza E, Bemfica F S, Disconzi M M and Noronha J 2021 arXiv:2104.02110 [hep-th]

Supplemental Material for: "Ideal spin hydrodynamics from Wigner function approach"

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Here we provide details of derivations for some important equations in the main text.

I. THE WIGNER FUNCTION

In absence of particle scatterings, the Wigner function satisfies the following kinetic equation

$$\left[\gamma_\mu\left(p^\mu+\frac{i}{2}\partial^\mu\right)-m\right]W(x,p)=0\,. \tag{1}$$

Multiplying this equation with $\gamma\cdot p+(i/2)\gamma\cdot\partial+m$ from the left, we obtain

$$\left[\left(p^2-\frac{1}{4}\partial^2-m^2\right)+ip\cdot\partial\right]W(x,p)=0\,. \tag{2}$$

The mass-shell condition and Vlasov equation for the Wigner function are obtained by linear recombinations of Eq. (2) and its Hermitian conjugate,

$$\begin{aligned}\left(p^2-\frac{1}{4}\partial^2-m^2\right)W(x,p)&=0\,,\\ p\cdot\partial W(x,p)&=0\,,\end{aligned} \tag{3}$$

where we have used the property $W^\dagger=\gamma^0W\gamma^0$. From the mass-shell condition we find that the momentum p^μ for the Wigner function is no longer on the traditional mass-shell $p^2=m^2$ at second order in space-time gradient.

To solve the Wigner function, we make a gradient expansion as

$$W(x,p)=\sum_{i=0,1,2}W_i(x,p)+\mathcal{O}(\mathrm{Kn}^3)\,, \tag{4}$$

where the subscript $i=0,1,2$ labels orders in space-time gradient. We note that the gradient expansion for the Wigner function is equivalent to an expansion with respect to the Knudsen number Kn, thus here and in the follows we use Kn to label orders in the gradient expansion. The zeroth order part satisfies the following equations

$$\begin{aligned}(\gamma\cdot p-m)W_0(x,p)&=0\,,\\ (p^2-m^2)W_0(x,p)&=0\,,\\ p\cdot\partial W_0(x,p)&=0\,,\end{aligned} \tag{5}$$

which are obtained from leading-order terms of Eqs. (1) and (3). A general solution of W_0 can be constructed by directly inserting the quantized field operators $\psi(x)$ and $\bar\psi(x)$ into the definition of the Wigner function, c.f., $W_0(x,p)$ in Eq. (4) in the main text. We then assume that higher order terms of the Wigner function are related to W_0 and take the trial solution as

$$W(x,p)=UW_0(x,p)\gamma^0U^\dagger\gamma^0+\mathcal{O}(\mathrm{Kn}^3)\,, \tag{6}$$

where the matrix operator U is assumed to be function of derivative operator

$$U\equiv 1+U_1\gamma\cdot\partial+U_2\partial^2+\mathcal{O}(\mathrm{Kn}^3) \tag{7}$$

In order to fulfill Eqs. (1), W_1 should satisfy

$$(\gamma\cdot p-m)W_1(x,p)+\frac{i}{2}\gamma\cdot\partial W_0(x,p)=0\,. \tag{8}$$

According to the assumption (6), W_1 is given by

$$W_1(x,p)=U_1\gamma\cdot\partial W_0(x,p)+W_0(x,p)\gamma\cdot\overleftarrow{\partial}\gamma^0U_1^\dagger\gamma^0\,. \tag{9}$$

Inserting W_1 into Eq. (8), we obtain

$$\begin{aligned}0=&(\gamma\cdot p-m)U_1\gamma\cdot\partial W_0(x,p)+\frac{i}{2}\gamma\cdot\partial W_0(x,p)\\ &+\left[(\gamma\cdot p-m)W_0(x,p)\right]\gamma\cdot\overleftarrow{\partial}\gamma^0U_1^\dagger\gamma^0\,.\end{aligned} \tag{10}$$

We then use the following relation to commute $\gamma\cdot p-m$ with $U_1\gamma\cdot\partial$ so that the result can be simplified with the first line in Eq. (5),

$$\begin{aligned}&(\gamma\cdot p-m)U\gamma\cdot\partial\\ &=\{\gamma\cdot p,U\gamma\cdot\partial\}-U\gamma\cdot\partial(\gamma\cdot p-m)-2mU\gamma\cdot\partial\,.\end{aligned} \tag{11}$$

Finally we obtain an equation for U_1,

$$\begin{aligned}U_1\gamma\cdot\partial W_0(x,p)=&\frac{i}{4m}\gamma\cdot\partial W_0(x,p)\\ &+\frac{1}{2m}\left\{\gamma\cdot p,\,U_1\gamma\cdot\partial\right\}W_0(x,p)\,,\end{aligned} \tag{12}$$

which can be solved by iterative method and the result reads

$$U_1=\frac{i}{4m}\,. \tag{13}$$

Inserting U_1 into Eq. (6) and taking out the second-order parts, we obtain

$$W_2 = \frac{1}{16m^2}\gamma\cdot\partial W_0\gamma\cdot\overleftarrow{\partial} + \partial^2\left[U_2W_0 + \partial^2 W_0\gamma^0U_2^\dagger\gamma^0\right], \tag{14}$$

which should fulfills Eq. (1),

$$(\gamma\cdot p - m)W_2(x,p) + \frac{i}{2}\gamma\cdot\partial W_1(x,p) = 0\,. \tag{15}$$

The calculation of the left-hand-side of Eq. (15) is straightforward

$$\begin{aligned}
&(\gamma\cdot p - m)W_2(x,p) + \frac{i}{2}\gamma\cdot\partial W_1(x,p)\\
&= \frac{1}{16m^2}(\gamma\cdot p - m)\gamma\cdot\partial W_0\gamma\cdot\overleftarrow{\partial}\\
&\quad +(\gamma\cdot p - m)\partial^2\left[U_2W_0 + W_0\gamma^0U_2^\dagger\gamma^0\right]\\
&\quad -\frac{1}{8m}\partial^2 W_0 + \frac{1}{8m}\gamma\cdot\partial W_0\gamma\cdot\overleftarrow{\partial}\,.
\end{aligned} \tag{16}$$

We further use the following relation

$$\begin{aligned}
&(\gamma\cdot p - m)\gamma\cdot\partial\\
&= \{\gamma\cdot p, \gamma\cdot\partial\} - \gamma\cdot\partial(\gamma\cdot p - m) - 2m\gamma\cdot\partial\\
&= 2p\cdot\partial - \gamma\cdot\partial(\gamma\cdot p - m) - 2m\gamma\cdot\partial\,,
\end{aligned} \tag{17}$$

together with the first line in Eq. (5) and obtain

$$(\gamma\cdot p - m)\partial^2\left[U_2W_0 + W_0\gamma^0U_2^\dagger\gamma^0\right] = \frac{1}{8m}\partial^2 W_0\,. \tag{18}$$

The matrix U_2 must have a singularity at $p^2 = m^2$, otherwise the left-hand-side of Eq. (18) vanish because $(\gamma\cdot p - m)W_0 = 0$ for an on-shell momentum p^μ. A simplest solution of Eq. (18) reads

$$U_2W_0 + W_0\gamma^0U_2^\dagger\gamma^0 = \frac{\gamma\cdot p + m}{8m(p^2-m^2)}W_0\,, \tag{19}$$

with

$$U_2 = \frac{\gamma\cdot p + m}{16m(p^2-m^2)}\,. \tag{20}$$

Note that $\gamma^0U_2^\dagger\gamma^0 = U_2$ and find $\{U_2, W_0\} = 0$, where the later one can be proved using the explicit expression of W_0. Inserting U_1 in Eq. (13) and U_2 in Eq. (20) into Eqs. (6) and (7), one obtains first order and second order parts of the Wigner function, c.f., Eq. (6) in the main text.

II. HYDRODYNAMICAL QUANTITIES IN LOCAL EQUILIBRIUM

This section we will show details for deriving the current density J^μ, the energy-momentum tensor (density) $T^{\mu\nu}$, the spin tensor (density) $S^{\lambda,\mu\nu}$, and the dipole-moment tensor (density) $D^{\mu\nu}$.

A. Wigner function

In order to calculate the physical quantities, we first look at the Wigner function $W = W_0 + \delta W$, where W_0 is the zeroth order part and δW contains the first and second order parts of the Wigner function. The explicit expressions of W_0 and δW are given in Eqs. (4) and (6) in the main text. One can easily derive the zeroth order part W_0 using explicit expressions of wave functions $u_r(\mathbf{p})$ and $v_r(-\mathbf{p})$,

$$W_0 = \frac{1}{4}(m + \gamma\cdot p)(V + \gamma^5\gamma^\mu n_\mu)\delta(p^2 - m^2)\,, \tag{21}$$

where we have defined

$$\begin{aligned}
V(x,p) &= \frac{2}{(2\pi)^3}\sum_{rs}\delta_{rs}\\
&\quad\times\left\{\theta(p^0)f_{rs}^+(x,\mathbf{p}) + \theta(-p^0)f_{rs}^-(x,-\mathbf{p})\right\}\,,\\
n^\mu(x,p) &= \frac{1}{(2\pi)^3 m}\sum_{rs}\left\{\theta(p^0)\bar{u}_s(\mathbf{p})\gamma^\mu\gamma^5 u_r(\mathbf{p})f_{rs}^+(x,\mathbf{p})\right.\\
&\quad\left.-\theta(-p^0)\bar{v}_s(-\mathbf{p})\gamma^\mu\gamma^5 v_r(-\mathbf{p})f_{rs}^-(x,-\mathbf{p})\right\}\,.
\end{aligned} \tag{22}$$

With the help of W_0, the calculation of δW is then straightforward with Eq. (6) in the main text. Using Eq. (9) in the main text, we obtain the following results in local thermal equilibrium

$$\begin{aligned}
V_{\rm eq} &= \frac{4}{(2\pi)^3}\left(1 + \frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\right)\\
&\quad\times\left[\theta(p^0)\exp(-\beta\cdot p + \xi) + \theta(-p^0)\exp(\beta\cdot p - \xi)\right]\,,\\
n^\mu_{\rm eq} &= -\frac{1}{(2\pi)^3 m}\epsilon^{\mu\nu\alpha\beta}p_\nu\omega_{\alpha\beta}\\
&\quad\times\left[\theta(p^0)\exp(-\beta\cdot p + \xi) - \theta(-p^0)\exp(\beta\cdot p - \xi)\right]\,,
\end{aligned} \tag{23}$$

where we have taken the Boltzmann limit and truncated the expansion series at $\mathcal{O}(\chi_s^2)$.

B. Current density and energy-momentum tensor

The current density J^μ and the energy-momentum tensor (density) $T^{\mu\nu}$ are given by the vector component of the Wigner function as shown as Eq. (11) and (12) in the main text. Here $\mathcal{V}^\mu$ is extracted from W by

$$\mathcal{V}^\mu = {\rm Tr}\left[\gamma^\mu(W_0 + \delta W)\right]\,. \tag{24}$$

The vector components of W_0 and δW are respectively given by

$$\begin{aligned}
{\rm Tr}\left(\gamma^\mu W_0\right) &= \delta(p^2 - m^2)p^\mu V(x,p)\,,\\
{\rm Tr}\left(\gamma^\mu\delta W\right) &= -\delta(p^2-m^2)\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\alpha\partial_\nu n_\beta(x,p)\\
&\quad -\left[\frac{\delta(p^2-m^2)}{4m^2} + \delta'(p^2-m^2)\right]\frac{1}{4}p^\mu\partial^2 V(x,p)\\
&\quad +\mathcal{O}({\rm Kn}^3)\,,
\end{aligned} \tag{25}$$

where $\delta'(x) \equiv d\delta(x)/dx = -\delta(x)/x$ is the first order derivative of the delta function. This result agree with Refs. [1, 2] at zeroth and first order in space-time gradient. At second order, we obtain two terms proportional to $\partial^2 V(x,p)$, the momentum p^μ in one term is fixed on the mass-shell by $\delta(p^2-m^2)$ while p^μ in the other term is not. In general we can redefine $V(x,p)$ and the delta-function as

$$\begin{aligned}&\delta(p^2-m^2-\delta m^2)\left[1+\delta V\right]V(x,p)\\ \equiv\ &\delta(p^2-m^2)V(x,p)\\ &-\left[\frac{\delta(p^2-m^2)}{4m^2}+\delta'(p^2-m^2)\right]\frac{1}{4}\partial^2 V(x,p)\,,\end{aligned} \tag{26}$$

where

$$\begin{aligned}\delta m^2 &\simeq \frac{1}{4V(x,p)}\partial^2 V(x,p)+\mathcal{O}(\mathrm{Kn}^3)\,,\\ \delta V &\simeq -\frac{1}{16m^2V(x,p)}\partial^2 V(x,p)+\mathcal{O}(\mathrm{Kn}^3)\,.\end{aligned}$$

In this way we can clearly see interpretations of the second order terms: the on-mass-shell part plays as a modification to the distribution function from long-range correlations, while the off-mass-shell part modifies the mass-shell of particles. We note that in this work, interactions between different particles are neglected, but long-range correlations still exist because particles are treated as wave-packets with finite space-time volume. The vector component of the Wigner function in local thermal equilibrium is obtained by substituting $V_{\rm eq}$ and $n^\mu_{\rm eq}$ in Eq. (23) into Eq. (25),

$$\begin{aligned}\mathcal{V}^\mu_{\rm eq} =\ &\frac{4}{(2\pi)^3}\delta\left(p^2-m^2-\frac{1}{4}\partial^2\right)\\ &\times\left(1-\frac{1}{16m^2}\partial^2\right)\left(1+\frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\right)p^\mu\\ &\times\left[\theta(p^0)\exp(-\beta\cdot p+\xi)+\theta(-p^0)\exp(\beta\cdot p-\xi)\right]\\ &+\frac{1}{2(2\pi)^3m^2}\delta(p^2-m^2)\epsilon^{\mu\nu\alpha\beta}p_\alpha\partial_\nu\epsilon_{\beta\rho\sigma\lambda}p^\rho\omega^{\sigma\lambda}\\ &\times\left[\theta(p^0)\exp(-\beta\cdot p+\xi)-\theta(-p^0)\exp(\beta\cdot p-\xi)\right]\\ &+\mathcal{O}(\mathrm{Kn}^3,\,\mathrm{Kn}^2\chi_s,\,\mathrm{Kn}\chi_s^2,\,\chi_s^3)\,,\end{aligned} \tag{27}$$

where the derivative operator acts on locally-defined parameters β^μ, ξ, and $\omega^{\mu\nu}$. Integrating $\mathcal{V}^\mu_{\rm eq}$ over momentum, we obtain the current density $J^\mu_{\rm eq}$, which is given in Eq. (14) in the main text. The energy-momentum tensor in local equilibrium is derived by substituting $\mathcal{V}^\mu_{\rm eq}$ in Eq. (27) into Eq. (12) in the main text. In the calculations, we have taken replacements $p^\mu \to -p^\mu$ for anti-particle parts and used the following rank-l moments

$$\begin{aligned}I^{\mu_1\mu_2\cdots\mu_l} \equiv\ &\frac{8}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\theta(p^0)\\ &\times p^{\mu_1}p^{\mu_2}\cdots p^{\mu_l}\exp(-\beta\cdot p)\,.\end{aligned} \tag{28}$$

Here $\beta^\mu \equiv \beta u^\mu = u^\mu/T$ is the thermal velocity. Such a moment expansion for distribution function is widely used for deriving hydrodynamics from kinetic theory [3, 4]. The rank-0, 1, 2 moments are given by

$$\begin{aligned}I &= K_0(\beta)\,,\\ I^\mu &= u^\mu K_1(\beta)\,,\\ I^{\mu\nu} &= u^\mu u^\nu K_2(\beta)+\frac{1}{3}\Delta^{\mu\nu}(m^2K_0-K_2)\,,\end{aligned} \tag{29}$$

and the rank-3 moment $I^{\mu\nu\alpha}$ is given in Eq. (17) in the main text. The projection operator is defined as $\Delta^{\mu\nu} \equiv g^{\mu\nu}-u^\mu u^\nu$. Here we have defined K_n as shown in Eq. (15) in the main text.

The recursive relation for K_n can be proved as follows: we first convert the definition of K_n to another equivalent form

$$K_n = \frac{16\pi}{(2\pi\hbar)^3}\int dp\,p^2(p^2+m^2)^{(n-1)/2}e^{-\beta\sqrt{p^2+m^2}}\,. \tag{30}$$

Then we try to calculate

$$\begin{aligned}&K_n-m^2K_{n-2}\\ &=\frac{16\pi}{(2\pi)^3}\int_0^\infty dp(p^2+m^2)^{(n-3)/2}p^4e^{-\beta\sqrt{p^2+m^2}}\\ &=-\frac{16\pi}{\beta(2\pi)^3}\int_0^\infty dp(p^2+m^2)^{(n-2)/2}p^3\frac{d}{dp}e^{-\beta\sqrt{p^2+m^2}}\\ &=\frac{16\pi}{\beta(2\pi)^3}\int_0^\infty dpe^{-\beta\sqrt{p^2+m^2}}\frac{d}{dp}\left[(p^2+m^2)^{(n-2)/2}p^3\right]\,,\end{aligned} \tag{31}$$

where in the last step we have used the method of integrating by parts. Completing the derivative with respect to p, we obtain

$$\begin{aligned}K_n-m^2K_{n-2} =\ &\frac{16\pi}{\beta(2\pi)^3}\int_0^\infty dpe^{-\beta\sqrt{p^2+m^2}}\\ &\times\left[3p^2(p^2+m^2)^{(n-2)/2}+(n-2)p^4(p^2+m^2)^{(n-4)/2}\right]\,.\end{aligned} \tag{32}$$

We further use $p^2=(p^2+m^2)-m^2$ for the second term and obtain

$$\begin{aligned}K_n-m^2K_{n-2} &= \frac{16\pi}{\beta(2\pi)^3}\int_0^\infty dpe^{-\beta\sqrt{p^2+m^2}}\\ &\quad\times\left[(n+1)p^2(p^2+m^2)^{(n-2)/2}\right.\\ &\quad\left.-m^2(n-2)p^2(p^2+m^2)^{(n-4)/2}\right]\\ &=\frac{n+1}{\beta}K_{n-1}-\frac{(n-2)}{\beta}m^2K_{n-3}\,.\end{aligned} \tag{33}$$

Note that for massless particles, the relations are simplified to $K_n=[n+1)/\beta]K_{n-1}$.

With the help of fluid velocity vector u^μ and the related projection operator $\Delta^{\mu\nu} \equiv g^{\mu\nu}-u^\mu u^\nu$, one can decompose $J^\mu_{\rm eq}$ as

$$\begin{aligned}J^\mu_{\rm eq} &= (u\cdot J_{\rm eq})u^\mu+\Delta^\mu_\nu J^\nu_{\rm eq}\\ &\equiv n_{\rm eq}u^\mu+\delta j^\mu\,.\end{aligned} \tag{34}$$

In analogue to the viscous hydrodynamics, we identify $n_{\rm eq} \equiv u \cdot J_{\rm eq}$ as the particle number density in the comoving frame of vector u^μ. The remaining part δj^μ is the diffusion current. In this work, δj^μ consists of two parts: one is the contribution of magnetization current, which is of order $\mathcal{O}({\rm Kn}\,\chi_s)$, while the other part contains second order space-time derivative and therefore is $\mathcal{O}({\rm Kn}^2)$.

On the other hand, the energy-momentum tensor in general can be decomposed into a symmetric part $T_S^{\mu\nu}$ and an anti-symmetric part $\delta T_A^{\mu\nu}$,

$$T_{\rm eq}^{\mu\nu} = T_S^{\mu\nu} + \delta T_A^{\mu\nu}\,. \tag{35}$$

From Eq. (14) in the main text, one can easily find that the anti-symmetric part will vanish for the spinless case with $\omega^{\mu\nu} = 0$. Therefore in our power counting scheme, $\delta T_A^{\mu\nu}$ is of order $\mathcal{O}({\rm Kn}\,\chi_s)$. The symmetric part can be further decomposed as

$$T_S^{\mu\nu} = \epsilon_{\rm eq} u^\mu u^\nu - P_{\rm eq}\Delta^{\mu\nu} + \delta T_S^{\mu\nu}\,, \tag{36}$$

in analogue to that in the viscous hydrodynamics. Similar to δj^μ, we also find that $\delta T_S^{\mu\nu}$ consists of two contributions. One of them depend on the spin potential $\omega^{\mu\nu}$ and will vanish when $\omega^{\mu\nu} = 0$. We identify this part as the heat flow and viscous tensor correction induced by the spin polarization. The other contribution arise from long-range correlations, which contains second order derivatives.

C. Spin tensor

The spin tensor is given by the axial-vector component of the Wigner function as shown in Eq. (13) in the main text. Using Eq. (21) and Eq. (6) in the main text, we obtain

$$\begin{aligned}\mathcal{A}^\mu &= {\rm Tr}\left[\gamma^\mu\gamma^5(W_0 + \delta W)\right] \\ &= \delta(p^2 - m^2) mn^\mu(x,p) + \mathcal{O}({\rm Kn}^2\chi_s)\,,\end{aligned} \tag{37}$$

where ${\rm Tr}\left(\gamma^\mu\gamma^5\delta W\right)$ is a term of $\mathcal{O}({\rm Kn}^2\chi_s)$ and thus does not contribute to $\mathcal{A}^\mu$. Using the equilibrium $n^\mu_{\rm eq}$ in Eq. (23), one can derive the spin tensor $S^{\lambda,\mu\nu}_{\rm eq}$ and the result is given in Eq. (21) in the main text.

The spin potential $\omega^{\mu\nu}$ is antisymmetric in its indices $\mu \leftrightarrow \nu$. In general, we can decompose it as

$$\omega^{\mu\nu} = (\omega^{\mu\alpha}u_\alpha)u^\nu - (\omega^{\nu\alpha}u_\alpha)u^\mu + \epsilon^{\mu\nu\alpha\beta}u_\alpha\tilde{\omega}_\beta\,. \tag{38}$$

Such a decomposition is in analogue to decomposing the electromagnetic field tensor in terms of the electric field and the magnetic field. Here we have defined

$$\tilde{\omega}^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\omega_{\alpha\beta}\,. \tag{39}$$

We now focus on the following term, which appears in $S^{\lambda,\mu\nu}_{\rm eq}$,

$$\begin{aligned}&u^\lambda\omega^{\mu\nu} + 2u^{[\mu}\omega^{\nu]\lambda} \\ &= u^\lambda(\omega^{\mu\alpha}u_\alpha)u^\nu - u^\lambda(\omega^{\nu\alpha}u_\alpha)u^\mu + u^\lambda\epsilon^{\mu\nu\alpha\beta}u_\alpha\tilde{\omega}_\beta \\ &\quad + u^\mu(\omega^{\nu\alpha}u_\alpha)u^\lambda - u^\mu(\omega^{\lambda\alpha}u_\alpha)u^\nu + u^\mu\epsilon^{\nu\lambda\alpha\beta}u_\alpha\tilde{\omega}_\beta \\ &\quad - u^\nu(\omega^{\mu\alpha}u_\alpha)u^\lambda + u^\nu(\omega^{\lambda\alpha}u_\alpha)u^\mu - u^\nu\epsilon^{\mu\lambda\alpha\beta}u_\alpha\tilde{\omega}_\beta \\ &= \left(u^\lambda\epsilon^{\mu\nu\alpha\beta} + u^\mu\epsilon^{\nu\lambda\alpha\beta} - u^\nu\epsilon^{\mu\lambda\alpha\beta}\right)u_\alpha\tilde{\omega}_\beta\,.\end{aligned} \tag{40}$$

We then use the Schouten identity [5],

$$\begin{aligned}&u^\lambda\epsilon^{\mu\nu\alpha\beta} + u^\mu\epsilon^{\nu\alpha\beta\lambda} \\ &\quad + u^\nu\epsilon^{\alpha\beta\lambda\mu} + u^\alpha\epsilon^{\beta\lambda\mu\nu} + u^\beta\epsilon^{\lambda\mu\nu\alpha} = 0\,,\end{aligned} \tag{41}$$

and obtain

$$\begin{aligned}u^\lambda\omega^{\mu\nu} + 2u^{[\mu}\omega^{\nu]\lambda} &= -\left(u^\beta\epsilon^{\lambda\mu\nu\alpha} + u^\alpha\epsilon^{\beta\lambda\mu\nu}\right)u_\alpha\tilde{\omega}_\beta \\ &= \epsilon^{\lambda\mu\nu\beta}\tilde{\omega}_\beta\,,\end{aligned} \tag{42}$$

where we have used $u^\mu u_\mu = 1$ and $u^\mu\tilde{\omega}_\mu = 0$ in the last step. Substitute the above equation into Eq. (21) in the main text, one can obtain Eq. (22) in the main text.

D. Dipole moment tensor

The dipole moment tensor is given by the tensor component of the Wigner function. The tensor component of W_0 is easy to write according to Eq. (21),

$${\rm Tr}(\sigma^{\mu\nu}W_0) = -\delta(p^2 - m^2)\epsilon^{\mu\nu\alpha\beta}p_\alpha n_\beta\,. \tag{43}$$

On the other hand, the tensor component of δW is obtained by substitute Eq. (21) into Eq. (6) in the main text and then projecting onto $\sigma^{\mu\nu}$,

$$\begin{aligned}{\rm Tr}(\sigma^{\mu\nu}\delta W) = &-\frac{1}{2m}\delta(p^2 - m^2)(p^\mu\partial^\nu - p^\nu\partial^\mu)V \\ &+\mathcal{O}({\rm Kn}^2\chi_s)\,,\end{aligned} \tag{44}$$

where we also truncate higher order terms. The physical interpretations of Eqs. (43) and (44) can be found in the rest frame with respect to a given p^μ. We identify ${\rm Tr}(\sigma^{\mu\nu}W_0)$ as the contribution form particle's intrinsic magnetic dipole moment, which is related to particle's spin polarization. Meanwhile, ${\rm Tr}(\sigma^{\mu\nu}\delta W)$ is the electric dipole moment contributed by inhomogeneity of particle distribution. The dipole moment tensor in local equilibrium is then given by

$$\begin{aligned}D^{\mu\nu}_{\rm eq}(x) &= \int d^4p\,{\rm Tr}\left[\sigma^{\mu\nu}(W_0 + \delta W)\right] \\ &= \frac{1}{2m}\omega^{\mu\nu}(K_2 - \beta^{-1}K_1)\sinh\xi \\ &\quad + \frac{1}{m}\omega^{\alpha[\mu}u^{\nu]}u_\alpha(K_2 + \beta^{-1}K_1)\sinh\xi \\ &\quad + \frac{1}{m}\partial^{[\mu}\left(u^{\nu]}K_1\sinh\xi\right) + \mathcal{O}({\rm Kn}^2\chi_s)\,,\end{aligned} \tag{45}$$

where we have used the equilibrium expressions of $V_{\rm eq}$ and $n^\mu_{\rm eq}$ in Eq. (23).

III. EOMS FOR THERMODYNAMICAL PARAMETERS

In this section, we will show how to derive the EOMs for thermodynamical parameters $\beta = 1/T$, ξ, u^μ, and $\omega^{\mu\nu}$, from conservation laws. Substituting the equilibrium current density $J^\mu_{\rm eq}(x)$ and energy-momentum tensor $T^{\mu\nu}_{\rm eq}(x)$ into the corresponding conservations laws, we derive

$$\begin{aligned}&\partial_\mu(u^\mu K_1 \sinh\xi) = 0\,,\\ &\partial_\mu\left\{\left[u^\mu u^\nu K_2 + \frac{1}{3}\Delta^{\mu\nu}(m^2K_0 - K_2)\right]\cosh\xi\right\} = 0\,.\end{aligned} \tag{46}$$

We note that contributions of $\omega^{\mu\nu}$ vanish in these equations. The EOM for $\omega^{\mu\nu}$ will be derived from the angular momentum conservation in later discussions. Since K_n are pure functions of β, one can find that

$$\partial_\mu K_n = -\left(\partial_\mu\beta\right) K_{n+1}\,. \tag{47}$$

Then conservation laws in (46) give

$$\begin{aligned}&\left(\theta K_1 - \dot\beta K_2\right)\sinh\xi + \dot\xi K_1\cosh\xi = 0\,,\\ &-\left[\frac{\theta}{3}\left(m^2K_0 - 4K_2\right) + \dot\beta K_3\right]\cosh\xi + \dot\xi K_2\sinh\xi = 0\,,\end{aligned} \tag{48}$$

and

$$\begin{aligned}\dot u^\nu\left(m^2K_0 - 4K_2\right)\cosh\xi - \frac{1}{3}(\nabla^\nu\beta)\left(m^2K_1 - K_3\right)\cosh\xi\\ +\frac{1}{3}(\nabla^\nu\xi)\left(m^2K_0 - K_2\right)\sinh\xi = 0\,.\end{aligned} \tag{49}$$

Here the first equation in (48) is the charge conservation and the second equation in (48) is obtained from the energy-momentum conservation by projecting onto u^μ, while Eq. (49) is obtained from the energy-momentum conservation by projecting onto the direction perpendicular to u^μ. Here dot represents the covariant time derivative $d/d\tau \equiv u_\mu\partial^\mu$ and $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ is the space derivative in comoving frame. By solving Eq. (48), we obtain $\dot\beta$ and $\dot\xi$, while $\dot u^\mu$ is obtained from Eq. (49),

$$\begin{aligned}\dot\beta &= \frac{-K_2\sinh^2\xi - \frac{1}{3}\left(m^2K_0 - 4K_2\right)\cosh^2\xi}{K_1K_3\cosh^2\xi - K_2K_2\sinh^2\xi}K_1\theta\,,\\ \dot\xi &= \frac{-\frac{1}{3}\left(m^2K_0 - 4K_2\right)K_2 - K_1K_3}{K_1K_3\cosh^2\xi - K_2K_2\sinh^2\xi}\theta\sinh\xi\cosh\xi\,,\\ \dot u^\mu &= -\frac{m^2K_0 - K_2}{3(m^2K_0 - 4K_2)}\tanh\xi\nabla^\mu\xi\\ &\quad+\frac{m^2K_1 - K_3}{3(m^2K_0 - 4K_2)}\nabla^\mu\beta\,.\end{aligned} \tag{50}$$

They can be further simplified using the recursive relation of K_n in Eq. (33). We choose to express K_0 and K_3 in terms of K_1 and K_2 as

$$\begin{aligned}K_0 &= \frac{1}{m^2}\left(K_2 - \frac{3}{\beta}K_1\right)\,,\\ K_3 &= \frac{3}{\beta}\left(K_2 + \frac{1}{\beta}K_1\right) + m^2K_1\,.\end{aligned} \tag{51}$$

Using these relations to substitute K_0 in $\dot\beta$, $\dot\xi$, and $\dot u^\mu$, as well as K_3 in $\dot u^\mu$, we obtain results shown in Eq. (28) in the main text.

On the other hand, the EOM of $\omega^{\mu\nu}$ is derived by inserting $T^{\mu\nu}_{\rm eq}$ and $S^{\lambda,\mu\nu}_{\rm eq}$ into the angular momentum conservation law, i.e., Eq. (29) in the main text. After a few steps of calculation, we obtain

$$\begin{aligned}m^2K_1\dot\omega^{\mu\nu}\cosh\xi = -m^2\omega^{\mu\nu}\left(\theta + \frac{d}{d\tau}\right)(K_1\cosh\xi)\\ -\partial_{x\lambda}\left[\cosh\xi\left(I^{\mu\lambda\rho}\omega^\nu{}_\rho - I^{\nu\lambda\rho}\omega^\mu{}_\rho\right)\right]\,.\end{aligned} \tag{52}$$

Since $\omega^{\mu\nu}$ can be decomposed as shown in Eq. (38), we can also decompose $\dot\omega^{\mu\nu}$ in a similar way using the fluid velocity vector u^μ,

$$\dot\omega^{\mu\nu} = \Delta^\mu_\alpha\Delta^\nu_\beta\dot\omega^{\alpha\beta} - u^\mu\dot\omega^{\nu\alpha}u_\alpha + u^\nu\dot\omega^{\mu\alpha}u_\alpha\,. \tag{53}$$

We can solve $\dot\omega^{\mu\nu}u_\nu$ (or $\Delta^\mu_\alpha\Delta^\nu_\beta\dot\omega^{\alpha\beta}$) by contracting Eq. (52) with u_ν (or with projection operators). The results are given in Eqs. (31) and (32) in the main text. In the calculations, we also used the recursive relation of K_n in Eq. (33) to simplify the coefficients. We express all coefficients in terms of K_1 and K_2 because they are respectively related to the particle number density and the energy density as $K_1 \simeq n_{\rm eq}/\sinh\xi + \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\chi_s, \chi_s^2)$ and $K_2 \simeq \epsilon_{\rm eq}/\cosh\xi + \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\,\chi_s, \chi_s^2)$.

[1] J.-H. Gao and Z.-T. Liang, Phys. Rev. **D100**, 056021 (2019), 1902.06510.

[2] N. Weickgenannt, X.-L. Sheng, E. Speranza, Q. Wang, and D. H. Rischke, Phys. Rev. **D100**, 056018 (2019), 1902.06513.

[3] S. R. De Groot, W. A. Van Leeuwen, and C. G. Van Weert, Relativistic Kinetic Theory. Principles and Applications (North-Holland, 1980).

[4] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev. D **85**, 114047 (2012), [Erratum: Phys.Rev.D 91, 039902 (2015)], 1202.4551.

[5] E. Remiddi and L. Tancredi, Nucl. Phys. B **880**, 343 (2014), 1311.3342.